A gauge-gravity toy model for heavy-ion collisions

Paul Romatschke, INT

in collaboration with D. Grumiller

arXiv:0803.3226

May 12, 2008
Outline

1. Motivation

2. Colliding shock waves in $AdS_5$

3. Numerical Gravity
1. Motivation

2. Colliding shock waves in $AdS_5$

3. Numerical Gravity

Paul Romatschke - A gauge-gravity toy model for heavy-ion collisions
Why would one want to have a gauge/gravity dual for heavy-ion collisions?
Thermalization in Weak Coupling is almost understood

Ignoring Plasma Instabilities: “Bottom-Up” Scenario (Baier, Mueller, Schiff and Son):

\[ \tau_{\text{therm}} \propto Q_s^{-1} \alpha_s^{-13/5} \]

With Plasma Instabilities: ??? (Same or faster)

Remaining Problem: don’t know \( \tau_{\text{therm}} \) when \( \alpha_s \) is not asymptotically small
Towards Thermalization (Isotropization) for Classical Yang-Mills

Towards Thermalization (Isotropization) for Classical Yang-Mills

- Direct Simulations of Classical Yang-Mills dynamics do show plasma instabilities
- Direct Simulations of Classical Yang-Mills dynamics do not show full isotropization
- Not (too) surprising: hard gluon splittings not included! (Very important in “Bottom-Up” scenario)
Gauge gravity duality allows direct study of strong coupling dynamics in (unphysical) gauge theories.

Thermalization corresponds to black hole formation.

Idea: study thermalization of $\mathcal{N} = 4$ SYM by investigating black hole formation in $AdS_5$. 

Thermalization in Strong Coupling
Why this could work: Janik + Peschanski 05 show that regular solution of boost-invariant classical gravity in $AdS_5$ is ideal hydro solution (thermal)

Also pose problem: starting with two pure shock waves

$$T^{++} = \mu_1 \delta(x^-), \quad T^{--} = \mu_2 \delta(x^+)$$

what is the full solution of $T^{\mu\nu}$ connecting to late time (ideal hydro) behavior?
Outline

1 Motivation

2 Colliding shock waves in $AdS_5$

3 Numerical Gravity
Motivation
Colliding shock waves in $AdS_5$
Numerical Gravity

Setup

- Single shock wave in $\mathcal{N} = 4$ SYM corresponds to $AdS_5$ line element

$$ds^2 = \frac{-2dx^+dx^- + \mu_1 z^4 \delta(x^-)dx^-}{z^2} + dx_\perp^2 + dz^2$$

- This is an exact solution to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 6g_{\mu\nu} = 0$$

- Two shock waves are exact solution before they collide
First try: solve Einstein equations for

\[ ds^2 = \frac{-2dx^+ dx^- + \mu_1 z^4 \delta(x^-) dx^-^2 + \mu_2 z^4 \delta(x^+) dx^+^2 + dx_{\perp}^2 + dz^2}{z^2} \]

around \( x^\pm = 0 \)

Difficult: if metric is \( C^{-1} \) because of distributions \( \delta(x^\pm) \) then Ricci is \( C^{-3} \)
First Problem Solved

Can do coordinate transform to Rosen-coordinates (Payne and D’Eath ’92) of one shock

\[ x^+ = u + \frac{1}{2} \mu_1 \tilde{z}^4 \theta(v), \quad x^- = v \]

\[ z = \tilde{z} + 2\mu_1 \tilde{z}^3 \theta(v) \]

Find

\[ ds^2 = -2dudv + dx_\perp^2 + \frac{[1 + 6\mu_1 \tilde{z}^2 v \theta(v)]^2 d\tilde{z}^2}{\tilde{z}^2[1 + 2\mu_1 \tilde{z}^2 v \theta(v)]^2} \]

which is \( C^1 \), but not Fefferman-Graham!
**Ansatz**

- Make ansatz for line element:

\[
\begin{align*}
\text{ds}^2 &= -\kappa \tilde{\tau}^2 + \lambda \tilde{\eta}^2 + \xi \text{d}x_\perp^2 + \nu \text{d}\tilde{z}^2 \\
&= \frac{-\kappa \tilde{\tau}^2 + \lambda \tilde{\eta}^2 + \xi \text{d}x_\perp^2 + \nu \text{d}\tilde{z}^2}{\tilde{z}^2[1 + 2\tilde{z}^2 \mu \tilde{\tau} \cosh(Y - \tilde{\eta})]^2}
\end{align*}
\]

in Milne coordinates $\tilde{\tau} = \sqrt{2}uv$, $\tilde{\eta} = \frac{1}{2} \ln \frac{u}{v}$, and $\mu = \sqrt{2\mu_1 \mu_2}$, $Y = \frac{1}{2} \ln \frac{\mu_1}{\mu_2}$

- $\kappa, \lambda, \xi, \nu$ and their derivative are known at $\tau = 0$

- Try to obtain early time behavior: power series ansatz for $\kappa, \lambda, \xi, \nu$
Early time solution

- Find solution of full Einstein equations to arbitrary power in \( \tilde{\tau} \) (exact in \( \tilde{z} \)), e.g.

\[
\kappa = 1 + c_1 \mu^2 \tilde{\tau}^2 \tilde{z}^4 + \frac{182 + 10c_1}{3} \mu^3 \tilde{\tau}^3 \tilde{z}^6 \cosh (Y - \tilde{\eta}) + O(\tilde{\tau}^4)
\]

where \( c_1 \) is constant of integration

- For early times, this is exact solution of \( AdS_5 \) line element

- To extract boundary \( T^{\mu\nu} \), need to do holographic renormalization
Holographic renormalization

- Have to bring metric back to Fefferman-Graham (FG) form
- Coordinate transform:

\[ \tilde{\tau} = \tau + \sum_{n=0}^{\infty} t_n(\tau, \eta) z^{4+2n} \]

(similarly for \( \tilde{\eta}, \tilde{z} \)) and determine \( t_n \) by requiring transformed metric to be FG
- Once metric is FG, can read off \( T^{\mu\nu} \) as coefficient of \( z^2 \) term
Motivation
Colliding shock waves in $\text{AdS}_5$
Numerical Gravity

Colliding shock waves in $\text{AdS}_5$: Early times

- Find result for $T^{\mu \nu}$:
  
  \[
  T_{\tau \tau} = \mu^2 \tau^2 - 3\mu^3 \tau^5 \cosh (Y - \eta) + \ldots
  \]
  
  \[
  \tau^{-2} T_{\eta \eta} = -3\mu^2 \tau^2 + 21\mu^3 \tau^5 \cosh (Y - \eta) + \ldots
  \]
  
  \[
  T_{xx} = T_{yy} = 2\mu^2 \tau^2 - 12\mu^3 \tau^5 \cosh (Y - \eta)
  \]

- Note: $T^{\mu \nu}$ does not depend on $c_1, \ldots$ and obeys
  
  $T^{\mu}_{\mu} = 0, \nabla_\mu T^{\mu \nu} = 0$

- Is not boost-invariant!
Why is boost-invariance violated?

Relevant part of line element before collision

\[ z^2 ds^2 \sim \delta(x^-)dx^{-2} + \delta(x^+)dx^{+2} \]

Note \( x^\pm = \tau e^{\pm \eta}/\sqrt{2} \) so line element is not boost-invariant to begin with!

For boost-invariant line element would need \( \delta'(x^-) \) or \( \delta(x^-)/x^- \) => not clear how to define!
Energy conservation

- Can also calculate $T^{\mu\nu}$ in $x^{\pm}$ or $t, z$ coordinates (to avoid problem with $\tau = 0$ coordinate singularity)
- Find

$$T^{00}(t, z) = \frac{\bar{\mu}(\delta(x^+)+\delta(x^-))}{2} - 2\bar{\mu}^2 \theta(x^+)\theta(x^-) \left( x^2 + x^{-2} - 4x^+x^- \right)$$

- Total energy

$$E(t) = \int_{-\infty}^{\infty} dz T^{00}(t, z) = \sqrt{2\bar{\mu}}$$

is conserved as it must be!
Colliding shock waves in $\text{AdS}_5$: $T_{\mu\nu}$ at early times

D. Grumiller+PR, arxiv:0803.3226
By inspecting power series for metric coefficients: for

\[ \tilde{z}^2 \sim \mu^{-1} \tilde{\tau}^{-1} \]

all terms become of same order. Radius of convergence?

Black hole formation is non-perturbative: expect power series to break down at horizon

Identify \( z \sim \mu^{-1/2} \tau^{-1/2} \) with BH horizon: how long until BH creation felt at boundary \( z = 0 \)?

\[ \tau_{\text{therm}} \sim \mu^{-1/3} \]
Outline

1. Motivation
2. Colliding shock waves in $AdS_5$
3. Numerical Gravity
Setup

- 5d Einstein-Hilbert action

\[ I = \int dx^5 \sqrt{-g^{(5)}} (R - 2\Lambda) \]

- Using symmetry to reduce to 3 dimensions

\[ I = V_2 \int dx^3 \sqrt{-g^{(3)}} \left( R^{(3)} - \frac{3}{2}(\nabla \phi)^2 - 2\Lambda e^{-2\phi} \right) \]

where \( \phi \) is a KK field
Use ADM 2+1 slicing: degrees of freedom are $\gamma_{ij}$, $K_{ij}$ and matter $\phi$, $\psi$. Gauge choice: $N$

\begin{align*}
\partial_{\tau} \gamma_{ij} &= -2NK_{ij}, \quad \partial_{\tau} \phi = \frac{N}{3\sqrt{\gamma}} \psi \\
\partial_{\tau} K_{ij} &= -D_i D_j N + N \left( R^{(2)}_{ij} + KK_{ij} - 2K_{ik}K_{j}^{k} \right) - N \left( \Xi_{ij} - \gamma_{ij}\Xi \right) \\
\partial_{\tau} \psi &= \sqrt{\gamma} N \left( 3D^2 \phi + 3\gamma^{ij} D_i \phi D_j \ln N + 4\Lambda e^{-2\phi} \right)
\end{align*}
Numerical Gravity Results (preliminary)
Summary

- Collisions of shock waves in $AdS_5$ may be an interesting toy model of heavy-ion collisions
- Studying black hole formation in $AdS_5$ may give new insights to thermalization in field theories
- Outlook: full numerical solutions may be possible