Topological entanglement, holography and confinement

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Outline

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• Topological entanglement entropy in 2+1 dimension from holography
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Topological order

There are interesting systems in 2+1 dimensions whose phases are not distinguished by local order parameters. Examples: Fractional Quantum Hall Effect states, discrete gauge theories.

The ground state might have topological order: long range correlations in the fields. These properties are reflected in the degeneracy of the ground state as a function of genus.

One way to distinguish phases/measure topological order it to compute topological entanglement entropy. [Levin, Wen; Kitaev, Preskill]
Entanglement entropy

Divide the system into subsystems A and B. Consider state described by a density matrix $\rho$. Let $\rho_A = \text{tr}_B \rho$. Entanglement entropy = Van Neumann entropy $S_A = -\text{tr} \rho_A \log \rho_A$. Example:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|1\rangle_A |1\rangle_B + |2\rangle_A |2\rangle_B)$$

For this state entanglement entropy $S_A = \log 2$. Measures entanglement between A and B.
Entanglement entropy: properties

In QFT on $\mathbb{R}^{d,1}$ A and B are complimentary subspaces of $\mathbb{R}^d$. For pure states we have complementarity and strong subadditivity

$$S_A = S_B,$$

$$S_A + S_B \geq S_{A \cup B} + S_{A \cap B}$$

For the subspace of large volume, $S_A$ is proportional to the area $S_A \approx Area(\partial A)/a^{d-1}$ where $a$ is the short distance cutoff.

For thermal state, $S_A \approx Volume(A)s_{thermal}$. 
Consider \((S_1 - S_2) - (S_3 - S_4) = -2\gamma\). [Levin, Wen]

The value of topological entropy \(\gamma\) can also be computed via the constant term in the entanglement entropy of a disk whose radius \(R \to \infty\):

\[
S = xR - \gamma + \ldots
\]
Consider $N$ D3 branes on a circle of radius $R_3$ at large t’Hooft coupling $\lambda = G_{YM}^2 N$. The metric of the gravity dual

$$ds^2 = L^2 \left[ \frac{dz^2}{z^2 h(z)} + \frac{dx_\mu dx^\mu}{z^2} + h(z) \frac{dx_3^2}{z^2} + d\Omega_5^2 \right]$$

where $\mu = 0, 1, 2$, $h(z) = 1 - z/2R_3$, $L^4 \sim \lambda$. Finite $R_3$ gives rise to strongly coupled 2+1 dimensional theory with confinement and a mass gap (the scale of the correlation length is set by $R_3$).
According to Ryu and Takayanagi, the value of entanglement entropy between a region and its complement is obtained by evaluating

\[ S_A = \frac{1}{4G^{(10)}_N} \int_\Gamma d^d\sigma \sqrt{G^{(8)}_{ind}} \]

on the minimal 8-dimensional hypersurface \( \Gamma \) at \( t = \text{const} \) which asymptotes to the border between the region and its complement at \( z = 0 \).
D3 branes: conformal case

Consider, as a warm-up exercise, conformal case:
$R_3 \to \infty$, $h(z) = 1$.

$z(r)$. Red curve [$z'(r = 0) = 0$] corresponds to the cylinder $x_1^2 + x_2^2 = R^2$ on the boundary at $z = 0$. Blue curve gives rise to two concentric cylinders at $z = 0$. 

D3 branes: conformal case

Action with parameterization $z(r)$:

$$S = \frac{4N_c^2 l}{15\pi} \int dr \, \frac{r}{z^3} \sqrt{1 + (z')^2}$$

Equation of motion:

$$\frac{d}{dr} \left( \frac{rz'}{z^3 \sqrt{1 + (z')^2}} \right) = -\frac{3r \sqrt{1 + (z')^2}}{z^4}$$

Near the boundary

$$z \simeq 2\sqrt{R} \sqrt{R - r}$$
D3 branes: conformal case

To compute entanglement entropy for the cylinder $x_1 + x^2 \leq R^2$ we need to subtract the divergent part. Solve equations of motion, and evaluate the action. The integral is cut off at $z = a$.

$$S = \frac{2N_c^2}{15\pi} \left( \frac{lR}{a^2} + \frac{l}{4R} \log \frac{a}{R} \right) - \frac{4N_c^2 l}{15\pi R} \tilde{\gamma}$$

where $l$ is the length of the cylinder. This gives $\tilde{\gamma} = 0.305$. This is similar to topological entanglement entropy, but is computed in the theory with infinite correlation length.
D3 branes on a circle

Consider finite correlation length $R_3$. rescale coordinates $z \rightarrow z/2R_3$

The space ends at $z = 1$. There are two solutions which asymptote to the circle at $z = 0$: disk topology (blue) and cylinder topology (red), where Kaluza-Klein circle shrinks to zero size.
D3 branes on a circle

Action

\[ S = \frac{4N_c^2}{15} \int_{z_0}^{R} dr \frac{r}{z^3} \sqrt{1 - z^4 + (z')^2} \]

Equation of motion

\[ \frac{d}{dr} \left( \frac{r z'}{z^3 \sqrt{1 - z^4 + (z')^2}} \right) = \frac{r(z^4 - 3[1 + (z')^2])}{z^4 \sqrt{1 - z^4 + (z')^2}} \]

Same small \( z \) behavior as in the conformal case
D3 branes on a circle

There are also other solutions, which asymptote to the annulus at \( z = 0 \). For \( R \ll R_3 \) the structure is similar to the conformal case. To compute entanglement entropy, we need to extract the UV-divergent part:

\[
S = \frac{4N_c^2}{15} \left( \frac{RR_3}{a^2} + \frac{R_3}{4R} \log \frac{a}{R} \right) + \frac{4N_c^2}{15} \tilde{S}
\]

Note that topological entropy is encoded in \( \tilde{S} \). In the conformal case hypersurface of disk topology gave non-vanishing \( \tilde{S} \)
D3 branes on a circle

\[ \tilde{S}(R) \]. Disk (cylinder) topology—blue (red) curve. For small (large) \( R \) disk-type (cylinder-type) solutions dominate the computation of entanglement entropy. For large \( R \), \( \tilde{S} = -R/4R_3 + \mathcal{O}(R^{-1}) \) which implies vanishing topological entropy!
D3 branes on a circle

Remarks:

- $\gamma = 0$ expected for QCD-like theory
- Phase transition picture, similar to the slab geometry case studied by Klebanov, Kutasov and Murugan, has a classic “swallowtail” shape.
- Phase transition is important to ensure $\gamma = 0$.
- The result can be viewed as a consistency check for holographic prescription and topological entanglement entropy.
- On both sides of the transition $S \sim N_c^2$
D3 branes on a circle

More remarks:

• In the vicinity of $R_c$, where the disk-type and cylinder-type solutions join together, there are two solutions of both topologies for a fixed $R$

• $R_c$ corresponds to the disk solution which starts with $z'(r = 0) = 0$ at $z_* \to 1$ and to the cylinder solution which starts at $r(z = 1) \to 0$.

• Near $R = R_c$ there is a self-similarity structure which we have not studied in detail.
D4 branes on a circle

Consider now D4 branes compactified on a circle of radius $R_4$. This theory reduces to 3+1 dimensional theory at low energies. Has confinement, mass gap—often used to model QCD. Gravity is a good description when $\lambda/R_4 \gg 1$

The UV behavior is very different from asymptotic freedom. Entanglement entropy is sensitive to the UV behavior!
D4 branes on a circle

Compute entanglement entropy for a ball in $R^3$. The phase structure similar to the 2+1 dimensional model. There is again 1st order phase transition.

\[ S = \lambda N_c^2 R^2 R_4 \left( \frac{\pi^2 \Lambda^4}{2} - \frac{1}{3^4 R_4^4} \right) + \gamma \]

\[ \gamma = -\frac{\lambda N_c^2}{144 R_4} \left( \frac{\pi}{3} + 2 + \log 3 - 9\pi \Lambda^2 R_4^2 \right) \]

Leading term behaves like $R^2$; Constant term at large $R$ is cutoff-dependent!
Conclusion/open problems

- Holographic prescription is consistent with vanishing topological entanglement entropy of a QCD-like theory.
- The structure of the phase transitions is important for this result.
- Many checks of the holographic prescription. Can it be [more or less] rigorously proven?
- Theories with nonvanishing topological entanglement entropy?
- Analogs in higher then 2+1 dimensions?