How similar are QCD and $\mathcal{N}=4$ SYM?

Work with Simon Caron-Huot, Kovtun, Starinets, Yaffe

- Value of Perturbative approach
- Leading-order comparisons:
  - Photon production
  - Shear viscosity
  - Heavy quarks
- Next-to-leading order comparison
- Conclusions
Analogy: $\mathcal{N}=4$ SYM and large $N_f$ QCD

Large $N_f$: QCD, playing with degrees of freedom

$N_f \to \infty$ with $N_c$, $g^2 N_c N_f [\sim m_D^2]$ fixed


**Bad**: lost asymptotic freedom, most nonabelian physics

$\mathcal{N}=4$ SYM: playing with degrees of freedom and $N_c$

3 Dirac fundamental $\to$ 4 Weyl adjoint, 6 real adjoint scalars
AND send $N_c \to \infty$, $g^2 N_c [\sim m_D^2]$ fixed [Harmless?]  

**Good**: solvable [mostly] at strong coupling

**Bad**: lost asymptotic freedom, other corrections unknown
Large $N_f$: what we learn

Can exactly solve for transport coeff, eg viscosity:

Suggests strong coupling $\frac{\eta}{s} \sim 0.4$
Large $N_f$: what we learn

Thermo also exactly solvable:

Corrections to free theory small ($N_f$ suppressed)
Weak coupling: right sign
Strong coupling: wrong sign

Seems to miss completely phase transition dynamics.
Wrong-sign corrections to $P$ strange and troubling.
$\mathcal{N}=4$ SYM: thermodynamics

Weak coupling: quite similar

Strong coupling: SYM misses phase transition....
\( \mathcal{N}=4 \) SYM: dynamics

Strong coupling: \( \mathcal{N}=4 \) SYM makes nice predictions:

- Shear viscosity \( \eta/s = 1/4\pi \), hep-th/0104066, hep-th/0405231
- Bulk viscosity \( \zeta = 0 \)
- Light “baryon” diffusion \( D = 1/2\pi T \) hep-th/0205052
- Photon and dilepton production hep-th/0607237

Comparison with QCD: if I knew, I wouldn’t be bothering
$\mathcal{N}=4$ SYM: dynamics

We *CAN* make the comparison at weak coupling. Perturbative approach valid, similar in both theories

If behaviors similar: gives us hope

If behaviors very different: $\mathcal{N}=4$ SYM at best a loose analog

Worth it to compare at weak coupling!
Weak coupling: shear viscosity

Comparison: terrible, equal $\lambda$. Good at equal $m_D^2$, IF scale by

Casimir ratio: $C_{\text{avg}}^{-1} \equiv \left( C_{\text{matt}}^{-1} g_{\text{matt}} + C_{a}^{-1} g_{\text{adj}} \right) / g_*$
Weak coupling: other quantities

Heavy quark diffusion: Vuorinen Chesler hep-ph/0607148

Diffusion constants different unless rescale by $m_D^2$:

$$\frac{D_{\text{QCD}}}{D_{\text{SYM}}} \rightarrow \frac{6N_c}{N_c + N_f/2}$$

Photon production Caron-Huot, Kvitun GM Starinets Yaffe hep-th/0607237

Factor $m_D^2$
Casimir scaling
systematic differences
Weak coupling: NLO comparison

Above may be deceptive: all probe same elastic scatt rate
Stronger coupling: that may not dominate or may change
Would be good to test at least 1 thing at NLO
Also valuable to know reliability of pert expansion

We should compute at least 1 thing at NLO
Quantities known at leading order:

- **Shear viscosity**

- **Bulk viscosity**  QCD: Arnold,Dogan,GM PRD 74:085021 2006. \( \mathcal{N}=4 \) SYM: \( \zeta = 0 \)

- **Photon production**

- **Light quark diffusion**  QCD: AMY, JHEP 0305:051 2003. \( \mathcal{N}=4 \) SYM: no one

- **Heavy quark diffusion and drag**
  \( \mathcal{N}=4 \) SYM: Chesler Vuorinen hep-ph/0607148
Quantities known at next-to-leading order (NLO):

Besides this work
One transport coefficient is simple

Heavy quarks: one particle just sits there

Heavy quark kinematics always same
4-momentum transfer: energy = 0
Radiation of no importance (at LO)
Only $\int$ over $P, Q$

Maybe we can do this one!
Scattering in vacuum

Coulomb potential: \( V \sim \alpha/r \) “conformal”

Phase shifts: equal \( \sim \alpha^2 \) scatt prob, \textbf{EVERY} partial wave

Total scattering rate

\[
\sigma \sim \frac{1}{E} \int \frac{d^2 q_\perp}{(2\pi)^2} \frac{g^4}{q_\perp^4}
\]

is IR (long range) \textit{QUADRATIC} divergent

Momentum diffusion goes as \( q_\perp^2 \)

\[
\kappa \equiv \int_Q d\sigma(Q) \ q_x^2 \sim \frac{1}{E} \int \frac{d^2 q_\perp}{(2\pi)^2} \frac{g^4 q_\perp^2}{q_\perp^4}
\]

“only” log divergent.
Scattering in medium

Other charges in medium cluster around quark, “screen” charge

Cross section reduced
Amount a function of radius
Roughly \( V \sim \alpha e^{-m_D r / r} \)

Momentum space: cross section becomes

\[
\sigma \sim \frac{1}{E} \int \frac{d^2 q_\perp}{(2\pi)^2} \frac{g^4}{(q^2_\perp + m^2_D)^2}
\]

Everything now finite and well behaved.
Leading order heavy quark

Just that—integrated over species in plasma, incl spin dependence of vertex (for $P \sim Q$)

$$\kappa^{LO} \equiv \frac{g^4 C_r}{12\pi^3} \int_0^\infty q^2 dq \int_0^{2q} \frac{p^3 dp}{(p^2 + m_D^2)^2}$$

$$\times \left\{ \begin{array}{l} N_f n_F(q)(1-n_F(q)) \left(2 - \frac{p^2}{2q^2}\right) \\ + N_c n_B(q)(1+n_B(q)) \left(2 - \frac{p^2}{q^2} + \frac{p^4}{4q^4}\right) \end{array} \right\}.$$ 

$$\approx \frac{C_r g^4 T^3}{18\pi} \left[ N_c \left( \ln \frac{2T}{m_D} + \xi \right) + \frac{N_f}{2} \left( \ln \frac{4T}{m_D} + \xi \right) \right]$$

with $\xi = \frac{1}{2} - \gamma_E + \frac{\zeta'(2)}{\zeta(2)} = -0.64718$. 

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Why there are $\mathcal{O}(g)$ NLO corrections

Strength of screening:
Any bosonic colored species: many soft-momentum charges.

$$n_{\text{glue}} \sim \int \frac{d^3p}{(2\pi)^3} n_B(p) \sim T^3 : \quad n_{\text{soft}} \sim \int_0^T \frac{d^3p}{(2\pi)^3} \frac{T}{p} \sim g^2 T^3$$

Small $p$ means easy to “bend”: deflection $\sim 1/p$.
$\mathcal{O}(g)$ fraction of Debye screening done by soft bosons
$\mathcal{O}(g)$ fraction of scatterings are with soft bosons
But these bosons themselves have $\mathcal{O}(1)$ dispersion shifts!
Example: scalar QED

Dispersion relation \((p^0)^2 = p^2 + m^2, m^2 \sim g^2 T^2\). Soft scalars \(p \sim gT\) large distortion. Two corrections:

- Screening: “massive” soft scalars screen less.
  \(mD^2 \sim g^2 T^2 + g^3 T^3\) correction

- Scattering: “slow” soft scalars scatter less.
  Scattering rate \(\sim g^4\) has \(\sim g^5\) correction

NLO corrections straightforward to calculate. [but we haven’t]
Nonabelian $\mathcal{O}(g)$ corrections

Color exchange:
In QCD, scattering can change particle color:

Color rotation occurs with each scattering
Rate of scattering $\Gamma \sim g^2 T$
Size of Debye sphere $\sim 1/gT$
Duration of scattering $\sim 1/gT$

Fraction of “overlapping scatterings” is therefore $\mathcal{O}(g)$.
Having color rotate in middle of scattering is big correction.
Nonabelian $O(g)$ corrections

Multiple scatterings again: Interference between

Diagram 1

Diagram 2

Diagram 3

Diagram 4

Diagram 5

Diagram 6

Diagram 7

where each vertex can be “clothed” by hard-screening

This is a mess. We need systematic approach!
Systematic approach

Heavy quark feels time varying force:

\[ \vec{p}_{\text{heavy}}(t) = \int_0^t dt' \vec{F}(t') \]

Momentum diffusion: mean squared accumulated momentum

\[ \kappa \equiv \frac{1}{3} \lim_{t \to \infty} \langle (p(t) - p(0))^2 \rangle = \frac{1}{t} \int_0^t dt' \int_0^t dt'' \vec{F}(t') \cdot \vec{F}(t'') \]

\[ \simeq \int_{-\infty}^\infty dt' \vec{F}(t') \cdot \vec{F}(0) \]

Force caused by electric field. Color: Wilson line

\[ \kappa = \frac{g^2}{3d_r} \int_{-\infty}^\infty dt \ Tr \ \langle W(-\infty, t) E_i^a t^a W(t, 0) E_i^b t^b W(0, -\infty) \rangle \]
Schwinger-Keldysh picture:

Wilson line along whole Schwinger-Keldysh contour ($\vec{x}$ fixed)

$E$ field (bend) inserted on each branch: $\int$ over $t$ difference

Not same as adjoint line (presence of quark,...)
Systematic approach

Right effective description:

• Hard Thermal Loop Effective Theory:
  * Systematically include self-energies on gauge bosons
  * Systematically add effective vertices to normal ones

• Real-time Schwinger-Keldysh method
  * $t$ from $-\infty \rightarrow \infty$ [1] and back [2]
  * Fields: 2 copies, [1] and [2]
  * Use average $r$ and difference $a$ variables:
    $G_{rr}$ correlation func, $G_{ra}$ retarded func $G_{aa} = 0$. 
Needed diagrams and rules

Double line: Wilson line:

• Integrate over times of vertices, but in time order shown
• All at same spatial point
• Vertices on ends: $p_i A_0$
• Vertices in middle: $A_0$
Rules: lines

Line rule:  \[ \ldots \quad = \quad \rightarrow \quad + \quad \leftarrow \quad + \quad \rightleftarrows \ ... \]

Arrow: timeflow for retarded. Doublebar: cut line

\[
G_{\mu \nu}^R(Q) = \frac{i\delta^\mu_0 \delta^\nu_0}{q^2 + \Pi^{00}(Q)} + \frac{-i(\delta^{ij} - \hat{q}^i \hat{q}^j)}{Q^2 + \Pi_T(Q)}
\]

\[
G_{rr}(Q) = \frac{T}{q^0}(G_R(Q) - G_R(-Q)) \quad \text{and} \quad \text{(defining} \quad \eta = q^0/q)\]

\[
\Pi^{00}(Q) = m_D^2 \left[ 1 - \frac{\eta}{2} \left( \ln \frac{|1+\eta|}{|1-\eta|} + i\pi \Theta(1-\eta^2) \right) \right]
\]

\[
\Pi_T(Q) = m_D^2 \left[ \frac{\eta^2}{2} + \frac{\eta(1-\eta^2)}{4} \left( \ln \frac{|1+\eta|}{|1-\eta|} + i\pi \Theta(1-\eta^2) \right) \right]
\]
**Vertices:** tree (1 outgoing line); Induced (HTL):

\[ J \text{ induces fields} \]

\[ J \text{ rotates as propagates} \]

\[ E \text{ field induces current } j \]

Beginning: \( ip^0 v^\mu \). Middle: \( -v^\mu f_{abc} \). End: \( iv^\mu \).

Hard line propagators: \( -i/v \cdot P^- \).

Additional \( m_D^2 \int \frac{d\Omega_v}{4\pi} \)

“current fluctuation”: \( 2\pi \delta(v \cdot P) \) propagator.
Details of calculation

Horrible. Example: diagram

2 contributions: real part of loop and imaginary part.

Real part: shift in screening \( m_D^2 \rightarrow m_D^2 + \delta m_D^2(Q) \)

\[
\delta m_D^2(Q) = -\frac{g^2 N_c T}{2\pi} \left[ m_D + \frac{q^2 - m_D^2}{q} \tan^{-1} \frac{q}{m_D} \right]
\]

reduction in screening: soft fields are less efficient. Calculable by Euclidean methods. “Easy”
Four distinct kinds of cuts:

For each cut, other contributions still complex (contour...)
4 distinct polarization contributions (LL, LT, two TT’s)
Some integrations easy. 4 are hard ($p$, $q$, $\theta_{pq}$, $\omega$)
All numeric integrals, some not absolutely convergent
Expressions for this diagram

\[ C_{(A), \text{main}} = 3\pi \int_p \frac{p^2}{(1 + p^2)^2} \int_Q \sum_i P_i \]
\[ \times \left\{ \rho^{Q_i}(Q)\rho^{R_i}(-R) \left[ (\Re M_i - \Re K_i)^2 - (\Im K_i)^2 \right] - (\Im K_{R_i})^2 + (\Re K_{i})^2 \right\} \]
\[ + 4\Im G^{Q_i}(Q)\rho^{R_i}(R) \left[ (\Re M_i - \Re K_i)\Im K_{Q_i} + \Re K_i \Im K_{R_i} \right] \]
\[ + 4\Re G^{Q_i}(Q)\Im G^{R_i}(R) \left[ (\Re M_i - \Re K_i)\Im K_{Q_i} + \Re K_i \Im K_{Q_i} \right] \]
\[ + 8\Re G^{Q_i}(Q)\Re G^{R_i}(R) \left[ (\Re M_i - \Re K_i)\Re K_{Q_i} + \Im K_{Q_i} \Re K_{R_i} \right] \]

\[ C_{(A), q^0 = 0} = 24\pi^2 \int_p \frac{p}{(1 + p^2)^2} \int_q \frac{p \cdot q}{q^4(r^2 + 1)} \]

\[ C_{(A), q^0 = q_\perp} = 3\pi^4 \int_p \frac{p^2}{(1 + p^2)^2} \int_Q \frac{\delta(q^0 - q_\perp)}{p^2 q_\perp} \sum_i P_i X_i^2 \]
\[ \times \left\{ \theta(-p \cdot q) - \theta(-p \cdot r) \left[ \Im G^{Q_i}(Q)\rho^{R_i}(R) + \rho^{Q_i}(Q)\Re G^{R_i}(R) \right] \right. \]
\[ + \theta(p \cdot q)\theta(p \cdot r) \left[ \Im G^{Q_i}(Q)\rho^{R_i}(R) - \rho^{Q_i}(Q)\Re G^{R_i}(R) \right] \]
\[
\Re M_i - \Re K_i \equiv -X_i \frac{\tan^{-1} \left( p \sqrt{\frac{q^2 - q_0^2}{p \cdot q}} \right) + \tan^{-1} \left( p \sqrt{\frac{q^2 - q_0^2}{p \cdot r}} \right)}{p \sqrt{q_\perp^2 - q_0^2}} - \frac{\pi}{2}
\]

\[
- Y_{Q_i} \frac{1}{2q} \ln \left( \frac{q + q^0}{q - q^0} \right) - Y_{R_i} \frac{1}{2r} \ln \left( \frac{r - q^0}{r + q^0} \right) + Z_i,
\]

\[
\Im K_{Q_i} \equiv Y_{Q_i} \frac{\pi}{2q}, \quad \Im K_{R_i} \equiv Y_{R_i} \frac{\pi}{2r}, \quad \Re K_i \equiv -X_i \frac{\pi}{2p \sqrt{q_\perp^2 - q_0^2}} \quad |q^0| < q_\perp
\]

\[
\Re M_i - \Re K_i \equiv -X_i \frac{\ln \left( \frac{|p \cdot q + \sqrt{q_0^2 - q_\perp^2}|}{|p \cdot q - \sqrt{q_0^2 - q_\perp^2}|} \right)}{2p \sqrt{q_0^2 - q_\perp^2}} - \frac{\pi}{2p \sqrt{q_0^2 - q_\perp^2}}
\]

\[
- Y_{Q_i} \frac{1}{2q} \ln \left( \frac{|q^0 + q|}{|q^0 - q|} \right) - Y_{R_i} \frac{1}{2r} \ln \left( \frac{|q^0 - r|}{|q^0 + r|} \right) + Z_i,
\]

\[
\Im K_{Q_i} \equiv X_i \frac{\pi \text{ sign}(p \cdot q)}{2p \sqrt{q_0^2 - q_\perp^2}} + Y_{Q_i} \frac{\pi}{2q} \theta(q^2 - q_0^2), \quad \Re K_i \equiv 0
\]

\[
\Im K_{R_i} \equiv -X_i \frac{\pi \text{ sign}(p \cdot r)}{2p \sqrt{q_0^2 - q_\perp^2}} + Y_{R_i} \frac{\pi}{2r} \theta(r^2 - q_0^2), \quad |q^0| > q_\perp
\]
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Results

NLO corrections look like $N_c g^2 T / m_D$:

$$\kappa_{NLO} = \frac{C_r g^4 T^3}{18 \pi} \left[ \left( N_c + \frac{N_f}{2} \right) \ln \frac{2 T e^\xi}{m_D} + \frac{N_f \ln 2}{2} + \frac{N_c m_D}{T} C \right]$$

where in 3-flavor, 3 color QCD, $C = 2.3302$. Of this:

- $\sim 1/3$ is from NLO evaluation of (simple) LO expression
- $\sim 1/3$ is from (simple) real Debye mass correction
- $\sim 1/3$ is from really complicated stuff!!
Coupling dependence: 3-flavor QCD

Perturbative expansion looks very bad.
What about $\mathcal{N}=4$ SYM?

Additional contributions:

- New Leading Order diagrams: direct scalar interactions

- Scalar participation in HTL’s. Soft scalar corrections.
\[
\kappa_{\text{SYM}} = \frac{\lambda^2 T^3}{6\pi} \left( \ln \frac{2T}{m_D} + \xi + \frac{1}{2} + \frac{\ln 2}{3} + \frac{\sqrt{2\lambda}}{6} C_{\text{SYM}} \right),
\]

\[C_{\text{SYM}} = 3.3984\] barely bigger than in QCD.
Comparison: QCD vs $\mathcal{N}=4$ SYM

Leading order: most similar expressed in terms of $m_D$:

QCD: $\kappa = \frac{C_F g^2 T m_D^2}{6\pi} \left( \ln \frac{T}{m_D} + 0.27701 + 1.5406 \frac{m_D}{T} \right)$

SYM: $\kappa = \frac{C_F g^2 T m_D^2}{6\pi} \left( \ln \frac{T}{m_D} + 0.7770 + 0.5664 \frac{m_D}{T} \right)$

NLO correction much ($3 \times$) bigger in QCD.

Larger even at equal values of $\lambda$ t’Hooft coupling.
Why??

Perturbatively, there are 3 scales:

- $T$ [most particles. Free propagation]
- $gT$ [screening and scattering processes]
- $g^2T$ [gluons only. Nonperturbative scale]

$gT$ larger in SYM due to many species [scalars+fermions]

- $T$, $gT$ overlap: propagation modified [SYM]
- $gT$, $g^2T$ overlap: scatt+screen are nonpert. [QCD]

Theories may be quite different
Conclusions

Compare QCD, SYM where you can: weak coupling.

- Leading order: many measurables say, match $m_D$ scales
  Reason: all depend mostly on Coulomb scattering!

- NLO: sensitive to sticky, gluey nonlinear physics

- Heavy quark diffusion: NLO can be done!

- Result: NLO corrections in QCD MUCH ($3\times$) bigger.

QCD, SYM not so much alike as we hoped!