Relativistic Field Theories in a Magnetic Background as Noncommutative Field Theories

E. Gorbar and V. M., Phys. Rev. D70, 105007 (2004); hep-th/0407219;

$[\mathbf{a}, \mathbf{B}]=i\mathbf{a}\mathbf{B}$, $\mathbf{a}\mathbf{B}=\frac{\mathbf{a}\mathbf{B}}{e\mathbf{B}}$

Magnetars: $B\sim 10^{12} - 10^{16}$ G
$1281 \sim 10^{3} M_{\odot}$
RFT in a magnetic background in spatial dimensions d>2 determine NCFT which are different from those considered in the literature.

The reason of that is an inner structure (i.e., dynamical form-factors) of neutral composites in these NCFT.

Nonlocal NCFT.

These NCFT are self-consistent. There is no UV/IR mixing. There exists well defined commutative limit.

NJL models

For a magnetic field with the maximal number of independent nonzero tensor component.
the NCFT are finite for even \(\mathcal{B}\) and their dynamics are quasi-\((1+1)\)-dimensional for odd \(\mathcal{B}\).

As all \(1\to\infty\), the theories are reduced either to a continuum set of \((1+1)\)-dim Gross-Neveu models labeled by \(d-1\) spatial coordinates (odd \(d\)) or to a set of quantum mechanical models labeled by \(d\) spatial coordinates (even \(d\)).
Magnetic catalysis phenomenon

A constant magnetic field is a strong catalyst of dynamical chiral symmetry breaking, leading to the generation of a fermion dynamical mass even at the weakest attractive interaction between fermions.


$D = 3 + 1$ and $D = 2 + 1$ cases.

Extension to $D = d + 1$, $d > 3$ in?

NJL Model in a Magnetic Field as NCFT: Effective Action

\[ L = \overline{\chi} i \gamma^\mu D_\mu \chi + \frac{g^2}{2N} \left( \overline{\chi} \gamma^5 \chi \right)^2 + \left( \overline{\chi} i \gamma^5 \chi \right)^2 \]

\( N \) is the number of "colors".

\( D_\mu = \partial_\mu - ie A^\mu \)

\( A^\mu = (0, \frac{\pi}{2}, \frac{\pi}{2}, 0) \)

Neutral composites ("dipoles"):

\( 0 \sim \overline{\chi} \chi, \quad 5 \sim i \overline{\chi} \gamma^5 \chi \)

\( \Gamma(0, 5) = -i T \ln \left( i (\chi^\dagger A_\mu - \overline{\chi} A^\mu \chi) \right) \)

\[ -\frac{N}{28} \int d^4 x \left( \chi^\dagger \gamma^5 \chi \right)^2 \quad N \gg 1 \]
Leading order in $1/N$
Magnetic catalysis
$U_c(1) \times U_p(0) \rightarrow U_4(0)$
for any value of $G$.

The gap equation for
$m = \langle 01110 \rangle$
\[
\frac{2}{36} \cdot V(s, \xi) / s = m, \xi = 0
\]
yields $m \neq 0$.

As $g = G_{11}m^2 \ll 1$, the LLL dominates
\[
m^2 = \exp(-m^2) = \exp(-N_{2019}) \ni (16, 16)\]
The LLL fermion propagator:

\[ S(x, y) = \exp\left[ i e \beta \epsilon_{ab} x^a y^b \right]. \]

\[ S(x, y); \quad a, b = 1, 2 \]

Schwinger phase

\[ \mathcal{S}(x) = 2i \epsilon_{12} e^{-\frac{\kappa^2}{16\pi}} \frac{\kappa x^0 - \frac{\kappa^2 x^2 + m}{\kappa^2 - \kappa^2 - m^2}}{\kappa^2 - \kappa^2 - m^2}. \]

\[ \pm (1 - i y^2 \text{sign}(x^2)) \]

projector
Factorization of the LLL propagator:

\[ S(x,y) = P(x,y) S_n(x_n-y_n) \]

\[ P(x,y) = \frac{e^{-\beta x y}}{2}\exp\left(-\frac{1}{\beta}(x^2 + y^2)\right) \exp\left(- \frac{1}{\beta}(x^2 - y^2)\right) \]

\[ S_n(x_n) = \frac{i}{(2\pi)^{N/2}} \int dx \exp(\frac{1}{\beta} (x^2 + y^2) - \frac{1}{2} \beta (x^2 - y^2)) \exp(\frac{i}{\beta} x_n x) \exp\left(-\frac{\beta}{2} x_n^2 \right) \frac{1}{1 - i \beta x_n y_n} \]

\[ \text{(1+1)-d fermion propagator:} P(x,y) \text{ is the projection operator on the LLL} \]
\[ P(x, y) = \frac{100}{2\pi} \exp\left(\frac{-100}{\pi} x^2 y^2 \right) \cdot \exp\left( -\frac{100}{\pi} (x - y)^2 \right) \]

\[ \text{density of states on the LLL} \]
Vertices

NCFT: n-point vertex

$$\frac{\Delta \Pi}{(2\pi)^n} \cdot \frac{d^2}{(2\pi)^2} \cdot \text{F}(\mathbf{K}) \cdot \text{S}^D(\Sigma \mathbf{K}) \exp \left[ \frac{1}{2} \sum \mathbf{K}_i \cdot \mathbf{N}_i \right],$$

$$\mathbf{K}_i \cdot \mathbf{N}_i = r^2 \theta^a \mathbf{K}_i \cdot \mathbf{N}_i,$$

$$[\mathbf{L}^a, \mathbf{S}^a] = i\theta^a.$$
NJI model in LLL approximation: Vertices
First, take \( \lambda_i = (x^0, x^3) = 0 \)
\[
\Gamma_{\text{tot}} = \frac{N! e^{B1}}{m^{n-2}} \int d\lambda_i d\lambda_j ... d\lambda_m \cdot \exp[\frac{1}{2} \sum \lambda_i \cdot \lambda_j / i \phi] \\
\exp[-\frac{i}{2} \sum \lambda_i \cdot \lambda_j / i \phi] \\
\lambda_i \cdot \lambda_j = \lambda_i^a \theta_i^a \lambda_j^a \\
\phi = \frac{e \phi}{eB} \\
\phi(\lambda_i) is \tilde{g}(\lambda_i) or \sigma(\lambda_i), \lambda_i \equiv \chi_i^2
New, "smeared," fields:

\[ \overline{\chi}(x) = \frac{\chi}{\sqrt{\text{area}}} \quad \Sigma(x) = \frac{\Sigma}{\sqrt{\text{area}}} \]

(fields with built-in form factors)

\[ \Gamma_{\phi\phi} = \frac{N_{\text{GB}}}{m^2} \int \frac{d^4k}{(2\pi)^2} \frac{d^4k'}{(2\pi)^2} \prod_{\phi} \prod_{\phi'} \frac{S(\Sigma(x)) \exp\left\{ \pm \frac{i}{\hbar} \Sigma(x) y \right\} \prod_{\phi} \prod_{\phi'} S(\Sigma(x))}{i} \]

\( \phi(x) \) is \( \chi(x) \) or \( \Sigma(x) \)
Propagators
Leading order in $1/N$:

\[ \begin{align*}
D^{(0)}_{\gamma}(k) &= \frac{4\pi^2}{N \pi^2 B/\Lambda^2} \frac{\text{Re} \lambda}{m^2} as \ k^2 \to 00 \\
D^{(0)}_{\gamma}(k) &\to \frac{4\pi^2}{N \pi^2 B/\Lambda^2} \frac{\text{Re} \lambda}{m^2} as \ k^2 \to 00 \\
D^{(0)}_{\gamma}(k) &= e^{-\frac{k^2}{4m^2}} D^{(0)}_{\gamma}(k') \\
D^{(0)}_{\gamma}(k) &\sim e^{-\frac{k^2}{4m^2}} as \ k^2 \to 00.
\end{align*} \]

$D^{(0)}_{\gamma}(k)$ is the propagator with a built-in form-factor.
Coordinate space
\[ \Gamma_{\mathcal{P}} = \frac{\text{Neck}}{m^2} \int dx_1 dx_2 \cdots [P \ast P \cdots \ast P(x)] \]

Star product:
\[ [P \ast P(x)] = \exp(i \theta \frac{a^2}{2m} \frac{2a}{3m} P \ast P(x)) \]
\[ a^2 = \frac{e^2}{\epsilon B} \]

\( \Phi(x) \) is \( \Sigma(x) \) or \( \Pi(x) \)
Noncommutative coordinate space \((\mathbb{C}^2)\)

\[\hat{\Phi}(\mathbf{x}) = \hat{W}(\mathbf{p}) = \frac{\partial}{\partial \mathbf{p}} \hat{W}(\mathbf{p}),\]
\[\hat{\Delta}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{x}} \hat{W}(\mathbf{p}) e^{i\omega_0 \mathbf{x}} e^{-i\omega_0 \mathbf{x}}\]

\[\hat{\gamma}_0 = C_{\alpha} \frac{\omega_0}{\hbar^2} \int d^4x \left( \hat{T}_F(\hat{\phi}_0 \cdots \hat{\phi}_n) \right)\]

\[\hat{T}_F(\hat{\phi}) = \int d^4x \hat{W}(\mathbf{p}), \quad \hat{T}_F(\hat{\Delta}(\mathbf{x})) = 1\]
$\Gamma_{FR} = 0$: Effective Action

\[
\Gamma = -\frac{N g^{4B}}{8\pi^2} \left[ \delta \varphi \left( \frac{Z^2 + i\tilde{r}^2}{R} \right) \cdot \left[ \log \left( \frac{Z^2 + i\tilde{r}^2}{R} \right) - 1 \right] + \frac{4\pi^2}{g^{2B} (6^2 + 5i)} \right] \\
\text{Noncommutative space} \\
\Gamma = -\frac{N g^{4B}}{8\pi^2} \left[ \delta \varphi \left( \frac{Z^2 + i\tilde{r}^2}{R} \right) \cdot \left[ \log \left( \frac{Z^2 + i\tilde{r}^2}{R} \right) - 1 \right] + \frac{4\pi^2}{g^{2B} (6^2 + 5i)} \right] \\
\hat{\varphi}(x) = \frac{\delta}{\delta \varphi(x)} \\
\hat{\varphi}^2 \varphi(x) = -(\partial^a)^2 \sum_{a=1}^5 \frac{1}{2} [\epsilon^a, \epsilon^b \varphi(x)] \\
\hat{\varphi} \varphi(x) = -i [\phi, \varphi(x)]$
\[ N_B = 0: \text{Vertices and Effective Action} \]

**Vertices**

\[
\Gamma_{\mathcal{V}} = - \frac{c \lambda^{\tau}}{2 \pi^2} \left[ \frac{1}{2} \sum_{B} \left( \frac{d \phi^B}{d t} \right)^2 + \frac{1}{2} \sum_{i} \left( \frac{d \phi^i}{d t} \right)^2 + \phi^B \left( \frac{d \phi^B}{d t} \right) + \phi^i \left( \frac{d \phi^i}{d t} \right) \right] \cdot \theta(\phi^B) \cdot \theta(\phi^i),
\]

\[
\phi_i(t) = \tilde{\phi}_i(t) + i \chi(t) \phi_i(t), \quad \tilde{\phi}_i(t) = \phi_i(t) - m
\]

**Action:**

\[
\Gamma = \frac{n \lambda^{\tau}}{2 \pi^2} \left[ - i T_{\mathcal{V}} \left[ \varepsilon \left( \frac{d \phi^B}{d t} \right) - \phi^B \right] \right] - \frac{\pi}{2 \pi^2} \left[ \varepsilon \left( \frac{d \phi^i}{d t} \right) - \phi^i \right] \cdot \theta(\phi^B) \cdot \theta(\phi^i),
\]

\[
\phi = \phi_i(t) + i \chi(t), \quad \phi_B = \phi_B(t)
\]

\[
\Gamma = \frac{n \lambda^{\tau}}{2 \pi^2} \left[ - i T_{\mathcal{V}} \left[ \varepsilon \left( \frac{d \phi^B}{d t} \right) - \phi^B \right] \right] - \frac{\pi}{2 \pi^2} \left[ \varepsilon \left( \frac{d \phi^i}{d t} \right) - \phi^i \right] \cdot \theta(\phi^B) \cdot \theta(\phi^i),
\]

\[
\phi = \phi_i(t) + i \chi(t), \quad \phi_B = \phi_B(t)
\]
Dynamics

Commutative limit: \( |eB| \to \infty \), 
\[ \epsilon \to \ell \to \epsilon_0 \to 0. \]

So, in fact, \( N \sim |eB| \to \infty, \ c \gg 1 \)
and \( m \) being fixed (continuum limit).

\[ \Theta_n = -\epsilon^{\frac{1}{2}} \bar{N} \int \ldots \int \frac{d^3 \mathbf{x}}{2 \pi} P(\mathbf{x}, \mathbf{x}_i) \cdots P(\mathbf{x}_n, \mathbf{x}_i) \frac{\partial}{\partial \mathbf{x}_i} \left( \frac{\epsilon}{\epsilon + \omega} \right) \frac{\partial}{\partial \mathbf{x}_i} \frac{\epsilon}{\epsilon + \omega}. \]

\[ \Theta_n = \frac{\epsilon}{\epsilon_0} \frac{\epsilon}{\epsilon + \omega}, \ \theta = \theta - \epsilon_0 \mu = \epsilon - \frac{\epsilon}{\epsilon_0}. \]

\[ P(x_i, x_m) = \frac{1+\epsilon}{\epsilon_0} \epsilon_{\frac{1}{2}}(\epsilon - \frac{1}{2}) \frac{1}{2\pi} e^{\frac{\epsilon}{\epsilon_0} \frac{\epsilon}{\epsilon + \omega}}. \]

\[ \mathbf{e} = \frac{1}{2}\epsilon_{\frac{1}{2}}(\epsilon - \frac{1}{2})^2. \]

The point with \( x_i = x_m, \ i = 1, \ldots, n \)
is both a saddle and stationary point.

As \( |eB| \to \infty \), one can put \( x_i = x_m \) in \( \mathbf{e} \).
\( \Gamma^{(os)} = - \frac{e^2}{\hbar} \int \prod_{n=1}^{N} d \phi_n \prod_{i<j}^{N} d x^i \ldots d x^j \)

\( \frac{1}{2} \left[ \sum_{n} (\partial_n x - \partial_n x^\mu) \right] \ldots \phi_n (x^\mu - x^\mu_n) \)

It is essentially a vertex in \((1+1)\)-dim. theory.

Action:
\( \Gamma^{(os)} = \frac{1}{2a} \left[ \frac{\phi^2 + \phi}{2} - i NF_n \gamma^\mu \mathcal{L}_n (x^\mu) + \right. \)
\[ - \left. \left( \phi + i \phi^\mu \right) \right] \frac{1}{2} \left[ \phi^2 + \phi \right] \]

It corresponds to a commutative field theory.

Action in the Gross-Neveu (GN) model:
\( \Gamma_{GN} = - i N F \mathcal{L}_n (x^\mu) \left( \phi^\mu - (\partial + m \gamma^\mu) \phi \right) \)
\[ - \frac{1}{2a} \left( \phi^2 + \phi \right), x = 0.1 \]
\( G \rightarrow 2m \Phi \)
\[ \frac{L_B l_s}{2 \pi} \] yields the number of the LLL states.

As \( |eB| \to \infty \), the model is reduced to a continuous set of independent \((1+1)\)-dim GN models labeled by \( \ell \).

With \( G \to 2 \pi G / |eB| \),
\[
\begin{align*}
\frac{eB}{2 \pi} & \to \frac{GN}{4 \pi^2} = \frac{GN}{16G} \\
\hbar^2 & = \hbar^2 e - \frac{4 \pi^2}{16G} \to m^2 = \hbar^2 e - \frac{4 \pi^2}{16G} \\
V(0, 0) & = \frac{eB l_s}{2 \pi} \to V(0, 0) = \frac{4 \pi}{2 \pi} (0^2 + \delta^2) \\
\left[ V(0, \delta^2) \right]^2 & = \frac{4 \pi}{2 \pi} (0^2 + \delta^2)
\end{align*}
\]
Dispersion relations

$\kappa^2 \ll 1/\kappa_0$

$\mathcal{E}_0 = \left[ \frac{m^2}{16\pi^2} \ln \left( \frac{16\pi^2}{|\kappa_0^2 + \kappa^2|} \right) \right]^{\frac{1}{2}}$

$\mathcal{E}_0 = \left[ \frac{m^2 + \frac{3m^2}{16\pi^2} \ln \left( \frac{16\pi^2}{|\kappa_0^2 + \kappa^2|} \right)}{16\pi^2} \right]^{\frac{1}{2}}$

$\mathcal{E}_0 = \frac{m}{16\pi^2} \ln \left( \frac{16\pi^2}{|\kappa_0^2 + \kappa^2|} \right) \sim O \left( \frac{m^2}{|\kappa_0^2|} \right)$

As $|\kappa_0| \to \infty$, $\sqrt{\mathcal{E}_0} \to 0$. 

As $|\kappa_0| \to \infty$, $\sqrt{\mathcal{E}_0} \to 0$. 
Truncated LLL model is a low energy theory of the NJL model in a magnetic field. Besides that, it is self-contained and self-consistent.

Take $k^2 \gg |eB|$: $E_n = \left[ 4m^2 \left( 1 - \frac{g^2 e^{-k^2 m^2}} {k^2 m^2} \right) + k^2 \right]^{1/2}$, $|E_1| \sim \frac{\sqrt{m}}{|eB|} e^{-k^2 m^2}$, $E_1(k^2 = 0) \approx 2m \left( 1 - \frac{g^2 e^{-k^2 m^2}} {k^2 m^2} \right) \approx 2m$ (box bound state)
Absence of UV/IR mixing

\[ S = \rho \max (\frac{1}{4} G_{\mu}^2 - \frac{1}{4} \beta^{\mu \nu \sigma} \frac{1}{2} \rho \max ) \]

\[ \Gamma_{\rho \rho} = \frac{\rho^{2}}{6} \left[ \Gamma_{\max }^{2} - \frac{1}{4} \beta^{\mu \nu \sigma} \frac{1}{2} \rho \max \right] \]

\[ \Gamma_{\rho \rho}^{(0)} = \frac{\rho^{2}}{4} \left( \frac{1}{4} \beta^{\mu \nu \sigma} \frac{1}{2} \rho \max \right) + O(\rho) \]

\[ \Gamma_{\rho \rho}^{(0)} \sim \rho^{4} \]

UV/IR mixing

S. Minwalla, A. Van Rees, and N. Seiberg
THED 0002, 020 (2000)
Truncated LLL model

One-loop nonplanar contribution to the $\phi_i$-propagator $D_{\phi_i}(x)$:

$$\frac{N_1 e B_1}{4 \pi^3} \int \frac{d^4 q}{(2\pi)^4} e^{i \eta (\omega^2 - \vec{p} \cdot \vec{q})} D_{\phi_i}(\vec{q}) \cdot I(\vec{p}, \vec{q}) D_{\phi_i}^{(0)}(\vec{p})$$

The $\vec{q}$-integral

$$\int \frac{d^4 q}{(2\pi)^4} e^{i \eta (\omega^2 - \vec{p} \cdot \vec{q})} D_{\phi_i}(\vec{q})$$

is finite for all $\vec{p}$, including $\vec{p} = 0$. Without $e^{-i \eta \vec{p} \cdot \vec{q}}$, it would diverge quadratically at $\vec{p} = 0$.

$D_{\phi_i}(\vec{p}) = e^{-i \eta \vec{p} \cdot \vec{q}} D_{\phi_i}(\vec{p})$
$D = d + 1, \quad d \geq 2$

NCFT: $E_{d+1}$

Canonical skew-diagonal form of $\Theta_{ab}$:

$$\Theta_{ab} = \begin{pmatrix}
0 & b' & 0 \\
-b' & 0 & b'' \\
0 & -b'' & 0
\end{pmatrix}$$

Skew-diagonal form of $F_{ab}$:

$$F_{ab} = \sum_{c=1}^{10} B_c (s_a b_c - s_b b_c)$$
Fermion propagator

$$S_{\alpha\beta}(x,y) = P \delta^{\alpha\beta}(x^0, y^0) S_{\mu}(x^\mu - y^\mu),$$

is equal to the direct product of $$P(x^{20}, x^{20})$$ in the $$x^{20}$$-planes with nonzero $$B^0$$.

$$S_{\mu}(x^\mu) = 2i \frac{k_\mu S + m}{k_\mu^2 - m^2} \sqrt{1 - i k_\mu / 2m} \delta_{\mu0} \text{sign}(k_0),$$

$$k_\mu = x^\mu \text{ for even } d \text{ and } k_\mu = (x^0, x^\mu) \text{ for odd } d.$$

The theories are finite for even $$d$$, and they describe gauge-confining dynamics for charged particles. For $$d = 2n + 1$$, their dynamics is quasi-$$(n+1)$$-dimensional.
As $|eB^0| \to \infty$, the theories are reduced either to a continuum set of $(1+1)$-dimensional CN models labeled by $d-1$ spatial coordinates (odd $d$) or to a set of quantum mechanical models labeled by $d$ spatial coordinates (even $d$).
Nonrelativistic model

\[ L = \frac{m}{2} \left( \dot{x}^2 + \dot{y}^2 \right) + \frac{eB}{c} \cos \left( \omega t - \omega x \right) \]

LLL dominance: \( B \to \infty \) or \( n \to 0 \)

\[ L_0 = 2eB \cos \left( \omega t - 2k \Delta x^2 \right) \]

\( \lambda = \frac{k^2}{2} \), \( \Delta = \frac{\lambda - i}{2} \)

\[ \Delta x = \frac{3a}{4} e^{\alpha} \text{, } \Delta y = \frac{3a}{8} e^{2\alpha} \]

\[ N = \frac{e^{\alpha} \lambda}{2eB} \]

Constraint: \( \Delta^2 = -\frac{e\alpha}{2eB} \)

Impurity: \( V(x) \)

\[ \langle N \rangle = \frac{2a}{eB} \text{, } V_0 = \frac{\langle N \rangle}{2} \]

\[ Q_\lambda = (e\alpha^2 x^2 / 4B, V_0) \]

\[ \text{V} \]
Solve now the problem for $m = 0$ and take limit $m \to 0$ in the solution. The result is different:

$$<\mathbf{r}^2 V_{\mathbf{q}}(\mathbf{p})> = \frac{1}{(2\pi)^3} \int d^3p \tilde{V}(p) S(\mathbf{r} - \mathbf{p}).$$

$$\mathbf{q}$$ is real, $\mathbf{p} \not= 2\pi \mathbf{x}.$

$$\Delta^2 = -\frac{c \alpha \hbar \sigma}{2\hbar B} \Rightarrow <\mathbf{r}^2 \Delta \mathbf{P}^2>_{\mathbf{L}_\mathbf{W}} =$$

$$= \left( c \alpha \hbar \sigma \mathbf{e} \cdot \mathbf{e} \right) \Delta^2 \mathbf{P}^2 - \frac{\sigma^2 (\mathbf{e} \cdot \mathbf{P})^2}{2\hbar B}.$$

$$\mathbf{e} \cdot \mathbf{P} = \frac{\sigma^2 \mathbf{e} \cdot \mathbf{e} \cdot \mathbf{P}}{2\hbar B}.$$
QED and QCD in a strong magnetic background

These dynamics determine nonlocal NCMFT which are much more complicated than those corresponding to the NJL model in a magnetic background. This is because while interactions in the NJL model are short-range, interactions in gauge theories are long-range.
Fermion propagator:
\[ S(x,y) = \exp \left[ i e B \cdot \nabla x \cdot y \right] . \]
\[ S(x-y), \alpha, \beta = 1, 2 \]
\[ \beta_{\text{ext}} = (0, \frac{Bz}{2}, -\frac{Bx}{2}, 0) - \text{symmetric gauge} \]
\[ S(x) = i \exp \left( -\frac{k^2}{2 eB} \right) \sum_{n=0}^{\infty} \frac{D_n(eB,x)}{\left( \frac{k^2}{m^2} - eB \right)^n} \]
\[ k_x = (k^0, k^1), \quad k_y = (k^0, k^2), \]
\[ D_n(eB,x) = 2 \left( k_0 x^n + m \right) \left[ P_{n-1}(\frac{k^2}{m^2}) - P_n(\frac{k^2}{m^2}) \right] + 4 \left( k^0 x^n + k^2 x^n \right) \]
\[ P_n \left( \frac{k^2}{m^2} \right) \]
\[ P_x = \frac{1}{2} \left( 1 \pm i k_y k^2 \text{sign}(eB) \right). \]
For \[ k_x, k_y, m << \text{v} \text{eB}, \] higher Landau levels with \[ n \gg 1 \] should decouple.
The LLL propagator:

\[ \tilde{S}_{LLL}(k) = 2i e^{\frac{-k^2}{4}} \frac{k_0^2 - k^2 + m}{k_0^2 - k^2 - m^2} \)

\[ \cdot \frac{1}{2} (1 - i e^{-i \Phi^2 \text{sign}(eB)}) \]

projector

Dimensional reduction:

\[ 3+1 \rightarrow 1+1 \quad 2+1 \rightarrow 0+1 \]

\[ k_0 = \pm \sqrt{k^2 + m^2} \quad k_0 = \pm m \]
Type I nonlocal NCFT (NJL-like models): There exists a field transformation that puts interaction vertices in the conventional form with a cost of introducing a form factor in field propagators (short-range interactions).

Type II nonlocal NCFT (QED and QCD): There is no such a field transformation. (Long-range interactions)
a-point vertex $\Gamma_{n_0}$ in nonlocal MCFT for $k_0 = 0$.

\[ \Gamma_{n_0} \sim \frac{\int d\mathbf{k}_1 \cdots d\mathbf{k}_n}{(2\pi)^{2n}} \cdot \prod_{i=1}^{n_0} S(\mathbf{q}_i) \cdot F^{(n_0)}(k_1, \ldots, k_n) \cdot \exp \left[ \frac{1}{2} \sum_{i<j} \mathbf{k}_i \cdot \mathbf{k}_j \right]. \]

NCIL model: $F^{(n_0)}(k_1, \ldots, k_n) = \prod_{0<i<n_0} \exp(-k_i^2/\mu_0)$.

QED, QCD: $F^{(n_0)}(k_1, \ldots, k_n)$ does not have a factorized form.
Nonlinear photodynamics in a strong magnetic background and noncommutative QED

n-point photon amplitudes

LLL approximation: $U(1) \to U(1)_{nc}$.

The $U(1)$ Ward identities are broken (LLL anomaly).

Nonrecycling phenomenon: Although a contribution of each of an infinite number of higher $LLT$ is suppressed in infrared, their cumulative contribution is not. Restoration of the $U(1)$. 
Nonlinear photodynamics in a strong magnetic field and noncommutative QED

\[
\Gamma (\phi) = - \frac{1}{2} \partial \phi^2 + \Gamma (\phi), \\
\Gamma_{\mu} (\phi) = - i N_c \text{Tr} \ln [\gamma^\mu (\phi) - \text{Id}], \\
B_\mu = B_\mu^{\text{ext}} + \hat{B}_\mu
\]

LLL approximation:

\[
2^0 (U) \rightarrow U(1)_{nc}
\]

\[
\begin{align*}
\phi_+ \rightarrow N (1) \star \phi_+ \star N (1) + \frac{2 \alpha s^* \phi^* \phi}{\beta} \\
\phi_+ \rightarrow N (1) \star \phi^+ \star N (1); \quad \phi_+ = (\Phi_{\mu}^\dagger) \Phi_{\mu} \\
E^\mu = \partial \phi^\mu - i \phi_{\mu}, \quad \phi_{\mu} = \phi (\mu), \quad \phi_{\mu} = \phi (\mu), \\
[\phi_+ \phi]_{\text{nc}} = \phi_+ \phi - \phi \phi_+.
\end{align*}
\]
Results

d) The n-point photon amplitude $T^\mathbf{A}_{\mathbf{k}_1,...,\mathbf{k}_n}$ is not transverse in the LLL approximation (LLL anomaly).

b) The LLL anomaly is cancelled by the contribution of all higher Landau levels (HLL). Although the contribution of each of HLL is suppressed as $\frac{1}{\nu^2}$ or $\frac{1}{\nu^4}$, the sum over all HLL planes unappreciably contribution. Nonanising effect.

c) The gauge invariant $T^\mathbf{a}_{\mathbf{k}_1,...,\mathbf{k}_n}$ is derived. The LLL approximation is reliable for $\mathbf{k}_1^2, \mathbf{k}_2^2 \ll \nu^2$ and $\mathbf{k}_3 \gg \nu^2$

d) Analogy with edge states in quantum Hall effect.
Conclusion

* Relativistic field theories in a strong magnetic background determine a new class of NCFT, nonlocal NCFT. These theories are self-consistent.

** There are type I and type II nonlocal NCFT, corresponding to dynamics with short-range interactions (mass-like models) and long-range interactions (gauge models), respectively.

*** Nonlinear photodynamics in a strong magnetic field yields sophisticated dynamics relevant for magnetars.