Hawking–Page transition at finite baryon, isospin and strangeness densities: bottom–up approach

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Plan

• The hard-wall model (EKSS, PR)
• Hawking–Page transition in a sliced AdS background (Herzog)
• Dense matter in the model: isospin matter
• HPt in dense matter
“In the bottom-up approach, one looks at QCD first and then attempts to guess its 5D-holographic dual.”
Let’s start from 2–flavor QCD at low energy and attempts to guess its 5D holographic dual, AdS/CFT dictionaries.

\[ Z_4[\phi_0(x)] = \int \mathcal{D}[\Phi] \exp\{iS_4 + i \int \phi_0(x)\mathcal{O}\}, \]

5D (classical) effective action: \[ \Gamma_5[\phi(x, z) = \phi_0(x)]; \phi_0(x) = \phi(x, z = 0). \]

AdS/CFT correspondence: \[ Z_4 = \Gamma_5. \]
AdS/CFT Dictionary

- 4D CFT (QCD) ↔ 5D AdS
- 4D generating functional ↔ 5D (classical) effective action
- Operator ↔ 5D bulk field
- [Operator] ↔ 5D mass
- Current conservation ↔ gauge symmetry
- Large Q ↔ small z
- Confinement ↔ Compactified z
- Resonances ↔ Kaluza–Klein states
5D field contents

**Operator** $\rightarrow$ **5D bulk field**

\[
\begin{align*}
\bar{q}_R q_L & \rightarrow \Phi(x, z) \\
\bar{q}_L \gamma^\mu q_L & \rightarrow L_M(x, z) \\
\bar{q}_R \gamma^\mu q_R & \rightarrow R_M(x, z)
\end{align*}
\]

[Operator] $\rightarrow$ **5D mass**

\[
(\Delta - p)(\Delta + p - 4) = m_5^2 \quad m_\phi^2 = -3
\]
5D Symmetry

Current conservation $\rightarrow$ gauge symmetry

$SU(2)_L \times SU(2)_R$ 5D gauge symmetry in $AdS_5$

Background: $AdS_5$

$$ds^2_5 = \frac{1}{z^2} \left( \eta_{\mu\nu} dx^\mu dx^\nu - dz^2 \right)$$
Confinement $\rightarrow$ IR cutoff in 5th direction

$\phi_0(x)$
\[\phi(x, z)\]
\[z = \epsilon \rightarrow 0\]
\[z = z_m\]

UV bulk IR

Polchinski & Strassler, 2000
Hard wall model


\[ S_1 = \int d^4 x dz \sqrt{g} \mathcal{L}_5 , \]
\[ \mathcal{L}_5 = \text{Tr} \left[ -\frac{1}{4g_5^2} (L_{MN}L^{MN} + R_{MN}R^{MN}) + |D_M \Phi|^2 - M_\Phi^2 |\Phi|^2 \right] , \]
\[
V_M \sim L_M + R_M, \quad A_M \sim L_M - R_M,
\]
\[
\Phi = S e^{iP/v(z)}, \quad v(z) \equiv \langle S \rangle,
\]
\[
\Phi \leftrightarrow X, \quad v(z) \leftrightarrow X_0.
\]

The model describes \( \rho, \ a_1, \ \pi, \ \sigma, \ldots \).
Where is the chiral condensate?

Klebanov and Witten, 1999

\[ \phi(x, z) \rightarrow z^{d-\Delta} \phi_0(x) + z^\Delta A(x) + \ldots, \quad z \rightarrow \epsilon, \]

where \( \phi_0(x) \) is the source term of 4D operator \( \mathcal{O}(x) \), and

\[ A(x) = \frac{1}{2\Delta - d} \langle \mathcal{O}(x) \rangle. \]

For example, \( \mathcal{O} = \bar{q}q \), \( \phi(x, z) = v(z) \):

\[ v(z) = c_1 z + c_2 z^3 \]

\[ c_1 \sim m_q, \quad c_2 \sim \langle \bar{q}q \rangle. \]

So, 4 unknowns in the model: \( g_5^2(M_5), \ z_m, \ m_q, \ \langle \bar{q}q \rangle. \)
### TABLE II. Results of the model for QCD observables.
Model A is a fit of the three model parameters to $m_{\pi}$, $f_{\pi}$, and $m_{\rho}$ (see asterisks). Model B is a fit to all seven observables.

<table>
<thead>
<tr>
<th>Observable</th>
<th>Measured (MeV)</th>
<th>Model A (MeV)</th>
<th>Model B (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{\pi}$</td>
<td>139.6 ± 0.0004 [8]</td>
<td>139.6*</td>
<td>141</td>
</tr>
<tr>
<td>$m_{\rho}$</td>
<td>775.8 ± 0.5 [8]</td>
<td>775.8*</td>
<td>832</td>
</tr>
<tr>
<td>$m_{a_1}$</td>
<td>1230 ± 40 [8]</td>
<td>1363</td>
<td>1220</td>
</tr>
<tr>
<td>$f_{\pi}$</td>
<td>92.4 ± 0.35 [8]</td>
<td>92.4*</td>
<td>84.0</td>
</tr>
<tr>
<td>$F_{\rho}^{1/2}$</td>
<td>345 ± 8 [15]</td>
<td>329</td>
<td>353</td>
</tr>
<tr>
<td>$F_{a_1}^{1/2}$</td>
<td>433 ± 13 [6]</td>
<td>486</td>
<td>440</td>
</tr>
<tr>
<td>$g_{\rho\pi\pi}$</td>
<td>6.03 ± 0.07 [8]</td>
<td>4.48</td>
<td>5.29</td>
</tr>
</tbody>
</table>

$$m_{\pi}^{2}f_{\pi}^{2} = (m_{u} + m_{d})\langle \bar{q}q \rangle = 2m_{q}\sigma.$$  

<table>
<thead>
<tr>
<th></th>
<th>Experiment</th>
<th>AdS$_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>0.4 ± 0.3</td>
<td>0.4</td>
</tr>
<tr>
<td>$L_2$</td>
<td>1.4 ± 0.3</td>
<td>0.9</td>
</tr>
<tr>
<td>$L_3$</td>
<td>−3.5 ± 1.1</td>
<td>−2.6</td>
</tr>
<tr>
<td>$L_4$</td>
<td>−0.3 ± 0.5</td>
<td>0.0</td>
</tr>
<tr>
<td>$L_5$</td>
<td>1.4 ± 0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>$L_6$</td>
<td>−0.2 ± 0.3</td>
<td>0.0</td>
</tr>
<tr>
<td>$L_9$</td>
<td>6.9 ± 0.7</td>
<td>5.4</td>
</tr>
<tr>
<td>$L_{10}$</td>
<td>−5.5 ± 0.7</td>
<td>−5.5</td>
</tr>
</tbody>
</table>

Chiral Lagrangian

\[ \mathcal{L}_2 = \frac{F^2}{4} \text{Tr} \left[ D_\mu U^\dagger D\mu U + U^\dagger \chi + \chi^\dagger U \right], \]

\[ \mathcal{L}_4 = L_1 \text{Tr}^2 [D_\mu U^\dagger D\mu U] + L_2 \text{Tr} [D_\mu U^\dagger D_\nu U] \text{Tr} [D^\mu U^\dagger D^\nu U] + L_3 \text{Tr} [D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U] \]

\[ + L_4 \text{Tr} [D_\mu U^\dagger D^\mu U] \text{Tr} [U^\dagger \chi + \chi^\dagger U] + L_5 \text{Tr} [D_\mu U^\dagger D^\mu U (U^\dagger \chi + \chi^\dagger U)] \]

\[ + L_6 \text{Tr}^2 [U^\dagger \chi + \chi^\dagger U] + L_7 \text{Tr}^2 [U^\dagger \chi - \chi^\dagger U] + L_8 \text{Tr} [\chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi] \]

\[ - i L_9 \text{Tr} [F_\mu^\nu D_\mu U D_\nu U^\dagger + F_\mu^\nu D_\mu U^\dagger D_\nu U] + L_{10} \text{Tr} [U^\dagger F_\mu^\nu U F_{L\mu\nu}]. \]

The LECs are determined by experiments, a way for ChPT to communicate with nature.
Deconfinement temperature:

Hawking–Page analysis in a cut–off $\text{AdS}_5$


\[ I = -\frac{1}{2\kappa^2} \int d^5 x \sqrt{g} \left( R + \frac{12}{L^2} \right). \]

\[ \kappa \sim 1/N_c, \quad F_\pi^2 \sim N_c \]

Gravitational action:\(\sim N_c^2\), Meson action:\(\sim N_c\)
1. thermal AdS:

\[ ds^2 = L^2 \left( \frac{dt^2 + d\vec{x}^2 + dz^2}{z^2} \right) \]

\[ \beta' : \text{the periodicity in the timelike direction, (undetermined)} \]

2. AdS black hole:

\[ f(z) = 1 - \frac{z^4}{z_h^4} \quad T = \frac{1}{\pi z_h} \]

\[ ds^2 = \frac{L^2}{z^2} \left( f(z) dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)} \right) \quad 0 \leq t < \pi z_h \]

Transition between two backgrounds $\leftrightarrow$ (De)confinement transition.
\[ R = -\frac{20}{L^2} \quad I = \frac{4}{L^2\kappa^2} \int d^5x \sqrt{g} \]

1. Cut–off thermal AdS:

\[ V_1(\epsilon) = \frac{4L^3}{\kappa^2} \int_0^{\beta'} dt \int_\epsilon^{z_0} dz \, z^{-5} \]

2. Cut–off AdS black hole:

\[ V_2(\epsilon) = \frac{4L^3}{\kappa^2} \int_0^{\pi z_h} dt \int_\epsilon^{\min(z_0, z_h)} dz \, z^{-5} \]
\[ \beta' = \pi z_h \sqrt{f(\epsilon)} \]

\[ \Delta V = \lim_{\epsilon \to 0} (V_2(\epsilon) - V_1(\epsilon)) \]
\[ = \begin{cases} 
\frac{L^3 \pi z_h}{\kappa^2} \frac{1}{2z_h^4} & z_0 < z_h \\
\frac{L^3 \pi z_h}{\kappa^2} \left( \frac{1}{z_0^4} - \frac{1}{2z_h^4} \right) & z_0 > z_h
\end{cases} \]

\[ T_c = 2^{1/4} / (\pi z_0) \]
No stable AdS black hole in confined phase. In confined phase, $t_{AdS}$ is a relevant background. So, no temperature dependence in confined phase. May be ok, if we are taking $N_c$ infinite.

A cartoon representation

T. Cohen@YKIS2006
1. 5D meson action.
2. Deformed AdS.
3. ...
Finite temperature as an IR BC

YK, H. K. Lee, hep-ph/08022409

\[
\left[ \partial_z^2 - \frac{3}{z} + \frac{3}{z^2} \right] X_0 = 0, \quad X_0 = c_1 z + c_2 z^3,
\]

Basic strategy is to impose the boundary conditions at a given temperature. Then we have a temperature dependent chiral condensate. We cannot, however, determine the temperature dependence of chiral condensate within the hard-wall model, since the chiral condensate is one of the integration constants to be fixed by the boundary conditions.
\[
\frac{<\bar{q}q>_T}{<\bar{q}q>_0} = 1 - c_1 \left( \frac{T^2}{8f^2_\pi} \right) - c_2 \left( \frac{T^2}{8f^2_\pi} \right)^2 - c_3 \left( \frac{T^2}{8f^2_\pi} \right)^3 \ln\left( \frac{\Lambda_q}{T} \right) + \mathcal{O}(T^8)
\]

\[
c_1 = \frac{2}{3} \frac{N_f^2 - 1}{N_f} \quad c_2 = \frac{2}{9} \frac{N_f^2 - 1}{N_f^2} \quad c_3 = \frac{8}{27} (N_f^2 + 1) N_f
\]

$\bar{\psi} \psi (T) / \bar{\psi} \psi (T=0)$

F. Karsch and Laermann, hep-lat/0305025
\[
\frac{\sigma^*}{\sigma_0} = 1 - \frac{T^2}{8f_{\pi}^2}
\]

\[
f_{\pi}^2 = -\frac{1}{g_5^2} \frac{\partial_z A(0, z)}{z} \bigg|_{z=z_0}, \quad \left[ \partial_z^2 - \frac{1}{z} \partial_z - 2\frac{\nu^2}{z^2} \right] A(0, z) = 0.
\]
Dense matter in the hard-wall model: isospin matter

\[ \mu_q \psi^\dagger \psi \quad (= \mu_q \bar{\psi} \gamma_0 \psi) \quad \longleftrightarrow \quad V_0(z) = \mu_q + \cdots, \text{ at the boundary } (z \to 0), \]

\[ \mu_I \bar{\psi} \tau^3 \psi \quad \longleftrightarrow \quad \tilde{V}_0(z) = \mu_I \cdot \text{diag}(1, -1) + \cdots. \]

For example, we generalize the SU(2) gauge symmetry to U(2).


\[ V_0 = \hat{V}_0 \hat{t} + V_0^a t^a \]

\[ \hat{V}_0 = c_1 + c_2 z^2 \]

\[ S_b = \frac{1}{2g_5^2} z \hat{V}_0 \partial_z \hat{V}_0 = \frac{1}{g_5^2} \mu_q c_2 \quad \rho_q = \frac{\partial S_b}{\partial \mu_q} = \frac{1}{g_5^2} c_2 \]
Isospin matter

D. T. Son and M. A. Stephanov, PRL 86, 592 (2001)

Isospin matter, where $\mu_I$ is finite and $\mu_B$ is zero, was proposed by Son and Stephanov as a useful setting to improve our understanding of cold dense QCD. Although the system of QCD with finite $\mu_I$ and zero $\mu_B$ hardly exists in nature, it has many interesting and useful features. The standard lattice QCD technique is applicable to the isospin matter, unlike to the QCD at finite baryon chemical potential. Furthermore, we can analytically study the system at very low $\mu_I$ using chiral perturbation theory and at very high $\mu_I$ by perturbative QCD.
\[ \rho_I = 0 : \text{normal phase with no pion condensation} \]

\[ \rho_I = f^2_{\pi} \mu_I (1 - m^4_{\pi} / f^4_{\pi}) : \text{pion condensed phase}. \]
FIG. 1. Phase diagram of QCD at finite isospin density.
Isospin matter in AdS/QCD


\[ \mathcal{L}_{V_{\pi\pi}} \sim \partial_\mu \pi [V_\mu, \pi] \quad \rightarrow \quad \mathcal{L}_{\pi\pi} \sim \partial_0 \pi [V_0, \pi] \]

Pions

\[ V_0^3 t^3 = \tilde{\mu}_I t^3. \quad D \equiv 1 - \partial_5 \left( \frac{1}{2a^3 v^2} \partial_5 a \right). \]

\[ D \left( m_\pm^2 + (2v^2 a^2 D) \pm 2\tilde{\mu}_I m_\pm \right) \pi^\pm = 0 \]
\[ D \left( m_0^2 + (2v^2 a^2 D) \right) \pi^0 = 0, \]
\[ m_\pm = \mp \bar{\mu}_I + \sqrt{\bar{\mu}_I^2 + m_0^2}, \]

In the chiral limit, \( m_0 = 0 \), we always have the pion condensation, since \( m_- \) is zero with any values of \( \mu_I \), which is consistent with the observation made in a top-down approach based on D4/D8, O. Aharony, K. Peeters and J. Sonnenschein, hep-th/07093948.

At small \( \bar{\mu}_I \), we have \( m_\pm \approx \mp \bar{\mu}_I + m_0 \), which is consistent with the observation made Son and Stephanov.
FIG. 1: The spatial component of the pion decay constant in the isospin matter. Here $R \equiv \frac{f_\pi^s(\mu_I)}{f_\pi(0)}$. 
HPt at finite baryon number density


\[ I = -\frac{1}{2\kappa^2} \int d^5x \sqrt{g} \left( R + \frac{12}{L^2} \right). \]

\[ S_{\text{matter}} = M_5 \int d^5x \sqrt{g} \text{Tr} \left[ \frac{1}{2} |D_\mu \Phi|^2 + \frac{1}{2} M_\Phi^2 |\Phi|^2 + \frac{1}{4} \left( F_L^2 + F_R^2 \right) \right] \]

\[ \hat{V}_0 = c_1 + c_2 z^2 \]
The final result for $z_h < z_{IR}$ is

$$\Delta V = \frac{L^3 \pi z_h}{\kappa^2} \left[ \frac{1}{z_{IR}^4} - \frac{1}{2z_h^4} - \frac{L^4 N_f c_2^2}{48 N_c} \left( z_{IR}^2 - z_h^2 \right) \right]$$
\[ R \equiv \frac{T_c}{T_0}, \quad \bar{\rho}_q \equiv \rho_q z_{I_R}^2 \]
Fodor and Katz, 2001
HPt at finite baryon, isospin and strangeness number density

4D:

$$\mathcal{L}^\mu_{QCD} = \mu_u u^\dagger u + \mu_d d^\dagger d + \mu_s s^\dagger s$$

$$= (\mu_q + \mu_I) u^\dagger u + (\mu_q - \mu_I) d^\dagger d + \mu_s s^\dagger s,$$

where $$\mu_q = \frac{1}{2}(\mu_u + \mu_d),$$ $$\mu_I \equiv \frac{1}{2}(\mu_u - \mu_d).$$

5D:

$$A_{L,R} = \hat{A}_{L,R} \tilde{t} + A^a_{L,R} \tilde{t}^a,$$

$$V_0 = 2\tilde{V}_0 \tilde{t} + \sqrt{2} \tilde{V}_0 \tilde{t} + V_0^3 t^3,$$

$$\tilde{t} = \frac{1}{2} \text{diag}(1,1,0), \quad \tilde{t} = \frac{1}{\sqrt{2}} \text{diag}(0,0,1), \quad t^3 = \frac{1}{2} \text{diag}(1,-1,0).$$
\[\tilde{V}_0 = \tilde{\mu}_q + c_q z^2, \quad \tilde{V}_0 = \tilde{\mu}_s + c_s z^2, \quad V_0^3 = \tilde{\mu}_I + c_I z^2.\]

\[
\Delta S \equiv S_{\text{BH}} - S_{\text{tAdS}} \\
= \frac{1}{\kappa^2} \pi z_h \left( \frac{1}{z_m^4} - \frac{1}{2z_h^4} \right) - \frac{1}{g_5^2} \pi z_h (z_m^2 - z_h^2) [4c_q^2 + 2c_s^2 + c_I^2]
\]
Figure 1: The critical temperature in a maximally asymmetric matter. Dashed line is for $c_I = 0$, while solid line is for the case of $c_I \neq 0$. Maximally asymmetric means $c_I = c_b$. Here $\tilde{c}_q \equiv c_q z_a^3$. 
Energy density criterion

\[
\frac{T_c}{T_0} = 1 - 0.021(2) \left( \frac{\mu_B}{T_0} \right)^2 - 0.039(1) \left( \frac{\mu_I}{T_0} \right)^2 \\
- 0.037(2) \left( \frac{\mu_S}{T_0} \right)^2 - 0.031(3) \frac{\mu_B \mu_S}{T_0^2} + \cdots ,
\]

Quark–antiquark criterion

\[
\frac{T_c}{T_0} = 1 - 0.017(1) \left( \frac{\mu_B}{T_0} \right)^2 - 0.109(4) \left( \frac{\mu_I}{T_0} \right)^2 \\
- 0.032(2) \left( \frac{\mu_S}{T_0} \right)^2 - 0.024(2) \frac{\mu_B \mu_S}{T_0^2} + \cdots ,
\]

in a hadron resonance gas model

The fractions of proton, neutron, $\Lambda$, $\Sigma$ and $\Xi$ of strange hadronic stars versus baryon density with $\beta$-equilibrium.

Figure 2: The strangeness number density of the critical temperature. Solid line is for $c_s \neq 0$, while dashed line is for the case of $c_s = 0$. 
In conclusion, the hard-wall model looks very simple, but it is telling us a lot about QCD.