Holographic entanglement entropy for time dependent states and disconnected regions

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VH, M. Rangamani, T. Takayanagi, arXiv:0705.0016

VH & M. Rangamani, arXiv:0711.4118
1 Motivation & background

2 Entanglement entropy with time-dependence
   - Candidate covariant constructions
   - Tests & applications

3 Entanglement entropy for disconnected regions
   - Proof for disconnected regions in 1 + 1 dimensions
   - Conjecture for disconnected regions in 2 + 1 dimensions

4 Summary and future directions
Entanglement Entropy

- Entanglement entropy of a given region $\mathcal{A}$ in the bdy CFT encodes the number of entangled/operative degrees of freedom.
- Applications to condensed matter systems, quantum information, ...
- It is the von Neumann entropy for reduced density matrix $\rho_{\mathcal{A}}$:

$$S_{\mathcal{A}} = -\text{Tr} \rho_{\mathcal{A}} \log \rho_{\mathcal{A}}$$

where $\rho_{\mathcal{A}}$ is trace of density matrix over the complement of $\mathcal{A}$.
- Depends on theory, state, and region $\mathcal{A}$.

FT motivation

Study operative DOF under time-dependence and for non-trivial (disconnected) regions
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AdS/CFT correspondence provides a useful framework for addressing questions in quantum gravity, e.g. emergence of spacetime. To make full use of this, we need to understand the AdS/CFT dictionary better.

Holographic dual of EE is a geometric quantity ⇒ we can use entanglement entropy to study bulk geometry.
Probing AdS/CFT

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QG motivation

Study bulk geometry in AdS/CFT

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Holographic Entanglement Entropy

- Area law of entanglement entropy: \( S_A \sim \text{area of } \partial A \)
  suggestive of a holographic relation...

- Holographic dual of entanglement entropy (for static bulk ST) = area of minimal co-dim. 2 surface \( S \) in bulk anchored at \( \partial A \) (e.g. in 3-D bulk, given by zero energy spacelike geodesics)

Ryu & Takayanagi
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In Lorentzian spacetime, \( \not\exists \) minimal area surface
(we can decrease area of spacelike surface by wiggling in time)

- We could bypass this by separating time & space directions
  (cf. in static spacetime, work on a const. \( t \) slice)
- But in a general, time-dependent background, there is no natural time slicing...

**Strategy:**

Seek covariantly-defined co-dim.2 surfaces which:
- reduce to minimal surface \( S \) in static backgrounds
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Set-up

- start with asymp. AdS bulk $\mathcal{M}$ and boundary $\partial\mathcal{M}$ (corresp. state)
- pick a time $t$ on the $\partial\mathcal{M}$ (well-defined: fixed static bgd)
- pick a region $\mathcal{A}$ on $\partial\mathcal{M}$ at time $t$ (bulk co-dim.2)
- we want to construct a co-dim.2, covariantly defined, surface in $\mathcal{M}$ anchored on $\partial\mathcal{A}$

We’ll consider 4 candidate surfaces, $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$. 
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Construction I:

- pick a region $A$ on $\partial M$ at time $t$
- find maximal-area spacelike slice $\Sigma$ in $M$ anchored at $t$ on $\partial M$
- $K_{\mu}{}^{\mu} = 0$
- on $\Sigma$, find minimal-area surface $X$ in $M$ with $\partial X = \partial A$

This is natural if EE pertains to a given time slice...
Minimal surface on maximal slice, $\mathcal{X}$

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Motivation | Time-dependent | Multi-region | Summary | Obstacle | Constructions | Tests | Applications

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   (if there are several solutions, pick the one with minimal-area)
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This is most natural extension of $S$, but no longer pertains to particular spacelike slice of bulk.
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Motivation Time-dependent Multi-region Summary

Obstacle Constructions Tests Applications

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Light-sheet construction, \( \mathcal{Y} \)

**Construction III:**
- pick a region \( A \) on \( \partial \mathcal{M} \) at time \( t \)
- consider light sheets in \( \mathcal{M} \) (non-positive null expansions \( \theta_{\pm} \))
- find the surface \( \mathcal{Y} \) in \( \mathcal{M} \) with \( \partial \mathcal{Y} = \partial A \) from which both expansions vanish

\[ \theta_+ = \theta_- = 0 \]

Turns out: equivalent to the extremal surface \( \mathcal{W} \);
Cf. Bousso’s Covariant Entropy bound...
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- pick a region $\mathcal{A}$ on $\partial \mathcal{M}$ at time $t$
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Simpler to compute, but does not necessarily reduce to $\mathcal{S}$ for static spacetimes... (may provide a bound for the EE)
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Relations between constructions

- $Z \neq S$ for some static bulk spacetimes
  (since uses null structure $\Rightarrow$ sensitive to $g_{tt}$)
  $\Rightarrow$ not a viable candidate for holographic EE

- $W = X = Y = S$ for static bulk spacetime
  $\Rightarrow$ a-priori all viable candidates for EE

- $W = Y$ in general
  (motivated via partition fn.: Lorentzian GKP-W relation)

- $X = Y$ (only) if $\Sigma$ is totally geodesic submanifold

conjecture:

Holographic dual of EE in general time-dependent spacetime is given by area of extremal co-dim.2 bulk surface, or equivalently surface with vanishing null expansions, anchored at $\partial A$. 
Motivation

Time-dependent Multi-region Summary

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Holographic Entanglement Entropy
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Test I: static BTZ

In 2-D CFT (finite $T$), we can calculate EE explicitly. For non-rotating BTZ: EE agrees w/ expression from $\mathcal{W}$.

$ds^2 = -(r^2 - r_+^2)\, dt^2 + \frac{dr^2}{(r^2 - r_+^2)} + r^2 \, dx^2$

$S_A = \frac{L_{\mathcal{W}}}{4\, G_N^{(3)}} = \frac{c}{3} \log \left( \frac{\beta}{\pi \varepsilon} \sinh \frac{2\pi h}{\beta} \right)$
Test II: rotating BTZ

CFT density matrix: \( \rho = e^{-\beta H + \beta \Omega P} \)

Trick for calculating EE: \( S_A = -\frac{\partial}{\partial n} \log \text{Tr} \rho_A^n \bigg|_{n=1} \)

\( \text{Tr} \rho_A^n \) obtained from 2-pt.fn. of twist opers w/ \( \Delta_n = \frac{c}{24} (n - \frac{1}{n}) \).

Both CFT and \( L_W/4G_N^{(3)} \) give

\[
S_A = \frac{c}{6} \log \left[ \frac{\beta_+ \beta_-}{\pi^2 \varepsilon^2} \sinh \left( \frac{\pi \Delta l}{\beta_+} \right) \sinh \left( \frac{\pi \Delta l}{\beta_-} \right) \right]
\]

BUT \( X \) does not coincide with \( W \):

\( \frac{\partial}{\partial t} \) is Killing but not hypersurface orthogonal

\( \Rightarrow \) while \( \Sigma \) is at const. \( t \), geods. \( W \) are not.

Conclusion: \( W = Y \) gives agreement w/ EE, \( X \) does not.
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Entanglement entropy during collapse

Example of extremal surfaces for collapse ($V$\textsubscript{Vaidya-AdS$_3$}):
Entanglement entropy increases in time-dependent background (satisfying the null energy condition)
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Entanglement entropy for disconnected regions

**Question:**

Given EE for any interval $A$ in some state of a 1+1 dim QFT, what is the EE of a disconnected region $X$ composed of two intervals?

**Naive guess:** $x \equiv S(X) = s_{12} + s_{34}$

Doesn’t work (cf. $p_2 \rightarrow p_3$: in general $s_{12} + s_{34} \neq s_{14}$)
Entanglement entropy for disconnected regions

**Answer from CFT:**
(cf. Calabrese & Cardy)

\[ x = s_{12} + s_{34} + s_{23} + s_{14} - s_{13} - s_{24}. \]

We will prove this using the holographic dual, via:

1. coincident upper & lower bounds on \( x \) from geometry
2. ansatz for \( x \) & consistency of limits \( p_i \to p_j \)

Useful tool:

**Strong subadditivity**
(cf. Lieb & Ruskai)

Given two overlapping boundary regions \( A \) and \( B \),

\[ S(A) + S(B) \geq S(A \cup B) + S(A \cap B) \]

\[ S(A) + S(B) \geq S(A \setminus B) + S(B \setminus A) \]
Geometric proof of strong subadditivity

Given two overlapping boundary regions $A$ and $B$,

consider $S(A) + S(B)$
Given two overlapping boundary regions $A$ and $B$,

compare with $S(A \cup B) + S(A \cap B)$
Given two overlapping boundary regions $A$ and $B$,

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(cf. Headrick & Takayanagi)
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Argument for two disconnected intervals in $1+1$ dim

Holographic EE as bulk minimal surface implies:

\[
\begin{align*}
S_{14} + S_{23} & \leq S_{13} + S_{24} \\
S_{12} + S_{34} & \leq S_{13} + S_{24} \\
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$$s_{12} + s_{34} \leq s_{13} + s_{24}$$

$$s_{ij} \leq s_{ik} + s_{jk}$$
for $i, j, k = 1, 2, 3, 4$

(12 redundant relations reduce to 4)

$$s_{13} \leq s_{12} + s_{23}$$
$$s_{13} \leq s_{14} + s_{34}$$
$$s_{24} \leq s_{12} + s_{14}$$
$$s_{24} \leq s_{23} + s_{34}$$
Argument for two disconnected intervals in $1 + 1$ dim

Holographic EE as bulk minimal surface implies:

\begin{align*}
S_{14} + S_{23} &\leq S_{13} + S_{24} \\
S_{12} + S_{34} &\leq S_{13} + S_{24} \\
S_{13} &\leq S_{12} + S_{23} \\
S_{13} &\leq S_{14} + S_{34} \\
S_{24} &\leq S_{12} + S_{14} \\
S_{24} &\leq S_{23} + S_{34}
\end{align*}
Argument for two disconnected intervals in $1+1$ dim

Strong subadditivity implies:

\[
\begin{align*}
    x & \leq s_{12} + s_{34} \\
    x & \leq s_{14} + s_{23} \\
    s_{12} + s_{14} - s_{13} & \leq x \\
    s_{14} + s_{34} - s_{24} & \leq x \\
    s_{12} + s_{23} - s_{24} & \leq x \\
    s_{23} + s_{34} - s_{13} & \leq x \\
    \text{where } x & \equiv S(X)
\end{align*}
\]
Argument I for two disconnected intervals in $1+1$ dim

Altogether, we have 6 relations among $s_{ij}$ and 6 constraints on $x$:

\[
\begin{align*}
S_{14} + S_{23} &\leq S_{13} + S_{24} & x &\leq S_{12} + S_{34} \\
S_{12} + S_{34} &\leq S_{13} + S_{24} & x &\leq S_{14} + S_{23} \\
S_{13} &\leq S_{12} + S_{23} & S_{14} + S_{34} - S_{24} &\leq x \\
S_{13} &\leq S_{14} + S_{34} & S_{12} + S_{23} - S_{24} &\leq x \\
S_{24} &\leq S_{12} + S_{14} & S_{23} + S_{34} - S_{13} &\leq x \\
S_{24} &\leq S_{23} + S_{34} & S_{12} + S_{14} - S_{13} &\leq x \\
\end{align*}
\]

Note: $\exists$ natural pairing between equations. Use these to get coincident upper & lower bounds on $x$, assuming two of the inequalities can be saturated:

\[
S_{12} + S_{23} + S_{34} + S_{14} - S_{13} - S_{24} \leq x \leq S_{12} + S_{23} + S_{34} + S_{14} - S_{13} - S_{24}
\]

$\Rightarrow \quad x = S_{12} + S_{34} + S_{23} + S_{14} - S_{13} - S_{24}$.
Argument 1 for two disconnected intervals in $1 + 1$ dim

Altogether, we have 6 relations among $s_{ij}$ and 6 constraints on $x$:

\begin{align*}
S_{14} + S_{23} &\leq S_{13} + S_{24} & X &\leq S_{12} + S_{34} \\
S_{12} + S_{34} &\leq S_{13} + S_{24} & X &\leq S_{14} + S_{23} \\
S_{13} &\leq S_{12} + S_{23} & S_{14} + S_{34} - S_{24} &\leq X \\
S_{13} &\leq S_{14} + S_{34} & S_{12} + S_{23} - S_{24} &\leq X \\
S_{24} &\leq S_{12} + S_{14} & S_{23} + S_{34} - S_{13} &\leq X \\
S_{24} &\leq S_{23} + S_{34} & S_{12} + S_{14} - S_{13} &\leq X
\end{align*}

Note: $\exists$ natural pairing between equations.
Use these to get coincident upper & lower bounds on $x$, assuming two of the inequalities can be saturated:

\begin{align*}
S_{12} + S_{23} + S_{34} + S_{14} - S_{13} - S_{24} &\leq X \leq S_{12} + S_{23} + S_{34} + S_{14} - S_{13} - S_{24} \\
\Rightarrow \quad X &= S_{12} + S_{34} + S_{23} + S_{14} - S_{13} - S_{24}
\end{align*}
Argument II for two disconnected intervals in \(1 + 1\) dim

Assume a linear ansatz:

\[ x = c_{12} s_{12} + c_{13} s_{13} + c_{14} s_{14} + c_{23} s_{23} + c_{24} s_{24} + c_{34} s_{34} \]

for some (universal) constants \(c_{ij}\).

Now consider special limits where we know \(x\) explicitly:

- \(p_1 \to p_2 \Rightarrow s_{12} \to 0, \ s_{13} \to s_{23}, \ s_{14} \to s_{24}, \ \text{and} \ x \to s_{34} \)
  \[ c_{34} = 1, \ c_{13} + c_{23} = 0, \ c_{14} + c_{24} = 0. \]

- \(p_2 \to p_3 \Rightarrow s_{14} \to 0, \ s_{12} \to s_{13}, \ s_{24} \to s_{34}, \ \text{and} \ x \to s_{14} \)
  \[ c_{14} = 1, \ c_{12} + c_{13} = 0, \ c_{24} + c_{34} = 0. \]

- \(p_3 \to p_4 \Rightarrow s_{34} \to 0, \ s_{13} \to s_{14}, \ s_{23} \to s_{24}, \ \text{and} \ x \to s_{12} \)
  \[ c_{12} = 1, \ c_{13} + c_{14} = 0, \ c_{23} + c_{24} = 0. \]

\[ c_{12} = c_{14} = c_{23} = c_{34} = -c_{13} = -c_{24} = 1 \]

\[ \Rightarrow x = s_{12} + s_{34} + s_{23} + s_{14} - s_{13} - s_{24} \ . \]

\[ \boxed{\text{QED}} \]
Multiply disconnected intervals in $1+1$ dim

Consider $X$ given by $m$ intervals specified by $n = 2m$ endpoints $p_i$. We have shown that for $n = 4$,

$$X = s_{12} + s_{34} + s_{23} + s_{14} - s_{13} - s_{24}.$$

We can easily generalise this to

$$X = \sum_{i,j=1}^{n} (-1)^{i+j+1} s_{ij}.$$

In the process we obtain higher-level ‘subadditivity’-type relations, e.g. for 6 endpoints $p_i < p_j < p_k < p_l < p_m < p_n$,

$$s_{ik} + s_{jm} + s_{ln} \geq s_{in} + s_{jk} + s_{lm}.$$
Higher dimensions

Natural question:
Can we generalize this geometric construction for entanglement entropy of disconnected regions to higher dimensions?

Note:
- ¬ QFT methods, but:
  - holographic entanglement entropy as given by area of extremal (minimal) surface holds in all dimensions
  - ⇒ strong subadditivity holds in all dimensions
- For translationally invariant (effectively 1-D) systems, the arguments are identical to the 1 + 1 dim. case

We offer a conjecture for 2 + 1 dim. QFT
Motivation

Time-dependent Multi-region Summary

$1 + 1$ dimensions

$2 + 1$ dimensions

Types of 2-dimensional regions

- cases (a) and (c) are generic, but they naturally define only three bulk surfaces
- case (b) is special intermediate case; but it naturally defines six bulk surfaces

$\Rightarrow$ we can apply previous arguments here:

$$X(b) = s_{12} + s_{34} + s_{23} + s_{14} - s_{13} - s_{24}$$

- however, case (b) can be ‘resolved’ into (a) or (c). This motivates a conjecture for the general cases.
Types of 2-dimensional regions

- cases (a) and (c) are generic,
  but they naturally define only three bulk surfaces
- case (b) is special intermediate case;
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  \[ X(b) = s_{12} + s_{34} + s_{23} + s_{14} - s_{13} - s_{24} \]

  however, case (b) can be ‘resolved’ into (a) or (c).
  This motivates a conjecture for the general cases.
Conjecture for 2-dimensional regions

Since we need at least 6 minimal surfaces in the bulk, let us introduce auxiliary curves $C_5$ and $C_6$ to lift (a) and (c) to configurations which admit 6 bulk surfaces:

Conjecture: Since $x$ cannot depend on our choice of the auxiliary curves $C_5$ and $C_6$, we minimize over all such configurations:

$$x(a) = \inf_{C_5, C_6} \left\{ s_{12} - s_{1536} + s_{1546} + s_{2536} - s_{2546} + s_{34} \right\}$$

$$x(c) = \inf_{C_5, C_6} \left\{ s_{1526} - s_{1536} + s_{14} + s_{23} - s_{2546} + s_{3546} \right\}$$
Support for the conjecture

\[ x(a) = \inf_{C_5, C_6} \{ s_{12} - s_{1536} + s_{1546} + s_{2536} - s_{2546} + s_{34} \} \]

\[ x(c) = \inf_{C_5, C_6} \{ s_{1526} - s_{1536} + s_{14} + s_{23} - s_{2546} + s_{3546} \} \]

- for limiting case (b), \( \{ s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34} \} \) are necessary to specify \( x(b) \); hence \( x(a,c) \) comprise the minimal ansatz

- infimum to satisfy subadditivity relations as \( C_5 \to C_6 \ldots \)

- UV divergence \( \propto C_1 + C_2 + C_3 + C_4 \) as required

- correct limiting behaviour as (a) \( \to \) (b) and (c) \( \to \) (b)

- correct single-region limits
Summary (part I)

- Holographic dual of entanglement entropy (for static bulk ST) = area of \textit{minimal} surface in bulk anchored at \( \partial \mathcal{A} \)

- For time-dependent bulk, use covariant generalization: = area of \textit{extremal} surface in bulk anchored at \( \partial \mathcal{A} \) also given by surface with vanishing null expansions

- Can be used to study EE for time-dependent bulk geometries e.g. EE increases in Vaidya satisfying energy conditions

- Since well-defined in time-dependent system, can analyze quantum systems far from equilibrium, zero-temperature quantum phase transitions, ...\n
\( \leadsto \) study condensed matter systems using AdS/CFT
Holographic dual of entanglement entropy (for static bulk ST) = area of minimal surface in bulk anchored at $\partial A$

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Summary (part II)

- For a static state in $1 + 1$ QFT, we derive the expression for entanglement entropy for disconnected regions:

$$x = \sum_{i,j=1}^{n} (-1)^{i+j+1} s_{ij}.$$

- This reproduces and generalizes the expression obtained by CFT methods.
- Geometrical picture immediately yields higher order generalizations of strong subadditivity.
- We presented conjecture & supporting evidence for EE of disconnected regions in $2 + 1$ QFT (no direct comparison w/ QFT results yet).
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Veronika Hubeny  Holographic Entanglement Entropy
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Future directions

- General proof for strong subadditivity in time-dependent cases
- Proof of conjecture for disconnected regions in $2 + 1$
- Generalizations to higher dimensions, multiple regions
- Relations between covariant constructions, physical interpretation of $\mathcal{X}$, $\mathcal{Z}$
- “Second Law” for entanglement entropy?
- Full bulk metric extraction?
- Study more general asymptopia
Details of Vaidya-AdS

Use to describe collapse to a black hole in AdS; 3-dim metric:

\[ ds^2 = -f(r, v) \, dv^2 + 2 \, dv \, dr + r^2 \, dx^2 \]

with \( f(r, v) \equiv r^2 - m(v) \) interpolating between AdS and Schw-AdS.

Energy-momentum tensor \( T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} + \Lambda \, g_{\mu\nu} \):

\[ T_{vv} = \frac{1}{2r} \frac{dm(v)}{dv} \]

Null energy condition holds \( \iff \) mass accretes: \( m'(v) \geq 0 \)
(NEC: \( T_{\mu\nu} k^\mu k^\nu \geq 0 \) for any null vector \( k^\mu \))
Details of coincident bounds argument

• Assume $x$ is given by a unique expression in terms of the $s_{ij}$ which satisfy the geometric constraints (for any state, etc.)

• Now use an extreme case of ‘allowed’ $s_{ij}$:

$$s_{14} + s_{23} = s_{13} + s_{24} \quad \text{and} \quad s_{13} = s_{12} + s_{23}$$

(one can check these are mutually consistent)

• Then $\forall$ constants $a_1$ and $a_2$,

$$s_{14} + s_{34} - s_{24} + (s_{12} + s_{23} - s_{13}) a_1 \leq x \leq$$

$$s_{12} + s_{34} + (s_{14} + s_{23} - s_{13} - s_{24}) a_2$$

• Now set $a_1 = a_2 = 1$.

$$\Rightarrow \quad x = s_{12} + s_{34} + s_{23} + s_{14} - s_{13} - s_{24}$$

• Any choice of saturating pair of eqs will yield the same answer.