Second order viscous hydrodynamics from AdS/CFT

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From Strings to Things
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[R. Janik, MPH], [+ A. Buchel, P. Benincasa]
Motivation: from QCD to $\mathcal{N} = 4$ SYM...

- heavy ion collisions @ RHIC - strongly coupled quark-gluon plasma (QGP)
- fully dynamical process - need for a new tool
- idea: exchange QCD in favor of $\mathcal{N} = 4$ SYM and use the gravity dual
- there are differences
  - SUSY
  - conformal symmetry at the quantum level
  - no confinement...
- ... but probably not very important at high temperature (at least for qualitative description)
Questions: ... from $\mathcal{N} = 4$ SYM to gravity dual

General questions

• what is gravity counterpart of gauge theory dynamics?
• how to construct the gravity dual?
• what is the role of gravitational singularities?

Hydrodynamics

• what is gravity counterpart of gradient expansion?
• what are the transport properties of plasma
  - viscosity?
  - relaxation time, $\lambda_1$?
Gauge theory setup

Snapshot from RHIC

Simplified description of QGP

- one-dimensional expansion along the collision axis $x^1$
- natural coordinates
  - proper time $\tau$ and rapidity $y$
  - $x^0 = \tau \cosh y$, $x^1 = \tau \sinh y$
- boost invariance (no rapidity dependence)

Bjorken, 1983
Boost-invariant energy-momentum tensors

• we work in the coordinates $x^\mu = (\tau, y, x_\perp) = (\tau, y, x_1, x_2)$

• imposing symmetries
  - boost-invariance
  - rotational, translational, $Z_2$ symmetries in $\perp$ plane

and conservation

$$\nabla_\mu T^{\mu\nu} = 0$$

gives the energy-momentum tensor of a form

$$T^{\mu\nu} = \begin{pmatrix}
  f(\tau) & 0 & 0 & 0 \\
  0 & -\tau^3 f'(\tau) - \tau^2 f(\tau) & 0 & 0 \\
  0 & 0 & f(\tau) + \frac{1}{2} \tau f'(\tau) & 0 \\
  0 & 0 & 0 & f(\tau) + \frac{1}{2} \tau f'(\tau)
\end{pmatrix}$$

• $T^{\mu\nu}$ fully expressed in terms of a single function $T^{\tau\tau} = f(\tau)$
AdS/CFT correspondence

Gauge-gravity duality is an equivalence between

\[ \mathcal{N} = 4 \text{ Supersymmetric Yang-Mills in } \mathbb{R}^{1,3} \]
- strong coupling
- non-perturbative results
- gauge theory operators

\[ \text{Superstrings in curved } \text{AdS}_5 \times S^5 \text{ 10D spacetime} \]
- (super)gravity regime
- classical behavior
- supergravity fields

AdS/CFT dictionary relates
energy-momentum tensor of \( \mathcal{N} = 4 \) SYM to 5D AdS metric
Elements of holographic renormalization

Gravity near the boundary

- we seek for the metric of a form

$$ds^2 = G_{MN}dx^Mdx^N = \frac{\tilde{g}_{\mu\nu}dx^\mu dx^\nu}{z^2} + \frac{dz^2}{z^2}$$

which solves 5-dimensional Einstein eqns with $\Lambda = -6$

$$R_{MN} + 4g_{MN} = 0$$

- 4-dimensional part of the metric $\tilde{g}_{\mu\nu}$ has an expansion around $z = 0$

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \left(\frac{N_c^2}{2\pi^2}\right)^{-1} \cdot \langle T_{\mu\nu} \rangle \cdot z^4 + O(z^6)$$

Boost-invariant metric

$$ds^2 = \frac{-e^{a(\tau,z)}d\tau^2 + \tau^2 e^{b(\tau,z)}dy^2 + e^{c(\tau,z)}dx^2}{z^2} + dz^2$$
How to get the physics right?

Holography

- let’s focus for a while on equation

\[ \tilde{g}_{\mu\nu} = \sum_{n=0}^{\infty} \tilde{g}_{\mu\nu}^{(2n)}(\tau) z^{2n} \]

- we can find \( \tilde{g}_{\mu\nu}^{(2n)} \) iterating

\[ R_{MN} + 4G_{MN} = 0 \]

starting from \( \tilde{g}_{\mu\nu}^{(0)} = \eta_{\mu\nu}, \tilde{g}_{\mu\nu}^{(2)} = 0 \) and \( \tilde{g}_{\mu\nu}^{(4)} = \langle T_{\mu\nu} \rangle \)

- GR constraints translate into \( \nabla_\mu < T_{\mu\nu} > = 0 \)

- Conserved \( T_{\mu\nu} \) is fixed uniquely once we know \( T^{\tau\tau} = f(\tau) \)

PROBLEM I: how to get physical \( f(\tau) \)?

PROBLEM II: how to get IR right \( (z \to \infty) \)?
Scaling variable and asymptotic solution

Scaling variable:

- the regime of interest (RHIC) is $\tau \to \infty$
- let’s guess the asymptotics

$$f(\tau) \sim \tau^{-s}, \text{ where } 0 < s < 4$$

- analysis of $\tilde{g}_{\mu \nu} = \sum_{n=0}^{\infty} \tilde{g}_{\mu \nu}^{(2n)}(\tau)z^{2n}$ for $f(\tau) = \frac{1}{\tau^s}$ suggest introduction of scaling variable $v = \frac{z}{\tau^{s/4}}$

so that $\tilde{g}_{\mu \nu}(\tau, z) = \tilde{g}_{\mu \nu}(\frac{z}{\tau^{s/4}}) + O(\frac{1}{\tau^\#})$

Comments:

- reorganizes holographic series
- reduces PDE’s in $(\tau, z)$ to ODE’s in $v = \frac{z}{\tau^{s/4}}$
- scaling variable solution gives IR physics right
Perfect fluid from nonsingularity

s-dependent asymptotic metric

\[ ds^2 = -e^{a_0(s)(v)} d\tau^2 + \tau^2 e^{b_0(s)(v)} dy^2 + e^{c_0(s)(v)} dx^2_{\perp} + O\left(\frac{1}{\tau^\#}\right) + \frac{dz^2}{z^2} \]

- functions \( a^{(s)}(v) \), \( b^{(s)}(v) \), \( c^{(s)}(v) \) exist for any \( 0 < s < 4 \)
- how to fix the physics, how to fix \( s \)?

Regularity of curvature invariants

- for generic \( s \), \( R_{MNOP}R^{MNOP} \) around \( v = \Delta(s)^{1/4} \) looks like

\[ R_{MNOP}R^{MNOP} = \frac{\#-4}{(v - \Delta(s)^{1/4})^4} + \cdots + \frac{\#-1}{(v - \Delta(s)^{1/4})} + \text{REGULAR} \]

- cancelation of poles leads to \( s = \frac{4}{3} \) (perfect fluid behavior)
Perfect fluid metric

Asymptotic metric:

\[ \text{d}s^2 = \frac{1}{z^2} \left[ -\frac{(1 - \frac{z^4}{3\tau^{4/3}})^2}{(1 + \frac{z^4}{3\tau^{4/3}})} \text{d}\tau^2 + \tau^2(1 + \frac{z^4}{3\tau^{4/3}})\text{d}y^2 + (1 + \frac{z^4}{3\tau^{4/3}})\text{d}x^2_\perp + \text{d}z^2 \right] \]

- the entropy is constant (perfect fluid - PF)
- temperature scales as \( T \sim \tau^{-1/3} \)

[hep-th/0512165]

Subasymptotic corrections

- the PF metric valid only in the limit \( \tau \to \infty \) while \( v = \frac{z}{\tau^{1/3}} \)
- there are subasymtotic corrections \( (O(\frac{1}{\tau^\#})) \)!!!
- they correspond to viscous terms on the CFT side
General remarks

- hydro = long-wavelength theory (gradient expansion)
- causal theory - second order viscous hydrodynamics

Gradient expansion

- each gradient $\nabla_\mu u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\mu\rho} u^\rho$ is suppressed by $\frac{1}{TL}$ ($L$ - characteristic length scale and $T$ - temperature)
- $\Gamma \sim \frac{1}{\tau}$ and $T \sim \tau^{-1/3}$ so expansion takes the form
  $$f(\tau) \sim \frac{1}{\tau^{4/3}} \left(1 + \#1 \frac{1}{\tau^{2/3}} + \#2 \left(\frac{1}{\tau^{2/3}}\right)^2 + \ldots\right)$$
- $\frac{1}{\tau^{4/3}}$ comes from dimensional analysis ($f(\tau) \sim T^4$)
- the rest ($\frac{1}{\tau^{2/3}}$) is the gradient expansion
Transport properties

- constants $#_1$ and $#_2$ have physical interpretation!
- they are related to transport coefficients of plasma

$$f(\tau) = \left( \frac{N_c^2}{2\pi^2} \right) \frac{1}{\tau^{4/3}} \left\{ 1 - 2\eta_0 \frac{1}{\tau^{2/3}} + \left[ \frac{3}{2} \eta_0^2 - \frac{2}{3} (\eta_0 \tau^0_\Pi - \lambda^0_1) \right] \frac{1}{\tau^{4/3}} + \cdots \right\}$$

where $(C = \frac{N_c^2}{2\pi^2})$
- shear viscosity $\eta = C \eta_0 \left( \frac{f}{C} \right)^{3/4}$
- relaxation time $\tau_\Pi = \tau^0_\Pi \left( \frac{f}{C} \right)^{-1/4}$
- new quantity $\lambda_1 = C \lambda^0_1 \left( \frac{f}{C} \right)^{1/2}$

Let’s AdS/CFT to get the constants
Gravity dual of gradient expansion

Reminder:

\[ ds^2 = -e^{a(\tau, z)} d\tau^2 + \tau^2 e^{b(\tau, z)} dy^2 + e^{c(\tau, z)} dx^2_{\perp} + dz^2 \]

Gravitational gradient expansion

- requiring nonsingularity gives \((R^2 = R_{MNOP}R^{MNOP})\)

\[
\begin{align*}
  a(\tau, z) &= a_0\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{2/3}} a_1\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{4/3}} a_2\left(\frac{z}{\tau^{1/3}}\right) + \ldots \\
  b(\tau, z) &= b_0\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{2/3}} b_1\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{4/3}} b_2\left(\frac{z}{\tau^{1/3}}\right) + \ldots \\
  c(\tau, z) &= c_0\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{2/3}} c_1\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{4/3}} c_2\left(\frac{z}{\tau^{1/3}}\right) + \ldots \\
  R^2(\tau, z) &= R^2_0\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{2/3}} R^2_1\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{4/3}} R^2_2\left(\frac{z}{\tau^{1/3}}\right) + \ldots
\end{align*}
\]

- this is gravitational counterpart of the gradient expansion

\[
f(\tau) \sim \frac{1}{\tau^{4/3}} \left(1 + \#_1 \frac{1}{\tau^{2/3}} + \#_2 \left(\frac{1}{\tau^{2/3}}\right)^2 + \ldots\right)
\]
First order and second order (shear viscosity)

First order solution \( \left( \frac{1}{\tau^{2/3}} \right) \)

\[
a_1(v) = 2\eta_0 \frac{(9 + v^4)v^4}{9 - v^8}
\]

\[
b_1(v) = -2\eta_0 \frac{v^4}{3 + v^4} + 2\eta_0 \log \frac{3 - v^4}{3 + v^4}
\]

\[
c_1(v) = -2\eta_0 \frac{v^4}{3 + v^4} - \eta_0 \log \frac{3 - v^4}{3 + v^4}
\]

[hep-th/0607123],

First order Riemann squared is regular

\[
R_1^2 = -\frac{41472\eta_0(3 - v^4)v^4}{(3 + v^4)^5}
\]

\( \eta_0 \) cannot be fixed here
solution in the second order has one integration constant (it corresponds to a linear combination of $\tau_0^0$ and $\lambda_1^0$)

- Riemann squared in the second order $R_2^2$ takes the form

$$R_2^2 = 36 \frac{\eta_0^2 - \frac{1}{6.3^{1/2}}}{(v - 3^{1/4})^4} + 24 \cdot 3^{1/4} \frac{\eta_0^2 - \frac{1}{6.3^{1/2}}}{(v - 3^{1/4})^3} - 6 \cdot 3^{1/2} \frac{\eta_0^2 - \frac{1}{6.3^{1/2}}}{(v - 3^{1/4})^2}$$

$$-18 \cdot 3^{1/4} \frac{\eta_0^2 - \frac{1}{6.3^{1/2}}}{v - 3^{1/4}} + \text{REGULAR}$$

- all poles are cancelled for $\eta_0 = \frac{1}{2^{1/2} 3^{3/4}}$

[hep-th/0610144]

- there are no contraints on $\tau_0^0$ and $\lambda_1^0$ in the second order (one has to go one order higher - generic behavior)
Third order - the breakdown of nonsingularity

- integration constant responsible for $\tau_0$ and $\lambda_1^0$ is fixed by requiring cancelation of all poles at $v = 3^{1/4}$

- there is always a logarithmic contribution

$$\mathcal{R}_{MNOP} \mathcal{R}^{MNOP} = \text{finite} + \frac{1}{t^2} 8^{1/2} 3^{3/4} \log (3^{1/4} - v) + O\left(\frac{1}{\tau^{8/3}}\right)$$

- $\log(3^{1/4} - v)$ shows in higher order curvature invariants (e.g. $\mathcal{R}_{MN} \mathcal{R}^{MN}$, $\mathcal{R}_{MNOP} \mathcal{R}^{OPQS} \mathcal{R}_{QS}^{MN}$)

- cannot be cancelled by a finite number of massive fields

- present in the case of
  - KK of type IIB to $\text{AdS}_5 \times S^5$ (1 massive field - $S^5$ radius)
  - Klebanov-Witten model (2 massive fields)

[hep-th/0703143], 0712.2025 [hep-th]
General remarks on singularities

The statement

- curvature singularities should not be present in SUGRA

Interpretations and possible resolutions

- they may signal plasma instabilities

- infinite tower of string states can do the job

- $\tau, z$ not adapted to probe the physics at the horizon (we don’t know where the point $v = 3^{1/4}$ is when $\tau \to \infty$)

[hep-ph/0412016]

0712.2025 [hep-th], 0803.3421 [hep-th]
Summary of results

Main result:
- gauge theory dynamics from AdS/CFT

\[ f(\tau) = \left( \frac{N_c^2}{2\pi^2} \right) \frac{1}{\tau^{4/3}} \left\{ 1 - 2\eta_0 \frac{1}{\tau^{2/3}} + \left[ \frac{3}{2} \eta_0^2 - \frac{2}{3} (\eta_0 \tau^0_1 - \lambda_1^0) \right] \frac{1}{\tau^{4/3}} + \cdots \right\} \]

Asymptotic behavior:
- perfect fluid behavior from nonsingularity
- temperature cools down as \( \frac{1}{\tau^{1/3}} \)

Viscosity coefficient:
- \( \eta_0 = \frac{1}{2^{1/2} 3^{3/4}} \)
- viscosity saturates the bound \( \left( \frac{\eta}{s} = \frac{1}{4\pi} \right) \)

Relaxation time stuff:
- \( \tau_\pi = \frac{2 - \log 2}{2\pi T} \)

0712.2451 [hep-th]
boosted black brane (BB) with slowly-varying $b$ and $u^\mu$
\[
ds^2 = -2u_\mu dx^\mu dr - r^2 \left(1 - \frac{1}{b^4 r^4}\right)u_\mu u_\nu dx^\mu dx^\nu + P_{\mu\nu} + O\left(\frac{1}{LT}\right)
\]

1. completely regular construction
2. hydrodynamics = long-wavelength distortions of BB horizon

0712.2456 [hep-th], 0803.2526 [hep-th]
Conclusions

What’s good:

• dynamics @ strong coupling
• agrees with near-equilibrium methods

What’s bad / Open problems:

• singularities in the bulk
  (possible signatures of plasma instabilities?)
• connection with Minwalla et.al. 0712.2456 [hep-th]
  (work in progress)

Future applications:

• Bjorken expansion of other conformal plasmas
  (A.Buchel, MPH, R.Myers, S.Vazquez - in preparation)
• other symmetric flows
  (work in progress)