Bosonic Symmetries of BEC in nuclei

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I.N.T. - Seattle - USA - 2007
1. In the first part I’ll briefly try to review a few concepts relevant to Bose-Einstein condensation, drawing a few comparisons with the nuclear case.

2. Next I will attempt a classification of the dynamical symmetries of a boson condensate, providing a general way to find energy formulae.

3. In the second part I will discuss a simple toy-model of interest to nuclear physics and an extension with a dynamical supersymmetry (that was presented at the Cluster ’07 conference in Sept.).
Characteristics of a Bose-Einstein condensate

- Below a certain critical temperature a gas of weakly interacting bosons condensate into a state with coherent wavefunctions.

- It is a highly ordered macroscopic (high N) quantum system. An entire population of bosons act like waves instead of particles.

Alkali atoms (Rb, Cs, Na, Li, …) have been used at ultra-low temperatures as well as polaritons in semiconductors at somewhat higher temperatures.
Critical Temperature for non-interacting $\alpha$

$$T_c = 2\pi \frac{\hbar^2}{m} \left( \frac{n}{\zeta(3/2)} \right)^{2/3}$$

Temperature is measured in MeV (the Boltzmann constant is not shown)

If one naively takes $n=N/V$ with $N= A/4$ and $V=4\pi r^3/3$ then the particle density does not depend on $A$.

I get that the Critical Temperature for alpha-condensation in nuclei is around 3.67 MeV.

$$\lambda_{dB} = \left( \frac{2\pi \hbar^2}{mk_BT} \right)^{1/2}$$

<table>
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<tr>
<th>Energy (MeV)</th>
<th>Wavelength (fm)</th>
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<td>8</td>
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<tr>
<td>10</td>
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Condensation occurs when the atomic de Broglie wavelength becomes comparable to the interparticle spacing.

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If we want to address the problem of boson condensation in nuclei we have to face the following problems:

• alpha particles **ARE NOT** genuine bosons (It is important to take into account Pauli exclusion). Nevertheless atoms, Cooper pair or polaritons are not true bosons too, but it works.

• alpha particle **ARE** interacting through strong and EM forces and are ‘very close’ (fermion and pion exchanges might represent a serious concern)

• we have always a small number of particles (in other words, we are far from statistical equilibrium)

Let’s anyhow proceed boldly and see how far we can go with bosonic symmetries of Bose-Einstein Condensates!
Each boson has an intrinsic spin (s,p,d,f,…), but they are all in a s-state \((\ell=0)\) of orbital angular momentum.

Trivially for each species one has boson annihilation and creation operators and form bilinear operators that close into a bosonic unitary algebra whose dimensions depend on \(\ell\).

\[
[b_i^{(\ell)}, b_{i'}^{(\ell')}\dagger] = \delta_{i,i'} \delta_{\ell,\ell'}
\]

\[
G_{i,i'}^{\ell,\ell'} = b_{i'}^{(\ell')}\dagger b_i^{(\ell)}
\]

\[
U(n) \supset U_i(2\ell + 1) \otimes U_{i'}(2\ell' + 1) \otimes ...
\]

It’s interesting to see what kind of symmetries we can expect and to use them to write down general energy formulae.

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A classification of Bosonic Symmetries of BEC

It is tempting to make a few definitions:

• (I) mono-bosonic condensate \( \Rightarrow U(2\ell + 1) \)

• (II) di-bosonic condensate

  (IIa) homo-spin (same \( \ell \)) \( \Rightarrow U_1(2\ell + 1) \otimes U_2(2\ell + 1) \)

  (IIb) hetero-spin (\( \neq \ell \)) \( \Rightarrow U_1(2\ell + 1) \otimes U_2(2\ell' + 1) \)

• (III) multi-bosonic condensate

  (IIIa) homo-spin

  (IIIb) hetero-spin

Notation: \( N_{spdf} \)

where \( N \) is the number of species and the indices are the \( \ell \)'s
A classification of Bosonic Symmetries of BEC

1_s \quad \text{mono-condensate of spin } \ell=0 \quad \Rightarrow \quad \text{U}(1)

Not very illuminating!
No rotational group allowed!!

1_p \quad \text{mono-condensate of spin } \ell=1 \quad \Rightarrow \quad \text{U}(3)

You start to have some algebraic structure!!
You can have rotational group!

\begin{align*}
\text{U}(3) & \supset \text{SO}(3) \supset \text{SO}(2) \\
| & \quad | & \quad | \\
n_p & \quad L & \quad M
\end{align*}

\[ E = E_0 + an_p + bn_p^2 + cL(L+1) \]

1_d \quad \Rightarrow \quad \text{fun begins} \quad (\text{fun for Piet, ask him,… not me})

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A classification of Bosonic Symmetries of BEC

$2_{ss'}$ di-condensate with two different $\ell=0$ species

(might be the case of mixtures of different atoms in a alkali BEC)

$U(2) \supset U_1(1) \otimes U_2(1)$  \text{ See Ch.1 of Alejandro Piet's book, $U(1)$ limit} \quad \text{\bf + } \ cN+dN(N+1)$

$2_{sp}$ hetero-spin di-condensate

( might be the case of alpha and d in nuclei )

$U(4) \supset U_1(1) \otimes U_2(3)$  \text{ See Vibron model}

\[ E = E_0 + an + bn^2 + cN + dN(N+1) \]

\[ E = E_0 + an_p + bn_p^2 + cL(L+1) + k_1N + k_2N(N+3) \]

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Inclusion of the higher algebra - hint

For each given problem \( \mapsto \)

there is a proper unitary algebra that contains all the others and whose Casimir corresponds to the total number of particles \( \mapsto \)

including this in the energy formula allows to write the total energy for a given number of bosons \( \mapsto \)

(might be relevant to atomic physics because it enters into a thermodynamic description of the gas: example in the partition function).

\[
Z = \text{tr}(e^{-\beta H})
\]

I still have to investigate this aspect (next life)

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Let me talk non-sense for one (more) slide

Suppose you are making a BEC out of alkali atoms, for example 7Li. The electronic configuration is $2S_{1/2}$ and the nuclear g.s. is $I=3/2$. Then you might have two possibilities for the (total atomic) boson: $\ell=1$ or $\ell=2$

This is OK as long as you have some mixture of aligned and antialigned states (if somebody among you knows it is not the case, then please tell it now or never…………)

Hetero-spin di-condensate: $2_{pd}$

$$U(8) \supset U_d(5) \otimes U_p(3) \supset SO_d(5) \otimes SO_p(3) \supset SO_d(3) \otimes SO_p(3)$$

$N \quad nd \quad np \quad \tau \quad Jd \quad Jp$
Back to nuclei: a simple toy-model...

Let’s try with a very simple toy model:
alpha-boson ($\ell = 0$) + d-boson ($\ell = 1$)

So we expect a U(4) algebra. We can construct a Hamiltonian with a linear combination of the Casimir operators:

$$H = C_1[ \text{U}(4)] + C_2[ \text{U}(4)] + C_1[ \text{U}(3)] + C_2[ \text{U}(3)] + C[\text{SO}(3)]$$

Then you have a dynamical symmetry (exactly solvable) and the eigenvalues may be read off directly as:

$$E = b_1N + b_2N(N+3) + c_1n_p + c_2n_p(n_p+2) + d J(J+1)$$
A simple toy-model...

If you have N bosons in total, then you have

\[ n_p = 0, 1, 2, \ldots, N \quad \text{and} \quad L = n_p, n_p - 2, \ldots, 1 \text{ or } 0 \]

\begin{align*}
N=1 & \quad n_p = 0, 1 \quad \text{these are the alpha and the deuteron g.s.} \\
N=2 & \quad n=0 \quad \text{is } 8\text{Be} \quad L=0 \\
 & \quad n=1 \quad \text{is } 6\text{Li} \quad L=1 \\
 & \quad n=2 \quad \text{is some d+d state in } 4\text{He} \text{ with } L=2, 0 \\
N=3 & \quad n=0 \quad \text{is } 12\text{C} \quad L=0 \\
 & \quad n=1 \quad \text{is } 10\text{B} \quad L=1 \\
N=4 & \quad n=0 \quad \text{is } 16\text{O} \quad L=0 \\
\end{align*}

\[ \text{.........................} \]
One must fit the 5 parameters to a number of states trying to confine ourselves as much as possible to the lightest nuclei with the idea then to extrapolate to heavier ones:

Ground states of 2H, 4He, 6Li, 8Be stop at mass 8
Excited 0+ at 20.21 MeV in 4He

\[ b_1 = 2241.23, \quad b_2 = 45.92, \quad c_1 = 14164.5, \quad c_2 = -1329.48, \quad d = 505 \text{ (in keV)} \]

\begin{align*}
N=3 & \quad n=0 & \quad L=0 & \quad 7.55 \text{ MeV} & \quad \text{Exp. 0+ 7.65 MeV (Hoyle)} \\
N=4 & \quad n=0 & \quad L=0 & \quad 14.98 \text{ MeV} & \quad \text{Exp. 0+ at 15.09 MeV} \\
(N=5 & \quad n=0 & \quad L=0 & \quad 20.08 \text{ MeV} & \quad \text{(Funaki, Schuck get 14.94!)}
\end{align*}
Too many states! For the moment would be difficult to compare unless experimentalists gives us more reliable spin-parity assignements. Anyway one can see some strength in that energy region for 16O+alpha reactions (not a proof though).
...with reasonable predictions!!
Certain (gas-like) (alpha-cluster) states, that many people call the Bose-Einstein Condensed states in nuclei, comes out at more or less the appropriate energies, just because of simple arguments based on dynamical symmetry principles (!)

(namely those states are the ones that NCSM, FMD, etc., have difficulties to reproduce)
A more ambitious model with supersymmetry
(and therefore more wrong ...)

With the simple toy-model one connects states of different nuclei that are built out of the same building blocks.

Together with bosonic degrees of freedom there are also fermionic ones. One way to build up a more general model is to insert (in the simplest way possible) also the fermions.
We propose an algebraic scheme based on Lie superalgebras with the aim to frame in a single model all the cluster states that can be constructed by using $d$, $\alpha$ and fermions as building blocks.

Super-partners $\Rightarrow$

Complementary with Prof. Cseh work, where they treat relative motion! Optimum would be to join the two approaches!! Too difficult for me at the moment.

Why? Because symmetry provides insight and gives us a reference frame to attack the problem of clusterization in nuclei.
How one builds the $u(4|4)$ Lie superalgebra - 1

We define creation and annihilation operators for bosons and fermions: the former commute the latter anticommute in the standard way.

\[
\begin{align*}
    b^\dagger_\nu, b_\nu & \quad \nu = 0, 1, 2, 3 \\
    a^\dagger_{s_z t_z}, a_{s_z t_z} & \quad s_z, t_z = \pm 1/2
\end{align*}
\]

Then one takes the bilinear combinations of all the operators forming a bosonic and a fermionic sector.

It can be shown that they close according to the $u(4|4)$ Lie superalgebra.

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One then splits the elements into a bosonic and a fermionic sector, $X_a$ and $Y_b$ respectively, then they satisfy:

$$
[X_a, X_b] = c_{ab}^c X_c , \quad c_{ab}^c = -c_{ba}^c
$$
$$
[X_a, Y_b] = d_{ab}^c Y_c , \quad d_{ab}^c = -d_{ba}^c
$$
$$
\{Y_a, Y_b\} = f_{ab}^c X_c , \quad f_{ab}^c = +f_{ba}^c
$$

where $c, d, f$ are graded structure constants. This property, together with generalized Jacobi relations defines the Lie super-algebra which admits super-representations, super-multiplets, super-Casimir operators and dynamical supersymmetries.
Dynamical super-symmetry

• Complicated lattice of subalgebras arising from the $u(4|4)$: we take the one passing through the maximal boson and fermion (= Wigner’s) unitary algebras and ending with the 3-dim rotational group

\[ u(4|4) \supset u^B(4) \oplus u^F(4) \supset u^B(3) \oplus su^F_{ST}(4) \supset SO^B(3) \oplus su^F_S(2) \oplus su^F_T(2) \supset SO_{BF}(3) \]

\[ \hat{H} = a_1 C_1 [u(4|4)] + a_2 C_2 [u(4|4)] + b_1 C_1 [u^B(4)] + b_2 C_2 [u^B(4)] + c_1 C_1 [u^B(3)] + c_2 C_2 [u^B(3)] + d C_2 [so_{BF}(3)] + e C_2 [su^F_T(2)] \]

\[ E = a_1 N + a_2 N(N - 1) + b_1 N + b_2 N(N + 3) + c_1 n_p + c_2 n_p(n_2 + 2) + d J(J + 1) + e T(T + 1) \]

• Dynamical super-symmetry: general hamiltonian written as a linear combination of Casimir operators

• Exact diagonalization and formula for eigenvalues (Mass formula)
Quantum numbers and supermultiplets

A generic state is labeled by the quantum numbers of the chain of subalgebras that we have chosen

\[ \mathcal{N} \to (N,0) \oplus (N-1,1) \oplus \cdots \oplus (N-4,4) \]

\[ N \to n_p = 0, 1, 2, \cdots, N \]

\[ M \to \text{see table I} \]

\[ n_p \to L = n_p, n_p - 2, n_p - 3, \cdots, 1 \text{ or } 0 \]

\[ L, S \to J = | \vec{L} - \vec{S} |, \cdots, | \vec{L} + \vec{S} | \]

\( \mathcal{N} \) is the number of superpartners: either bosons or fermions.
The \( \mathcal{N} = 1 \) supermultiplet contains all the building blocks, \( p, n, d \) and \( \alpha \)
The \( \mathcal{N} = 2 \) supermultiplet contains two of them, \( \alpha \alpha, \alpha d, \alpha p, \alpha n, dd, dp, dn, pn \), and so on.....
Example (to fix ideas): magnetic substates

\[ J \begin{cases} \text{M} = +J \\ \text{M} = -J \end{cases} \]

\[ \text{O}(3) \quad \text{O}(2) \]

\[ \mathcal{N} = 2 \begin{cases} (N, M) = (2, 0) \\ (N, M) = (1, 1) \\ (N, M) = (0, 2) \end{cases} \]

\[ \text{U}(4|4) \quad \text{U}^B(4) \otimes \text{U}^F(4) \]
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<tr>
<th>$\mathcal{N}$</th>
<th>$N$</th>
<th>$M$</th>
<th>$n_p$</th>
<th>$L$</th>
<th>$S$</th>
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<td>1</td>
<td>0</td>
<td>$d^3$</td>
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</tbody>
</table>

There are states in different IRREP’s which refer to different partitions of the same isotope!!

And then there are states which refer to the same partition, but with different angular momentum.

AUF BAU
Fit to experimental masses ... 

\[ E = a_1 \mathcal{N} + a_2 \mathcal{N}(\mathcal{N}-1) + b_1 N + b_2 N(N+3) + c_1 n_p + c_2 n_p(n_p+2) + \\
+ d J(J+1) + e T(T+1) \]

We need to fix the 8 parameters to a number of states trying to confine ourselves as much as possible to the lightest nuclei with the idea to extrapolate then to heavier ones:

- Ground states of \(^2\text{H}, \(^4\text{He}, \(^6\text{Li}, \(^8\text{Be}\)
- Average over pairs of isotopes: \(\text{p+n} \quad \text{\(^5\text{He}+\text{\(^5\text{Li}\)}}\)
- Two resonant states: \(^4\text{He}(0^+ \text{at 20.21}) \quad \text{\(^8\text{Be} (2^+ \text{at 3.04})} \)

I used a trial-and-error procedure, a least-square fitting on a larger data set would be desirable (maybe you can help...)

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... still the fitted ones ...
... and results for even nuclei (boson symmetry) ...
... and for odd nuclei (Supersymmetry)
Finally, new result obtained here at the conference!

A proof of the goodness of this type of meetings is that you have a chance to see, learn, talk, check, add and improve!!

The $(2^+)$ state at 10.3 in $^{12}$C was mentioned by M. Itoh (1st day) as a dilute $^8$Be$(2^+) + \alpha$ state: one of the three alphas is in a D-orbit. This, in my algebraic model, would correspond to the $|NH N_{np} JT\rangle = |3\ 3\ 0\ 2\ 0\rangle$ state which comes out at

Theory: 10.591 MeV  Experiment: 9.9 MeV  
(10.3 MeV)

This is a hint, based purely on (dynamical) symmetry arguments, that the state might indeed be a $2^+$ !!

Another point is the suggestion of the 14.94 MeV 4-$\alpha$ condensed state by Funaki. I obtain 15 MeV!

Then… who knows, the conference is not finished yet...

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Cluster spectroscopic factors

By evaluating the action of specific operators on the basis states (i.e. non-null matrix elements), one can calculate various types of transfer probabilities and from them give predictions on specific reactions. Example:

\[
\langle [N], n_p - 1, L - 1, M' \mid [s^\dagger \times \hat{p}]^{(1)}_m \mid [N], n_p, L, M \rangle = \sqrt{\frac{(N - n_p + 1)(n_p + L + 1)L}{2L - 1}} \cdot \Sigma CG
\]

\[
\frac{\sigma(^6 Li \rightarrow ^8 Be)}{\sigma(d \rightarrow \alpha)} = 2
\]

This is strictly valid provided that one ensures the same kinematical conditions. There can then be corrections due to higher order terms.
Conclusions

There is indeed some evidence for a dynamical supersymmetry that:

• obeys a simple algebraic scheme
• links together many states in different isotopes
• is broken (in a twofold way!) with the same degree of the Wigner’s SU(4)
• makes predictions that can be compared with experiments: spectra and cluster spectroscopic factors

Outlook:

• theorists may upgrade, enlarge or refine the algebraic model
• experimentalists may improve cluster spectroscopic factors measurements and detection (these things are especially crucial!!)
THANKS TO THE ORGANIZERS AND THE I.N.T. STAFF FOR THIS VERY LIVELY AND SUCCESSFUL WORKSHOP!
Appendix:

### Table of quantum numbers for the fermion sector

<table>
<thead>
<tr>
<th>( u_{ST}^F(4) )</th>
<th>( su_S(2) \uplus su_T(2) )</th>
<th>N. of states</th>
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<tr>
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<td>([1^2])</td>
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<td>[1^3] \equiv [1]</td>
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<tr>
<td>[1^4] \equiv [0]</td>
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**TABLE I**: Branching rules for the fermion spin-isospin quantum numbers. They are, of course, similar to the Wigner ones.