The collective model from a Cartan-Weyl perspective

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1. The geometrical collective model
   - The basics of the geometrical collective model
   - The physics motivation

2. The collective variables in the Cartan-Weyl scheme
   - An algebraic description...
   - ... within the Cartan-Weyl scheme

3. Test application in quantum shape phase transitions

4. Conclusions & outlook
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What’s so geometrical about the geometrical model?

- Macroscopically, the atomic nucleus can be compared to a charged liquid drop.
- Deviations from the sphere are developed in multipole orders. Up to 2nd order

Radius

\[ R(\theta, \phi) = R_0[1 + \alpha \cdot Y_2(\theta, \phi)] \]

- \( \alpha^2 \) \( ^aL = 2 \) tensor
A gallery of shapes

Radius

\[ R(\theta, \phi) = R_0 [1 + \alpha \cdot Y_2(\theta, \phi)] \]

- Rotation to the *intrinsic* system

\[
\begin{align*}
\alpha_0' &= \beta \cos \gamma \\
\alpha_2' &= \alpha_{-2}' = \beta / \sqrt{2} \sin \gamma \\
\alpha_1' &= \alpha_{-1}' = 0
\end{align*}
\]

- \( \beta \) is a measure for the *deformation*, \( \gamma \) for the *triaxiality*
The nuclear chart around Z=82 shell closure

- Collective excitation modes are very important in the low-energy spectra
- Renewed interest due to the quantum shape phase transitions
All kinds of shapes

neighbouring isotope chains

- Os :: triaxial nuclei
- Pt :: the $\gamma$-softie
- Pb :: 3 coexisting families

- Hartree-Fock mean field calculation
- A minimum can be found at $\gamma \neq 0$


Calculations in the framework of analytically solvable potentials

All kinds of shapes

neighbouring isotope chains

- Os :: triaxial nuclei
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- Potential Energy Surface calculation
- Very flat $\gamma$ dependence

All kinds of shapes

neighbouring isotope chains

- Os :: triaxial nuclei
- Pt :: the γ-softie
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- Interacting Boson Model calculation
- Extension to coexisting configurations points towards spherical-oblate-prolate structure
The Bohr Hamiltonian

\[ \hat{H} = \hat{T} + V(\alpha) \]

- \( \hat{T} \) is the kinetic energy of the surface
- \( \frac{1}{B_2} \pi \cdot \pi + B_3 [\pi \alpha]^2 \cdot \pi + \ldots \)

\( \alpha \) describes small deformations

- Use a Taylor expansion

\[ V(\alpha) = c_2 (\alpha \cdot \alpha) + c_3 ([\alpha \alpha]^2 \cdot \alpha) \]
\[ + c_4 (\alpha \cdot \alpha)^2 + c_5 ([\alpha \alpha]^2 \cdot \alpha)(\alpha \cdot \alpha) + \ldots \]
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sijn de baerdemacker (ghent university)
The rules of the game

Commutation relations fix the structure

\[
[\pi_\mu, \alpha_\nu] = -i\hbar \delta_{\mu\nu}, \quad [\alpha_\mu, \alpha_\nu] = 0, \quad [\pi_\mu, \pi_\nu] = 0
\]

- The algebraic structure of the geometrical model is contained in the following recoupling formula

\[
(\alpha \cdot \alpha)(\pi^* \cdot \pi^*) = (\alpha \cdot \pi^*)(\alpha \cdot \pi^*) + 3i\hbar(\alpha \cdot \pi^*)
- 2([\alpha\pi^*]^{(1)} \cdot [\alpha\pi^*]^{(1)} + [\alpha\pi^*]^{(3)} \cdot [\alpha\pi^*]^{(3)})
\]

- It comprises the generators of the direct product group

\[
\begin{align*}
\hat{Z}_1 &= \alpha \cdot \alpha \\
\hat{Z}_2 &= \pi^* \cdot \pi^* \\
\hat{Z}_3 &= \alpha \cdot \pi^*
\end{align*}
\]

\[
SU(1, 1) \times O(5) \left\{ \begin{array}{c}
\frac{i\hbar}{\sqrt{10}} L_M = [\alpha\pi^*]^{(1)}_M \\
\frac{i\hbar}{\sqrt{10}} O_M = [\alpha\pi^*]^{(3)}_M
\end{array} \right.
\]
Why $SU(1, 1) \times O(5)$?

The Hamiltonian

\[
\hat{H} = \frac{1}{2B_2} \pi \cdot \pi + B_3 \pi \cdot [\pi \pi]^2 + c_2 (\alpha \cdot \alpha) + c_3 ([\alpha \alpha]^2 \cdot \alpha) + c_4 (\alpha \cdot \alpha)^2 + c_5 ([\alpha \alpha]^2 \cdot \alpha)(\alpha \cdot \alpha) + c_6 (\alpha \cdot \alpha)^3 + d_6 ([\alpha \alpha]^2 \cdot \alpha)^2 + \ldots
\]

$SU(1, 1)$ basic block

\[\alpha \cdot \alpha = \beta^2\]

- The "radial" dependence
- Basis is known

$O(5)$ basic block

\[[\alpha \alpha]^2 \cdot \alpha = \sqrt{\frac{2}{7}} \beta^3 \cos 3\gamma\]

- The "angular" dependence
- Cartan-Weyl basis
The Cartan-Weyl basis of $O(5)$

**Cartan’s theorem**

Every semi simple algebra of dimension $n$ and rank $r$ can be rotated to a natural basis for which

$$[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha_i E_\alpha,$$

$$[E_\alpha, E_\beta] = N_{\alpha+\beta} E_{\alpha+\beta} \quad [E_\alpha, E_{-\alpha}] = \alpha^i H_i \quad \{i, j\} \in r \quad \{\alpha, \beta\} \in n - r$$

- Rotation
  $$\{L_m, O_{m'}\} \rightarrow \{X_i, Y_j, T_{\mu\nu}\}$$

- Group reduction is clear
  $$O(5) \supset O(4) \cong SU(2) \times SU(2)$$

- A natural Cartan basis emerges
  $$|vX(M_X, M_Y)\rangle$$
Action of the $O(5)$ generators on the natural basis (i)

- $\{X_\pm, X_0\}$ and $\{Y_\pm, Y_0\}$ span standard $SU(2)$ algebras
- According Racah: $T_{\mu,\nu}$ acts as a bispinor of character $\frac{1}{2}$ in the $SU(2) \times SU(2)$ space

$$[X_0, T_{\frac{1}{2} \frac{1}{2}}] = \mu T_{\frac{1}{2} \frac{1}{2}}$$
$$[X_\pm, T_{\frac{1}{2} \frac{1}{2}}] = \sqrt{(\frac{1}{2} \pm \mu)(\frac{1}{2} \pm \mu + 1)} T_{\mu \pm 1 \nu}$$

- The action of $T_{\mu,\nu}$ on a basis state $|vX(M_X, M_Y)\rangle$ is thus

$$T_{\frac{1}{2} \frac{1}{2}} |vXM_X M_Y\rangle = a_+ |v, X + \frac{1}{2}, M_X + \mu, M_Y + \nu\rangle$$
$$+ a_- |v, X - \frac{1}{2}, M_X + \mu, M_Y + \nu\rangle$$

- Applying the Wigner Eckart theorem twice

$$a_\pm = (-)^k \begin{pmatrix} X + \frac{1}{2} & \frac{1}{2} & X \\ -M_X - \mu & \mu & M_X \end{pmatrix} \begin{pmatrix} X + \frac{1}{2} & \frac{1}{2} & X \\ -M_Y - \nu & \nu & M_Y \end{pmatrix} \langle vX \pm \frac{1}{2} || T || vX \rangle$$
Action of the $O(5)$ generators on the natural basis (ii)

- For every $v$ and $X$ :: two unknown matrix elements ☞ two conditions needed

$$C_2[O(5)] = 2(X^2 + Y^2 - 2[TT]^{(00)})$$

$$[T_{\mu\nu}, T_{\mu'\nu'}] = c_m^X X_0 + c_m^Y Y_0 \quad m = \{\mu\nu, \mu'\nu'\}$$

- norm ☞ selection rules ($X = 0 \ldots \frac{v}{2}$)

- intermediate state method renders double reduced matrix elements

**Example: Action of $T_{1/2,1/2}$**

$$T_{\frac{1}{2} \frac{3}{2}} |vXM_X M_Y \rangle = \frac{\sqrt{(X+M_X+1)(X+M_Y+1)(v-2X)(v+2X+3)}}{2\sqrt{(2X+1)(2X+2)}} |vX + \frac{1}{2} M_X + \frac{1}{2} M_Y + \frac{1}{2} \rangle$$

$$- \frac{\sqrt{(X-M_X)(X-M_Y)(v-2X+1)(v+2X+2)}}{2\sqrt{(2X)(2X+1)}} |vX - \frac{1}{2} M_X + \frac{1}{2} M_Y + \frac{1}{2} \rangle$$
Tensor character of the $\alpha$ variables

### Basis block of the potential

$$\beta^3 \cos(3\gamma) \sim [\alpha\alpha]^{(2)} \cdot \alpha$$

Need for matrix elements

$$\langle vX(M_X, M_Y)|\alpha_\mu|v'X'(M'_X, M'_Y)\rangle$$

- $\{\alpha_0\}$ and $\{\alpha_{-2}, \alpha_{-1}, \alpha_1, \alpha_2\}$ have bispinor 0 and $\frac{1}{2}$ character respectively

$$\alpha_0 \rightarrow \alpha_{00}^{00}, \quad \{\alpha_{-2}, \alpha_{-1}, \alpha_1, \alpha_2\} \rightarrow \alpha_{\mu\nu}^{\frac{1}{2}\frac{1}{2}}$$

Wigner Eckart

$$\langle vXM_X M_Y|\alpha^{\lambda\lambda}_{\mu\nu}|v'X' M'_X M'_Y\rangle$$

$$= (-)^k \begin{pmatrix} X & \lambda & X' \\ -M_X & \mu & M'_X \end{pmatrix} \begin{pmatrix} X & \lambda & X' \\ -M_Y & \nu & M'_Y \end{pmatrix} \langle vX||\alpha^\lambda||v'X'\rangle$$
Closed expressions for the \( \alpha \) matrix elements

**What is used**

\[
[T_{\mu \nu}, \alpha^{\frac{1}{2} \frac{1}{2}}_{\mu', \nu'}] = \frac{(-)^{\mu - \nu}}{\sqrt{2}} \delta_{-\mu, \mu'} \delta_{-\nu, \nu'} \alpha^{00}_{00} \\
[T_{\mu \nu}, \alpha^{00}_{00}] = \frac{1}{\sqrt{2}} \alpha^{\frac{1}{2} \frac{1}{2}}_{\mu, \nu} \\
[\alpha^{\lambda \lambda}_{\mu, \nu}, \alpha^{\lambda' \lambda'}_{\mu', \nu'}] = 0 \\
\alpha \cdot \alpha = \beta^2 = Z_1
\]

- Seniority selection rules :: \( \alpha \) is a \( \nu = 1 \) tensor
- Bispinor \( \{ \frac{1}{2} \frac{1}{2} \} \) can be expressed in terms of biscalar \( \{00\} \) matrix elements
- **Closed expressions** for the matrix elements result

**What is obtained: matrix elements of \( \alpha \)**

\[
\langle \nu X M_{X} M_{Y} | \alpha^{00}_{00} | \nu + 1, X M_{X} M_{Y} \rangle = \beta \sqrt{\frac{(\nu-2X+1)(\nu+2X+3)}{(2\nu+3)(2\nu+5)}} \\
\langle \nu X M_{X} M_{Y} | \alpha^{00}_{00} | \nu - 1, X M_{X} M_{Y} \rangle = \beta \sqrt{\frac{(\nu-2X)(\nu+2X+2)}{(2\nu+1)(2\nu+3)}}
\]
Projection to the physical basis

- Experimental spectra have good angular momentum quantum number $L$
- The Hamiltonian is a scalar with respect to the angular momentum algebra $O(3)$
  \[
  [L \cdot L, C_G] \neq 0 \\
  [L_0, C_G] = 0
  \]
- Only the angular momentum projection is a good quantum number in the Cartan basis
- A rotation brings the natural- to the physical basis
All ingredients are ready

- $\alpha$ Matrix elements in $O(5)$ basis are derived
- Inclusion of $SU(1, 1)$ basis is straightforward in a similar fashion
- Diagonalising = choosing a basis
- Harmonic oscillator = choosing $\hbar\omega$
- $H = \frac{1}{2B_2} \pi \cdot \pi + \xi V_{\text{vib}} + (1 - \xi)V_{\gamma\text{-ind}}$

- Computer code is now under continuous development to diagonalise general collective potentials.
- Present status: upto $\beta^4$
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Test application in quantum shape phase transitions

- Quantum shape phase transitions cover a large part of the model Hilbert space
- Ideal testground for the method
- Upto $\beta^4$, 3 meaningful limits result

3 limits

- vibrational limit
- $\gamma$-independent rotor
- axial deformed rotor
3 limits

- Vibrational limit
- $\gamma$-independent rotor
- Axial deformed rotor

- Trivial limit
- Large degeneracies
- $B(E2)$ addition rule
3 limits

- vibrational limit
- $\gamma$-independent rotor
- axial deformed rotor

- Remaining seniority symmetry
- $\beta$-excitation band
3 limits

- Vibrational limit
- \(\gamma\)-independent rotor
- Axial deformed rotor

Highly pronounced bands

Rotation-Vibration model
Along the transition path: energy spectrum

vibrator $\rightarrow \gamma$-independent rotor $\rightarrow$ axial rotor $\rightarrow$ vibrator
Along the transition path: quadrupole moments

**Quadrupole moments**

\[ Q = \langle \hat{Q}_\mu \rangle_{2_1} = \frac{3ZR_0^2}{4\pi} \langle \alpha_\mu \rangle_{2_1} \]

- \( \alpha_\mu \) is seniority \( \nu = 1 \) tensor
- \( Q \equiv 0 \) for vib. \( \rightarrow \gamma \)-ind. rotor transition
- Other observables \( (B(E2), \rho(E0)) \)

\( \gamma \)-ind. rotor \( \rightarrow \) axial rotor \( \rightarrow \) vibrator
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Conclusions & outlook

- Collectivity accounts for a lot of the physics around $Z = 82$ shell closure.
- All necessary matrix elements of a more general type of potential can be calculated in a Cartan-Weyl scheme.
- First test applications in the framework of quantum shape phase transitions renders reliable results.
- Further terms need to be included to study more general collective structures (e.g. triaxiality, shape coexistence), needed for the collectivity around the $Z = 82$ closed shell.
- Possible extension to higher rank algebras ($O(7)$ octupole degrees of freedom and beyond).
- ...
Thank you for your attention!