Ward identities, $O(a)$ improvement & twisted mass QCD (lecture IV)

Stefan Sint

Trinity College Dublin

Seattle, August 20, 2007
Some (more or less) pedagogical references

1. R. Sommer, “Non-perturbative renormalisation of QCD”, Schladming Winter School lectures 1997, hep-ph/9711243v1; “Non-perturbative QCD: Renormalization, O(a) improvement and matching to heavy quark effective theory” Lectures at Nara, November 2005 hep-lat/0611020


For twisted mass QCD and chirally twisted Schrödinger functional see

1. A. Shindler “Twisted mass lattice QCD” review article, July 2007 (arXiv:0707.4093 [hep-lat])

2. S. Sint, “Lattice QCD with a chiral twist” Lectures at Nara, November 2005 hep-lat/0702008
Symmetries and Ward identities
Wilson quarks and chiral Ward identities
Chiral symmetry and $O(a)$ improvement
Wilson quarks with a chirally twisted mass term
Equivalence to standard QCD
By-passing lattice specific renormalisation problems
Continuum vs. lattice symmetries

On the lattice symmetries are typically reduced with respect to the continuum. Examples are

1. **Space-Time symmetries**: the Euclidean O(4) rotations are reduced to the O(4,\mathbb{Z}) group of the hypercubic lattice. Other lattice geometries are possible, even random lattices have been tried.

2. **Supersymmetry**: only partially realisable on the lattice (cf. lectures by S. Catterall)

3. **Chiral and Flavour symmetries**:
   - staggered quarks: only a U(1) × U(1) symmetry remains
   - Wilson quarks: an exact SU(N_f)_V
   - twisted mass Wilson quarks: various U(1) symmetries (both axial and vector)
   - overlap/Neuberger quarks: complete continuum symmetries!
   - Domain Wall quarks: (negligibly ?) small violations of axial symmetries; consequences are analysed like for Wilson quarks

In the following: chiral and flavour symmetries with Wilson like quarks
Exact lattice Ward identities (1)

Euclidean action $S = S_f + S_g$:

$$S_f = a^4 \sum_x \bar{\psi}(x) \left( D_W + m_0 \right) \psi(x), \quad S_g = \frac{1}{g_0^2} \sum_{\mu, \nu} \text{tr} \{ 1 - P_{\mu\nu}(x) \}$$

$$D_W = \frac{1}{2} \left\{ (\nabla_\mu + \nabla_\mu^*) \gamma_\mu - a \nabla_\mu^* \nabla_\mu \right\}$$

Non-singlet vector transformations ($N_f = 2$, $\tau_{1,2,3}$ are the Pauli matrices):

$$\psi(x) \rightarrow \psi'(x) = \exp \left( i \theta(x) \frac{1}{2} \tau^a \right) \psi(x) = (1 + \delta^a_V(\theta) + O(\theta^2)) \psi(x),$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = \bar{\psi}(x) \exp \left( -i \theta(x) \frac{1}{2} \tau^a \right) \psi(x) = (1 + \delta^a_V(\theta) + O(\theta^2)) \bar{\psi}(x)$$

Perform change of variables in the functional integral and expand in $\theta$

$$\langle O[\psi, \bar{\psi}, U] \rangle = Z^{-1} \int D[\psi, \bar{\psi}] D[U] e^{-S} O[\psi, \bar{\psi}, U].$$

Due to $D[\psi, \bar{\psi}] = D[\psi', \bar{\psi}']$ one finds the vector Ward identity

$$\langle \delta^a_V(\theta) O \rangle = \langle O \delta^a_V(\theta) S \rangle$$
Exact lattice Ward identities (2)

Variation of the action:

$$\delta^a_V(\theta) S = -ia^4 \sum_x \partial^*_\mu \tilde{V}^a_\mu(x)$$

Noether current:

$$\tilde{V}^a_\mu(x) = \bar{\psi}(x)(\gamma_\mu - 1) \frac{\tau^a}{4} U(x, \mu) \psi(x + a\hat{\mu}) + \bar{\psi}(x + a\hat{\mu})(\gamma_\mu + 1) \frac{\tau^a}{4} U(x, \mu) \dagger \psi(x)$$

Choose region $R$ and $\theta$:

$$R = \{x : t_1 \leq x_0 \leq t_2\}, \quad \theta(x) = \begin{cases} 1 & \text{if } x \in R \\ 0 & \text{otherwise} \end{cases}$$

if $O = O_{\text{ext}}$ is localised outside $R$:

$$0 = \langle O_{\text{ext}} \delta^a_V(\theta) S \rangle = -ia \sum_{x_0=t_1}^{t_2} a^3 \sum_x \langle O_{\text{ext}} \partial^*_\mu \tilde{V}^a_\mu(x) \rangle = a \sum_{x_0=t_1}^{t_2} \partial^* \langle O_{\text{ext}} Q^a_V(x_0) \rangle$$

There is a conserved charge, $Q^a_V(t_1) = Q^a_V(t_2)$ reflecting the exact vector symmetry on the lattice
Exact lattice Ward identities (3)

Choosing $O = O_{\text{ext}} \bar{V}_\mu^b(y)$, with $y \in R$:

\[
i \varepsilon^{abc} \left\langle O_{\text{ext}} \bar{V}_k^c(y) \rightangle = \left\langle O_{\text{ext}} \bar{V}_k^b(y) [Q^a_V(t_2) - Q^a_V(t_1)] \right\rangle
\]

\[
i \varepsilon^{abc} \left\langle O_{\text{ext}} Q_V^c(y_0) \rightangle = \left\langle O_{\text{ext}} Q_V^b(y_0) [Q^a_V(t_2) - Q^a_V(t_1)] \right\rangle
\]

Euclidean version of charge algebra!

- implies that the Noether current $\bar{V}_\mu^a$ is protected against renormalisation; if we admit a renormalisation constant $Z_{\bar{V}}$ it follows that $Z_{\bar{V}}^2 = Z_{\bar{V}}$ hence $Z_{\bar{V}} = 1$; its anomalous dimension vanishes!

- Any other definition of a lattice current, e.g. the local current

\[
V_\mu^a(x) = \overline{\psi}(x) \gamma_\mu \gamma_5 \psi(x), \quad (V_R)_\mu^a = Z_V V_\mu^a
\]

can be renormalised by comparing with the conserved current. Its anomalous dimension must vanish, i.e.

\[
Z_V = Z_V(g_0) \xrightarrow{g_0 \to 0} 1 + \sum_{n=1}^\infty Z_V^{(n)} g_0^{2n}.
\]
Continuum chiral WI’s as normalisation conditions

- For chiral symmetry there is no conserved current with Wilson quarks.
- However: expect that chiral symmetry can be restored in the continuum limit!
- [Bochicchio et al ’85]: use continuum chiral Ward identities and impose them as normalisation condition at finite $a$
- Define chiral variations:
  \[ \delta_A^a(\theta)\psi(x) = i\gamma_5 \frac{1}{2} \tau^a \theta(x) \psi(x), \quad \delta_A^a(\theta)\bar{\psi}(x) = \bar{\psi}(x) i\gamma_5 \frac{1}{2} \tau^a \theta(x) \]
- Derive formal continuum Ward identities assuming that the functional integral can be treated like an ordinary integral:
  \[ \langle \delta_A^a(\theta)O \rangle = \langle O \delta_A^a(\theta)S \rangle, \]
  \[ \delta_A^a(\theta)S = -i \int d^4x \theta(x) \left( \partial_\mu A_\mu^a(x) - 2mP^a(x) \right) \]
  \[ A_\mu^a(x) = \bar{\psi}(x)\gamma_\mu \gamma_5 \frac{1}{2} \tau^a \psi(x), \quad P^a(x) = \bar{\psi}(x) i\gamma_5 \frac{1}{2} \tau^a \psi(x) \]
Simplest chiral WI: the PCAC relation

- Shrink the region $R$ to a point:
  \[
  \langle O_{\text{ext}} \delta^a_A(\theta) S \rangle = 0 \\
  \Rightarrow \langle \partial_\mu A^a_\mu(x) O_{\text{ext}} \rangle = 2m \langle P^a(x) O_{\text{ext}} \rangle
  \]

- The PCAC relation implies that chiral symmetry is restored in the chiral limit.

- Impose PCAC on Wilson quarks at fixed $a$: define a bare PCAC mass:
  \[
  m = \frac{\langle \partial_\mu A^a_\mu(x) O_{\text{ext}} \rangle}{\langle P^a(x) O_{\text{ext}} \rangle}
  \]

- A renormalised quark mass can thus be written in two ways
  \[
  m_R = Z_A Z_P^{-1} m = Z_m (m_0 - m_{\text{cr}}) \Rightarrow m = Z_m Z_P Z_A^{-1} (m_0 - m_{\text{cr}})
  \]

- The critical mass can be determined by measuring the bare PCAC mass $m$ as a function of $m_0$ and extra/interpolation to $m = 0$.

- Note: $m$ is only defined up to $O(a)$; any change in $O_{\text{ext}}$ will lead to $O(a)$ differences.
Determination of the critical mass

PCAC quark mass from SF correlation functions:

\[ m = \frac{\partial_0 f_A(x_0)}{2f_P(x_0)} \]

$8^3 \times 16$ lattice, quenched QCD, $a = 0.1 \text{ fm}$
More chiral WI’s: axial current normalisation

- At \( m = 0 \) we can derive the Euclidean charge algebra:

\[
i \varepsilon^{abc} \left\langle O_{\text{ext}} Q^c_V(y_0) \right\rangle = \left\langle O_{\text{ext}} Q^b_A(y_0) [Q^a_A(t_2) - Q^a_A(t_1)] \right\rangle
\]

- Imposing this continuum identity on the lattice (at \( m = 0 \)) fixes the normalisation of the axial current:

\[
(A_R)^a_\mu = Z_A(g_0) A^a_\mu, \quad Z_A(g_0) \xrightarrow{g_0 \to 0} 1 + \sum_{n=1}^{\infty} Z^{(n)}_A g_0^{2n}. \]

- Note: When changing the external fields \( O_{\text{ext}} \), the result for \( Z_A \) will change by terms of \( O(a) \).

- The PCAC relation and the charge algebra become operator identities in Minkowski space. Changing \( O_{\text{ext}} \) corresponds to looking at different matrix elements of these operator identities. On the lattice these must be equal up to \( O(a) \) terms.
Need for $O(a)$ improvement of Wilson quarks

$O(a)$ artefacts can be quite large with Wilson quarks:

PCAC quark mass from SF correlation functions:

$$m = \frac{\partial_0 f_A(x_0)}{2 f_P(x_0)}$$

$8^3 \times 16$ lattice, quenched QCD, $a = 0.1$ fm, 2 different gauge background fields.
On-shell O($a$) improvement

Recall Symanzik’s effective continuum theory from lecture 1

$$S_{\text{eff}} = S_0 + aS_1 + a^2 S_2 + \ldots, \quad S_0 = S_{\text{cont}}^{\text{QCD}}$$

$$S_k = \int d^4x \mathcal{L}_k(x)$$

where $\mathcal{L}_1$ is a linear combination of the fields:

$$\bar{\psi} \sigma_{\mu\nu} F_{\mu\nu} \psi, \quad \bar{\psi} D_\mu D_\mu \psi, \quad m \bar{\psi} D_\mu \psi, \quad m^2 \bar{\psi} \psi, \quad m \text{tr} \{F_{\mu\nu} F_{\mu\nu}\}$$

The action $S_1$ appears as insertion in correlation functions

$$G_n(x_1, \ldots, x_n) = \langle \phi_0(x_1) \ldots \phi_0(x_n) \rangle_{\text{con}}$$

$$+ a \int d^4y \langle \phi_0(x_1) \ldots \phi_0(x_n) \mathcal{L}_1(y) \rangle_{\text{con}}$$

$$+ a \sum_{k=1}^n \langle \phi_0(x_1) \ldots \phi_1(x_k) \ldots \phi_0(x_n) \rangle_{\text{con}} + \mathcal{O}(a^2)$$
On-shell $O(a)$ improvement (1)

Basic idea:
- Introduce counterterms to the action and composite operators such that $S_1$ and $\phi_1$ are cancelled in the effective theory.
- As all physics can be obtained from on-shell quantities (spectral quantities like particle energies or correlation function where arguments are kept at non-vanishing distance) one may use the equations of motion to reduce the number of counterterms.
- The contact terms which arise from having $y \approx x_i$ can be analysed in the OPE and are found to be of the same structure as the counterterms anyway contained in $\phi_1$; this amounts to a redefinition of the counterterms in $\phi_1$.
- After using the equations of motion one remains with:

$$\bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \quad m^2\bar{\psi}\psi, \quad m\text{tr}\left\{F_{\mu\nu}F_{\mu\nu}\right\}$$
On-shell $O(a)$ improvement (2)

1. On-shell $O(a)$ improved Lattice action
   
   The last two terms are equivalent to a rescaling of the bare mass and coupling ($m_q = m_0 - m_{cr}$):

   \[ \tilde{g}_0^2 = g_0^2(1 + b_g(g_0)a m_q), \quad \tilde{m}_q = m_q(1 + b_m(g_0)a m_q) \]

   The first term is the Sheikholeslami-Wohlert or clover term

   \[ S_{\text{Wilson}} \rightarrow S_{\text{Wilson}} + i ac_{sw}(g_0)a^4 \sum_x \overline{\psi}(x)\sigma_{\mu\nu} \hat{F}_{\mu\nu}(x)\psi(x) \]

2. On-shell $O(a)$ improved axial current and density:

   \[ (A_R)^a_{\mu} = Z_A(\tilde{g}_0^2)(1 + b_A(g_0)a m_q) \left\{ A^a_{\mu} + c_A(g_0)\tilde{\partial}_{\mu}P^a \right\} \]

   \[ (P_R)^a = Z_P(\tilde{g}_0^2, a_{\mu})(1 + b_P(g_0)a m_q)P^a \]
On-shell $O(a)$ improvement (3)

- There are 2 counterterms in the massless theory $c_{sw}, c_A$, the remaining ones ($b_g, b_m, b_A, b_P$) come with $am_q$.
- Note: all counterterms are absent in chirally symmetric regularisations!

⇒ turn this around: impose chiral symmetry to determine $c_{sw}, c_A$
non-perturbatively:
  - define bare PCAC quark masses from SF correlation functions

$$m_R = \frac{Z_A(1 + b_A am_q)}{Z_P(1 + b_P am_q)} m, \quad m = \frac{\tilde{\partial}_0 f_A(x_0) + c_A a \partial^*_0 \partial_0 f_P(x_0)}{f_P(x_0)}$$

- At fixed $g_0$ and $am_q \approx 0$ define 3 bare PCAC masses $m_{1,2,3}$ (e.g. by varying the gauge boundary conditions) and impose

$$m_1(c_{sw}, c_A) = m_2(c_{sw}, c_A), \quad m_1(c_{sw}, c_A) = m_3(c_{sw}, c_A) \Rightarrow c_{sw}, c_A$$

SF b.c.'s ⇒ high sensitivity to $c_{sw}$ & simulations near chiral limit
On-shell $O(a)$ improvement (4)

Before and after $O(a)$ improvement (PCAC masses from SF correlation functions, $8^3 \times 16$ lattice)

![Graph 1](image1)

![Graph 2](image2)
The RGI charm quark mass can be defined in various ways
- starting from the subtracted bare quark mass $m_{q,c} = m_{0,c} - m_{cr}$
- starting from the average strange-charm PCAC mass $m_{sc}$
- starting from the PCAC mass $m_{cc}$ for a hypothetical mass degenerate doublet of quarks.

Tune the bare charm quark masses to match the $D_s$ meson mass

Obtain the corresponding $O(a)$ improved RGI masses:

$$r_0 M_c |_{m_{sc}} = Z_M \left\{ 2r_0 m_{sc} \left[ 1 + (b_A - b_P) \frac{1}{2}(am_{q,c} + am_{q,s}) \right] \
- r_0 m_s \left[ 1 + (b_A - b_P) am_{q,s} \right] \right\},$$

$$r_0 M_c |_{m_c} = Z_M r_0 m_c \left[ 1 + (b_A - b_P) am_{q,c} \right],$$

$$r_0 M_c |_{m_{q,c}} = Z_M Zr_0 m_{q,c} \left[ 1 + b_m am_{q,c} \right].$$

N.B.: all $O(a)$ counterterms are known non-perturbatively in the quenched case!
Continuum extrapolation:

\[ r_0 M_c = A + B \left( \frac{a^2}{r_0^2} \right) \]
\[ r_0 = 0.5 \text{ fm} \]

\[ M_c = 1.654(45) \text{ GeV} \]

\[ \overline{m}_c^{\text{MS}}(\overline{m}_c) = 1.301(34) \text{ GeV} \]
Summary On-shell O(\(a\)) improvement

After O(\(a\)) improvement:

- The ambiguity in \(m_{cr}\) is reduced to O(\(a^2\))
- Axial current normalisation can be defined up to O(\(a^2\))
- Results exist for \(c_{sw}, c_A\) for quenched and \(N_f = 2, 3\) and different gauge actions
- On-shell O(\(a\)) improvement seems to work; rather economical for spectral quantities (e.g. hadron masses): just need \(c_{sw}\)!
- Quark bilinear operators are still tractable
- Four-quark operators are probably impractical
- Non-degenerate quark masses: rather complicated, proliferation of counterterms [Bhattacharya et al '99]; Not all can determined by chiral symmetry, due to violation of on-shell condition in Ward identities at finite mass
- However: for small quark masses and fine lattices \(am_q\) is small (a few percent at most) and perturbative estimates of improvement coefficients may be good enough!
Twisted mass QCD, continuum considerations (1)

Consider the continuum action of a doublet of massless quarks

\[ S_f = \int d^4x \overline{\psi}(x) \partial_\mu \gamma_\mu \psi(x) \]

The massless action is symmetric under chiral transformations

\[ \psi \rightarrow \psi' = \exp(i \omega^a_A \gamma_5 \tau^a / 2) \psi \]
\[ \overline{\psi} \rightarrow \overline{\psi}' = \overline{\psi} \exp(i \omega^a_A \gamma_5 \tau^a / 2) \]

When introducing a quark mass term the choices \( \overline{\psi} \psi \) or

\[ \overline{\psi}' \psi' = \overline{\psi} \exp(i \omega^a_A \gamma_5 \tau^a) \psi = \cos(\omega_A) \overline{\psi} \psi + i \sin(\omega_A) u^a_A \overline{\psi} \gamma_5 \tau^a \psi \]

are equivalent!

(\( \omega_A \) is the modulus of \( (\omega^1_A, \omega^2_A, \omega^3_A) \) and \( u^a = \omega^a_A / \omega_A \) a unit vector)

- The choice of a mass term \( \overline{\psi} \psi \) is a mere convention; in general one may pick any other direction in chiral flavour space
- The form of symmetry transformations depends on this choice:
Twisted mass QCD, continuum considerations (2)

- by definition, the flavour (isospin) symmetry leaves the mass term invariant:

\[
\begin{align*}
\psi & \rightarrow \exp(-i\omega^a_A \gamma_5 \tau^a/2) \exp(i\omega^b_V \tau^b/2) \exp(i\omega^c_A \gamma_5 \tau^c/2) \psi \\
\bar{\psi} & \rightarrow \bar{\psi} \exp(i\omega^a_A \gamma_5 \tau^a/2) \exp(-i\omega^b_V \tau^b/2) \exp(-i\omega^c_A \gamma_5 \tau^c/2) \gamma_0
\end{align*}
\]

- similarly for parity:

\[
\begin{align*}
\psi(x) & \rightarrow \gamma_0 \exp(i\omega^a_A \gamma_5 \tau^a) \psi(x_0, -x), \\
\bar{\psi}(x) & \rightarrow \bar{\psi}(x_0, -x) \exp(i\omega^a_A \gamma_5 \tau^a) \gamma_0
\end{align*}
\]

**Question**: why should one deviate from the standard convention for the quark mass term?
Twisted Mass Lattice QCD (1)

Lattice action for a doublet $\psi$ of mass degenerate light Wilson quarks [Aoki '84]:

$$S_f = a^4 \sum_x \overline{\psi}(x) \left( D_W + m_0 + i\mu_q \gamma_5 \tau^3 \right) \psi(x)$$

$D_W$: Wilson-Dirac operator with/without Sheikholeslami-Wohlert (clover)

$\mu_q$: bare twisted mass parameter

Properties:

- regularisation of QCD with $N_f = 2$ mass degenerate quark flavours (see below)
- $\mu_q \neq 0 \Rightarrow$ no unphysical zero modes:

$$\det \left( D_W + m_0 + i\mu_q \gamma_5 \tau^3 \right) = \det \begin{pmatrix} \gamma_5(D_W + m_0) + i\mu_q & 0 \\ 0 & \gamma_5(D_W + m_0) - i\mu_q \end{pmatrix} = \det \left( [D_W + m_0]^\dagger [D_W + m_0] + \mu_q^2 \right) > 0$$
positive and selfadjoint transfer matrix provided $\mu_q$ is real and $|\kappa| < 1/6,$
\[ \kappa = (2am_0 + 8)^{-1} \Rightarrow \text{unitarity} \]
The flavour symmetry is reduced to U(1) with generator $\tau^3/2$
Discrete symmetries: C, axis permutations, reflections with flavour exchange, e.g.
\[
\psi(x) \rightarrow \gamma_0 \tau^1 \psi(x_0, -x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x_0, -x) \gamma_0 \tau^1
\]
Equivalence between tmQCD and QCD (1)

Classical continuum limit of twisted mass lattice QCD:

\[ S_f = \int \! \! d x \, \bar{\psi}(x) \left( \mathcal{D} - m + i \mu_q \gamma_5 \tau^3 \right) \psi(x). \]

Perform a global chiral (non-singlet) rotation of the fields:

\[ \psi' = R(\alpha) \psi, \quad \bar{\psi}' = \bar{\psi} R(\alpha), \quad R(\alpha) = \exp \left( i \alpha \gamma_5 \frac{\tau^3}{2} \right). \]

For \( \tan \alpha = \frac{\mu_q}{m} \) the action reads:

\[ S'_f = \int \! \! d x \, \bar{\psi}'(x)(\mathcal{D} + M)\psi'(x), \quad M = \sqrt{m^2 + \mu_q^2} \]

\[ \bar{\psi}' \psi' = \bar{\psi} \exp(i \alpha \gamma_5 \tau^3) \psi = \cos(\alpha) \bar{\psi} \psi + i \sin(\alpha) \bar{\psi} \gamma_5 \tau^3 \psi \]

corresponds to \( \omega^a_A = \alpha \delta^{3a} \) in the previous discussion.
Equivalence between tmQCD and QCD (2)

Introduce polar mass coordinates \( m = M \cos(\alpha) \), \( \mu_q = M \sin(\alpha) \), and consider the formal functional integral

\[
\langle O[\psi, \overline{\psi}] \rangle_{(M,\alpha)} = \mathcal{Z}^{-1} \int D[U, \psi, \overline{\psi}] \, O[\psi, \overline{\psi}] \, e^{-S[m,\mu_q]}
\]

The change of variables leads to the identity:

\[
\langle O[\psi, \overline{\psi}] \rangle_{(M,0)} = \langle O[R(\alpha)\psi, \overline{\psi}R(\alpha)] \rangle_{(M,\alpha)}
\]

For a member \( \phi^{(r)}_A \) of a chiral multiplet in the representation \( r \),

\[
\phi^{(r)}_A [R(\alpha)\psi, \overline{\psi}R(\alpha)] = R^{(r)}_{AB}(\alpha) \phi^{(r)}_B [\psi, \overline{\psi}]
\]

The identity for \( n \)-point functions of such fields becomes

\[
\left\{ \prod_{i=1}^n R^{(r_i)}_{A_iB_i}(\alpha) \right\} \langle \phi^{(r_1)}_{A_1}(x_1) \cdots \phi^{(r_n)}_{A_n}(x_n) \rangle_{(M,0)} = \langle \phi^{(r_1)}_{B_1}(x_1) \cdots \phi^{(r_n)}_{B_n}(x_n) \rangle_{(M,\alpha)}
\]
Equivalence between tmQCD and QCD (3)

Examples: chiral multiplets \((A^a_\mu, V^a_\mu)\) and \((\frac{1}{2} S^0, P^a)\)

\[
\begin{align*}
A^a_\mu &= \overline{\psi} \gamma_\mu \gamma_5 \frac{\tau^a}{2} \psi, \\
V^a_\mu &= \overline{\psi} \gamma_\mu \frac{\tau^a}{2} \psi, \\
P^a &= \overline{\psi} \gamma_5 \frac{\tau^a}{2} \psi, \\
S^0 &= \overline{\psi} \psi.
\end{align*}
\]

With \(\psi' = R(\alpha)\psi\), \(\overline{\psi}' = \overline{\psi} R(\alpha)\), \(O' \equiv O[\psi', \overline{\psi}']\), \(c \equiv \cos(\alpha)\), \(s \equiv \sin(\alpha)\):

\[
\begin{align*}
A^1_\mu' &= c A^1_\mu + s V^2_\mu, \\
A^2_\mu' &= c A^2_\mu - s V^1_\mu, \\
A^3_\mu' &= A^3_\mu, \\
P^a' &= P^a, \quad (a = 1, 2), \\
V^1_\mu' &= c V^1_\mu + s A^2_\mu, \\
V^2_\mu' &= c V^2_\mu - s A^1_\mu, \\
V^3_\mu' &= V^3_\mu, \\
P^3' &= c P^3 + is \frac{1}{2} \overline{\psi} \psi.
\end{align*}
\]

For instance:

\[
\begin{align*}
\langle A^1_\mu(x) P^1(y) \rangle_{(M,0)} &= \cos(\alpha) \langle A^1_\mu(x) P^1(y) \rangle_{(M,\alpha)} \\
&\quad + \sin(\alpha) \langle V^2_\mu(x) P^1 \rangle_{(M,\alpha)}
\end{align*}
\]
The PCAC and PCVC relations,

\[ \partial_\mu A^a_\mu = 2mP^a + \delta^a_3 i \mu qS^0, \quad \partial_\mu V^a_\mu = -2 \mu q \varepsilon^{3ab} P^b, \]

take their standard form in the primed basis

\[ \partial_\mu A'^a_\mu = 2MP'^a, \quad \partial_\mu V'^a_\mu = 0. \]

Remarks:

- We refer to the basis of primed fields as “physical” because the mass term takes its standard form in this basis.
- We still need to explain how the relationship between QCD with a standard mass term and twisted mass QCD works out beyond the formal continuum theory.
- If tmQCD is regularized with Ginsparg-Wilson quarks the same identities can be derived in the bare theory.
- If the renormalization procedure respects the chiral multiplet structure and the multiplicative renormalization constants do not depend on $\alpha$ (e.g. mass independent renormalization schemes) $\Rightarrow$ the formal continuum relations hold between renormalized theories. 
  
  **N.B.:** no reference to perturbation theory! Assuming universality the correspondence is established non-perturbatively. In PT it works out order by order in the loop expansion.
- The angle $\alpha$ is given by the ratio between renormalized PCVC and PCAC masses: $\tan \alpha = \mu_R / m_R$. 

Stefan Sint

Ward identities, $O(a)$ improvement & twisted mass QCD (lecture IV)
Lattice tmQCD with Wilson quarks

1. restore the chiral multiplets in the massless bare theory by imposing the chiral flavour Ward identities, e.g. \((Z_A A^a_\mu, \tilde{V}^a_\mu)\).

2. If necessary renormalize a given chiral multiplet by imposing a renormalization condition on one of its members. Choose a mass independent renormalization scheme!

3. Renormalization of the parameters:

\[
g^2_R = Z_g g^2_0, \quad m_R = Z_m (m_0 - m_c), \quad \mu_R = Z_\mu \mu_q,
\]

From the exact PCVC relation

\[
\partial^*_\mu \tilde{V}^2_\mu = 2\mu_q P^1 = 2\mu_R (P_R)^1 \Rightarrow Z_\mu Z_P = 1.
\]

⇒ to define \(\alpha\) measure a bare PCAC mass \(m\)

\[
m = \frac{\langle \partial_\mu A^1_\mu (x) O \rangle}{\langle P^1 (x) O \rangle} \quad \Rightarrow \quad \tan \alpha = \frac{\mu_R}{m_R} = \frac{Z_P^{-1} \mu_q}{Z_P^{-1} Z_A m} = \frac{\mu_q}{Z_A m}.
\]

the definition of \(\alpha\) requires \(Z_A\), except for \(\alpha = \pi/2\), where \(m = 0\).
The freedom of introducing more general mass terms can be used to avoid lattice renormalization problems:

1. $F_\pi$ can be obtained from the 2-point function

$$
\langle (A_R)_0^1(x)(P_R)^1(y) \rangle_{(M_R,0)} = \cos(\alpha) \langle (A_R)_0^1(x)(P_R)^1(y) \rangle_{(M_R,\alpha)} + \sin(\alpha) \langle \tilde{V}_0^2(x)(P_R)^1(y) \rangle_{(M_R,\alpha)}.
$$

At $\alpha = \pi/2$ one has $\cos(\alpha) = 0$ and $F_\pi$ is obtained from the vector current. The determination of $Z_A$ is avoided!

2. The chiral condensate:

$$
\langle (S_R)^0(x) \rangle_{(M_R,0)} = \cos(\alpha) \langle (S_R)^0(x) \rangle_{(M_R,\alpha)} + 2i \sin(\alpha) \langle (P_R)^3(x) \rangle_{(M_R,\alpha)}
$$

At $\alpha = \pi/2$ the chiral condensate is represented by $P^3$ which only renormalizes multiplicatively in the chiral limit!
**Application to $B_K$:** The $B_K$ parameter is defined in QCD with dynamical $u, d, s$ quarks:

$$\langle \bar{K}^0| O^{\Delta S=2}_{(V-A)(V-A)}|K^0\rangle = \frac{8}{3} F_K^2 m_K^2 B_K$$

The local operator

$$O^{\Delta S=2}_{(V-A)(V-A)} = \sum_{\mu} [\bar{s}\gamma_{\mu}(1 - \gamma_5)d]^2$$

is the effective local interaction induced by integrating out the massive gauge bosons and $t, b, c$ quarks in the Standard Model.

- only the parity-even part contributes to $B_K$

$$O_{(V-A)(V-A)} = O^{VV+AA}_{\text{parity-even}} - O^{VA+AV}_{\text{parity-odd}}$$

- Operator mixing problem with Wilson quarks [Bernard et al., ’88]:

$$[O_{VV+AA}]_R = Z_{VV+AA} \left\{ O_{VV+AA} + \sum_{i=1}^{4} z_i \ O_{i}^{d=6} \right\}$$

$$[O_{VA+AV}]_R = Z_{VA+AV} O_{VA+AV}$$
⇒ parity-odd component renormalizes multiplicatively!

**Question:** Can we avoid the mixing problem by using the multiplicatively renormalized operator $O_{VA+AV}$ to compute $B_K$?

- consider continuum theory for a light quark doublet $\psi$ and the $s$-quark:

\[
\mathcal{L}_f = \bar{\psi} \left( \not{D} + m + i\mu_q \gamma_5 \tau^3 \right) \psi + \bar{s} \left( \not{D} + m_s \right) s
\]

\[
\Rightarrow \quad O'_{VV+AA} = \cos(\alpha) O_{VV+AA} - i \sin(\alpha) O_{VA+AV}
\]

\[
= -iO_{VA+AV} \quad (\alpha = \pi/2)
\]
Conclusions

- Wilson quarks break all chiral/axial symmetries which leads to additive quark mass renormalisation, non-trivial axial current normalisation and $O(a)$ effects; can be “cured” by imposing chiral continuum Ward Identities.

- Twisted mass QCD with Wilson type quarks is a regularisation which
  - is equivalent to standard QCD with $N_f = 2, 4, \ldots$
  - has an additional unphysical parameter, the twist angle $\alpha$. This angle determines the physical interpretation (flavour vs. chiral symmetries, parity) and can be used to circumvent certain lattice specific renormalization problems: $F_\pi$ without $Z_A$, the chiral order parameter without cubic divergence, $B_K$ without mixing . . . .
  - enjoys automatic $O(a)$ improvement at $\alpha = \pi/2$ (cf. lecture V)
  - breaks flavour and parity symmetries; expect that these are restored in the continuum limit (just as axial symmetry with standard Wilson quarks).