HEAVY QUARKS ON THE LATTICE 2:

Heavy-Quark Discretization Effects

Recapitulation of Lecture 1

Symanzik EFT for Heavy Quarks

HQET/NRQCD as Theory of Cutoff Effects

Applications: Continuum and Lattice

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Recapitulation of Lecture I

Effective Field Theories for Heavy Quarks

In first lecture we discussed how heavy-quark systems can be described by an effective Lagrangian:

$$\mathcal{L}_{\mathrm{QCD}} \doteq \mathcal{L}_{\mathrm{HQ}}, \qquad \text{read } \doteq \text{ as “has the same physics as”.} \tag{1}$$

The physical equivalence holds for energy-momentum insufficient to produce real $Q\bar{Q}$ pairs. Therefore, these singularities are absent, and an (asymptotic) expansion in E/m is possible.

The effective Lagrangian $\mathcal{L}_{\mathrm{HQ}}$ is, thus, built from two-component spinors, $h^{(\pm)}$, instead of Dirac spinors, q. The rest mass term commutes with everything else, so mass-dependence of wavefunctions enters only through the NR kinetic term, and other higher-dimension terms.

Interactions in $\mathcal{L}_{\mathrm{HQ}}$ are classified by a power counting scheme; different for heavy-light (HQET) and heavy-heavy (NRQCD).

An elegant and not incomplete way to set up $\mathcal{L}_{\mathrm{HQ}}$ is to write down all possible interactions $\mathcal{O}_{s,i}$ and write

$$\mathcal{L}_{\mathrm{HQ}} = \sum_{s,i} \mathcal{C}_{s,i}(\mu)\, \mathcal{O}_{s,i}(\mu); \quad \mu = \text{separation scale}; \tag{2}$$

obtaining coefficients $\mathcal{C}_{s,i}$ via matching calculations with the underlying theory (QCD).

Note that the effective theory will have the same $\mathcal{O}_{s,i}$ for any theory with heavy-quark symmetry and physics describable by two-component spinors. The Wilson coefficients depend on the short-distance structure of the underlying theory.

External electroweak operators (currents, 4-quark operators for mixing) can be built up the same way:

$$\bar{q}\gamma^{\mu}q \doteq \sum_{s,i} \bar{C}_{s,i}(\mu) V^{\mu}_{s,i}(\mu). \tag{3}$$

Framework can be established order-by-order in perturbative QCD, and is thought to hold non-perturbatively as well.

Symanzik EFT and Heavy Quarks

Massive Wilson Quarks

Thinking with the gut, one might expect lattice gauge theories with very massive quarks to break down. Let's see if they do.

Consider Wilson fermions in the hopping parameter notation:

$$S = \sum_n \left\{ \bar{\psi}_n \psi_n - \kappa \sum_\mu \left[\bar{\psi}_n (1 - \gamma_\mu) U_{n,\mu} \psi_{n+\mu} + \bar{\psi}_n (1 + \gamma_\mu) U^\dagger_{n-\mu,\mu} \psi_{n-\mu} \right] \right\}, \qquad (4)$$

where $2\kappa = (4 + m_0 a)^{-1}$. For really heavy quarks, $m_0 a > 1$ and $\kappa \ll 1$.

The propagator has a hopping parameter expansion: it starts with the shortest path(s), *e.g.*, a straight temporal line. It becomes static, plus small corrections of order κ^n.

The approach to this limit is smooth, and can be establish order-by-order in perturbative QCD.

So LGT does not break down (although we've not yet lent it physical sense).

Symanzik EFT

Stefan Sint has introduced you to the Symanzik effective theory. We can write is as follows

$$\mathcal{L}_{\mathrm{LGT}} \doteq \mathcal{L}_{\mathrm{Sym}} = \frac{1}{2g^2}\operatorname{tr}[F^{\mu\nu}F_{\mu\nu}] - \bar{q}(\not{D}+m)q + \quad \mathcal{L}_I \qquad (5)$$
$$= \qquad \text{QCD} \qquad + \text{ small corrections}$$

where the right-hand side is a continuum theory, and the coupling g^2 and mass m are renormalized at subtraction point μ: $\Lambda_{\mathrm{QCD}} \ll \mu \lesssim a^{-1}$.

The extra piece describes discretization effects

$$\mathcal{L}_I = \sum_{\mathcal{O}} a^{\dim \mathcal{O}-4} K_{\mathcal{O}}(g^2, ma; c_{\mathrm{SW}}; \mu a)\, \mathcal{O}_R(\mu), \qquad (6)$$

where the $K_{\mathcal{O}}$ are a short-distance coefficients (*aka* Wilson coefficients).

For light quarks, m introduces a long distance, and it is sensible to expand $K_{\mathcal{O}}$ in powers of ma, and attach the powers of m to "new" operators.

For heavy quarks, m introduces a short distance, so it is essential to retain m in the short-distance coefficients. If $ma \not\ll 1$ it is a mistake to expand K_O in powers of ma.

The last statement may seem obvious, but many people find the urge to expand everything in powers of a irresistible.

Treating the full mass dependence of K_O is conceptually simple enough, though potentially cumbersome in practice.

A more serious problem is that some of the operators in $\mathcal{L}_I$ have the form $(n > 2)$

$$a^n O_{X,n} = \bar{q} X \sum_\mu (-\gamma_\mu D_\mu a)^n q \sim \bar{q} X (-\gamma_4 D_4 a)^n q \sim (ma)^n \bar{q} X q \tag{7}$$

so some of the would-be small corrections are not small.

But don't give up: we know (for Wilson, and any other fermion method with a good heavy-quark limit) that $\mathcal{L}_{\mathrm{Sym}}$ as a whole should be bounded, because $\mathcal{L}_{\mathrm{LGT}}$ is.

Analogously to the way we massaged the HQ effective Lagrangian, we can use field transformations systematically to purge $\mathcal{L}_I$ of higher powers of D_4.

This leads to (nested) commutators of D_4 and $\boldsymbol{D}$, which lead to chromoelectric fields and derivatives thereof.

It also leads to explicit powers $(ma)^n$, multiplying operators with no explicit D_4. Most of this is a big rearrangement of $\mathcal{L}_I$, but the leading part of $\mathcal{L}_{\text{Sym}}$ is modified too:

$$\mathcal{L}_{\text{LGT}} \doteq \mathcal{L}_{\text{Sym}} = \frac{1}{2g^2}\text{tr}[F^{\mu\nu}F_{\mu\nu}] - \bar{q}\left(\gamma_4 D_4 + \sqrt{\frac{m_1}{m_2}}\boldsymbol{\gamma}\cdot\boldsymbol{D} + m_1\right)q + \mathcal{L}'_I, \tag{8}$$

which is not “QCD + small corrections” if (as it generically turns out)

$$\sqrt{\frac{m_1}{m_2}} \not\approx 1, \tag{9}$$

although $\mathcal{L}'_I$ is built from $\boldsymbol{D}$, $\boldsymbol{E}$, and $\boldsymbol{B}$ only (and no $D_4 q$).

NB: Eq. (8) is on the same footing as Eq. (5).

Synopsis I

What we have seen is as follows:

Lattice gauge theory (with Wilson fermions, at least) does not break down for $ma \not\ll 1$.

The Symanzik field theory also does not break down, but (generically) its interpretation as "QCD + small corrections" does break down.

For a quark of momentum $\boldsymbol{p}$ (in a free theory) the energy is given by

$$E^2 = m_1^2 + \frac{m_1}{m_2}\boldsymbol{p}^2 + \cdots \quad \Leftrightarrow \quad E = m_1 + \frac{\boldsymbol{p}^2}{2m_2} + \cdots, \tag{10}$$

so m_1 is called the rest mass and m_2 the kinetic mass.

Comment: $\sqrt{m_1/m_2}$ is often called the "speed of light" by analogy with $E^2 = c^4m^2 + c^2\boldsymbol{p}^2$. From the first term, we then see $m = m_2$. But a photon still has $E = pc$, with $c = 1$, so the jargon is not especially appealing.

Exercise: Start with the effective Lagrangian

$$\bar{q}\,[\gamma_4 D_4 + m_1 + z\boldsymbol{\gamma}\cdot\boldsymbol{D} + aK_{D_4^2}(\gamma_4 D_4 + m_1)^2 + aK_{\boldsymbol{D}^2}(\gamma\cdot\boldsymbol{D})^2 \\ + aK_{\Sigma.B}\bar{q}i\boldsymbol{\Sigma}\cdot\boldsymbol{B} + aK_{\alpha.E}\bar{q}i\boldsymbol{\alpha}\cdot\boldsymbol{E}]\,q, \tag{11}$$

where $\alpha_i = \gamma_4\gamma_i$, $i\Sigma_i = \frac{1}{2}\varepsilon_{ijk}[\gamma_j, \gamma_k]$. Show that changes of variable

$$q \to [1 + ac(\gamma_4 D_4 + m_1) + ad\boldsymbol{\gamma}\cdot\boldsymbol{D}]\,q, \tag{12}$$

$$\bar{q} \to \bar{q}\,[1 + ac(\gamma_4 D_4 + m_1) + ad\boldsymbol{\gamma}\cdot\boldsymbol{D}] \tag{13}$$

can be adjusted so that the coefficients of $\bar{q}(\gamma_4 D_4 + m_1)^2 q$ and $\bar{q}(\gamma\cdot\boldsymbol{D})^2 q$ vanish.

Exercise: Show that all higher-dimension interactions in which $\gamma_4 D_4$ or, equivalently, $\gamma_4 D_4 + m_1$ acts on q or $\bar{q}$ can be removed by changes of variable. Use induction: you just showed it for dimension 5. Commute with $(\gamma_4 D_4 + m_1)\boldsymbol{D} = \boldsymbol{E} + \boldsymbol{D}(\gamma_4 D_4 + m_1)$ until any given term has the form $\bar{q}[(\gamma_4 D_4 + m_1)X + X(\gamma_4 D_4 + m_1)]q$.

HQET/NRQCD as Theories of Cutoff Effects

Heavy Quark Expansion

Although the heavy-quark version of the Symanzik LEL doesn't look like "QCD + small corrections," it is amenable to yesterday's manipulations, leading again to a heavy-quark effective Lagrangian:

$$\mathcal{L}_{\mathrm{Sym}} = -\bar{q}\left(\gamma_4 D_4 + m_1 + \sqrt{\frac{m_1}{m_2}}\boldsymbol{\gamma}\cdot\boldsymbol{D}\right) q + \mathcal{L}'_I \tag{14}$$

$$\doteq \mathcal{L}_{\mathrm{HQ}} = -\bar{h}^{(+)}\left[D_4 + m_1 - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{2m_2}\right] h^{(+)} + \mathcal{L}''_I + \text{anti-quark terms} \tag{15}$$

We must now introduce a power counting scheme:

$$\left.\begin{array}{r} pa \sim \lambda \\ p/m \sim \lambda \\ ma \sim 1 \end{array}\right\}, \quad \text{where } p \sim \Lambda_{\mathrm{QCD}},\ mv \text{ (quarkonium)} \tag{16}$$

Can be imposed on Eq. (14) as well as Eq. (15), with extra power of λ for Dirac matrices that anti-commute with γ_4.

Comparison

At dimension five, the original $\mathcal{L}_I$ treating all four directions the same looks like

$$\mathcal{L}_{I,5} = aK_{\sigma\cdot F}\bar{q}i\sigma^{\mu\nu}F_{\mu\nu}q \tag{17}$$

from which a redundant term $\bar{q}(\not{D}+m)^2q$ has been removed by a redefinition of the fields q and $\bar{q}$. (Allowed for on-shell Green functions.)

Treating the 4 direction as special

$$\mathcal{L}'_{I,5} = aK'_{\Sigma\cdot B}\bar{q}i\boldsymbol{\Sigma}\cdot\boldsymbol{B}q + aK'_{\alpha\cdot E}\bar{q}i\boldsymbol{\alpha}\cdot\boldsymbol{E}q \tag{18}$$

from which two redundant terms, $\bar{q}(\gamma_4 D_4 + m_1)^2 q$ and $\bar{q}(\boldsymbol{\gamma}\cdot\boldsymbol{D}+m)^2q$, have been removed by a redefinition of the fields q and $\bar{q}$.

Finally,

$$\mathcal{L}''_{I,5} = aK''_{\sigma\cdot B}\left(\bar{h}^{(+)}i\boldsymbol{\sigma}\cdot\boldsymbol{B}h^{(+)} + \bar{h}^{(-)}i\boldsymbol{\sigma}\cdot\boldsymbol{B}h^{(-)}\right) \tag{19}$$

from which the redundant terms $\bar{h}^{(\pm)}(\pm D_4 + m_1)^2 h^{(\pm)}$ can be omitted.

What happened to the chromoelectric operator? In HQ systems

$$\bar{q}\boldsymbol{\alpha}\cdot\boldsymbol{E}q \doteq \frac{1}{2m_2}\left(\bar{h}^{(+)}[\boldsymbol{\sigma}\cdot\boldsymbol{D},\boldsymbol{\sigma}\cdot\boldsymbol{E}]h^{(+)} - \bar{h}^{(-)}[\boldsymbol{\sigma}\cdot\boldsymbol{D},\boldsymbol{\sigma}\cdot\boldsymbol{E}]h^{(-)}\right) \quad (20)$$

$$\bar{q}\{\boldsymbol{\gamma}\cdot\boldsymbol{D},\boldsymbol{\alpha}\cdot\boldsymbol{E}\}q \doteq \bar{h}^{(+)}[\boldsymbol{\sigma}\cdot\boldsymbol{D},\boldsymbol{\sigma}\cdot\boldsymbol{E}]h^{(+)} - \bar{h}^{(-)}[\boldsymbol{\sigma}\cdot\boldsymbol{D},\boldsymbol{\sigma}\cdot\boldsymbol{E}]h^{(-)} \quad (21)$$

which shows why Dirac structures, like $\boldsymbol{\alpha}$, that anti-commute with γ_4 receive an extra power in the power counting.

Follows immediately from Foldy-Wouthuysen transformations.

This phenomenon is seen in the mass spectrum: splittings generated by the chromoelectric interaction are commensurate with this extra power. Such splittings include those of the χ_{cJ} and χ_{bJ} states in quarkonium. (States with $L = 1$, $S = 1$ leading to total $J = 0, 1, 2$.)

HQET as a Theory of Cutoff Effects

In yesterday's lecture we saw that, for heavy-light systems (omitting $h^{(-)}$)

$$\mathcal{L}_{\mathrm{QCD}} \doteq \frac{1}{2g^2}\mathrm{tr}[F^{\mu\nu}F_{\mu\nu}] + \bar{h}^{(+)} iv\cdot D h^{(+)} - m\bar{h}^{(+)}h^{(+)} + \sum_{s>0,i} \mathcal{C}_{s,i}^{\mathrm{QCD}}(\mu)\, \mathcal{O}_{s,i}(\mu) \quad (22)$$

Today we saw that, for heavy-light systems

$$\mathcal{L}_{\mathrm{LGT}} \doteq \frac{1}{2g^2}\mathrm{tr}[F^{\mu\nu}F_{\mu\nu}] + \bar{h}^{(+)} iv\cdot D h^{(+)} - m_1\bar{h}^{(+)}h^{(+)} + \sum_{s>0,i} \mathcal{C}_{s,i}^{\mathrm{LGT}}(\mu)\, \mathcal{O}_{s,i}(\mu) \quad (23)$$

The leading parts are the same, except that one need not identify m_1 with m.

Since the rest mass doesn't alter wavefunctions, we see that cutoff effects of LGT are embodied in an overall mass shift $m - m_1$ of the spectrum and $\mathcal{C}_{s,i}^{\mathrm{QCD}} - \mathcal{C}_{s,i}^{\mathrm{LGT}}$.

At least for Wilson fermions, coefficient mismatches are all bounded.

The leading corrections provide an illuminating example:

$$\frac{\bar{h}^{(+)}\boldsymbol{D}^2 h^{(+)}}{2m} + Z_{\sigma B}\frac{\bar{h}^{(+)} i\boldsymbol{\sigma}\cdot\boldsymbol{B} h^{(+)}}{2m} \tag{24}$$

$$\frac{\bar{h}^{(+)}\boldsymbol{D}^2 h^{(+)}}{2m_2} + Z_{\sigma B}^{\mathrm{LGT}}\frac{\bar{h}^{(+)} i\boldsymbol{\sigma}\cdot\boldsymbol{B} h^{(+)}}{2m_2} \tag{25}$$

We should adjust the lattice gauge theory so that

$$m_2 = m, \quad Z_{\sigma B}^{\mathrm{LGT}} = Z_{\sigma B} \tag{26}$$

to normalize correctly the $1/m$ contributions to heavy-light matrix elements.

How well this is achieved in practice is a subject for tomorrow's lecture, which will cover the methods actually used in simulations.

Caveat: There are higher-dimension $O_{s,i}$ allowed by lattice spacetime symmetries that are forbidden by continuum spacetime symmetries. For them we simply say that $C_{s,i}^{\mathrm{QCD}} = 0$, and consider it to be a special case.

NRQCD as a Theory of Cutoff Effects

Similarly, for quarkonium (omitting $h^{(-)}$)

$$\mathcal{L}_{\mathrm{QCD}} \doteq \cdots + \bar{h}^{(+)} iv \cdot D h^{(+)} - m \bar{h}^{(+)} h^{(+)} + \frac{\bar{h}^{(+)} D_\perp^2 h^{(+)}}{2m} + \cdots \tag{27}$$

$$\mathcal{L}_{\mathrm{LGT}} \doteq \cdots + \bar{h}^{(+)} iv \cdot D h^{(+)} - m_1 \bar{h}^{(+)} h^{(+)} + \frac{\bar{h}^{(+)} D_\perp^2 h^{(+)}}{2m_2} + \cdots \tag{28}$$

The leading parts are the same, but one must identify m_2 with m.

Since the rest mass m_1 doesn't alter wavefunctions, we see that cutoff effects of LGT are again embodied in an overall mass shift $m_2 - m_1$ of the spectrum and the difference in $\mathcal{C}_{s,i}^{\mathrm{QCD}} - \mathcal{C}_{s,i}^{\mathrm{LGT}}$ for interactions higher in the velocity counting.

Synopsis II

For $ma \not\ll 1$, the useful analysis tool of the Symanzik EFT develops, generically, a "bug:" the factor $\sqrt{m_1/m_2}$ multiplying the spatial kinetic term.

One is not obliged to use this analysis tool: as an alternative one can use the effective Lagrangian for heavy quarks to study cutoff effects.

As long as the lattice theory is consistent with the heavy-quark symmetries— has, for example, no extra high-energy poles in the propagator—this description is adequate and useful. (Why wouldn't you have heavy-quark symmetry?)

It is essential (efficient) to identify m_2 with m in quarkonium (heavy-light hadrons).

For example, LGT with Wilson fermions is "QCD + small corrections + mass shift".

The key to all of this is to treat m_Q^{-1} as a short distance, commensurate with a. Unless you have c and b sea quarks, you are making this assumption already.

Some Specifics

The plan is to introduce the general framework for understanding the cutoff effects of lattice gauge theory with heavy quarks, and postpone discussion of specific methods until Monday.

Since I've mentioned (and, perhaps, advocated) the possibility of Wilson-like quarks, it is useful to see some specific results in this case.

The rest and kinetic mass come from the propagator

$$
\begin{aligned}
G(t,\boldsymbol{p}) &= \int \frac{dp_4}{2\pi} \frac{e^{ip_4t}}{i\sum_\mu \gamma_\mu \sin(p_\mu a) + m_0a + \sum_\mu [1-\cos(p_\mu a)]} \\
&= \frac{ae^{-Et}}{2\sinh(Ea)} \frac{\gamma_4 \sinh(Ea) - i\sum_i \gamma_i \sin(p_i a) + m_0 a + \frac{1}{2}a^2\hat{\boldsymbol{p}}^2 + 1 - \cosh(Ea)}{1 + m_0 a + \frac{1}{2}a^2 \hat{\boldsymbol{p}}^2}
\end{aligned}
\tag{29}
$$

where

$$\cosh(Ea) = 1 + \frac{1}{2}\frac{\sum_i \sin^2(p_i a) + (m_0 a + \frac{1}{2}a^2\hat{\boldsymbol{p}}^2)^2}{1 + m_0 a + \frac{1}{2}a^2\hat{\boldsymbol{p}}^2} \tag{30}$$

Exercise: Carry out the p_4 integration via contours to obtain the end result.

One finds (tree level)

$$m_1 a = \ln(1 + m_0 a), \quad \frac{1}{m_2 a} = \frac{2}{m_0 a(2 + m_0 a)} + \frac{1}{1 + m_0 a}. \tag{31}$$

which agree through $O(a^2)$, but for large $m_0 a$ are rather different.

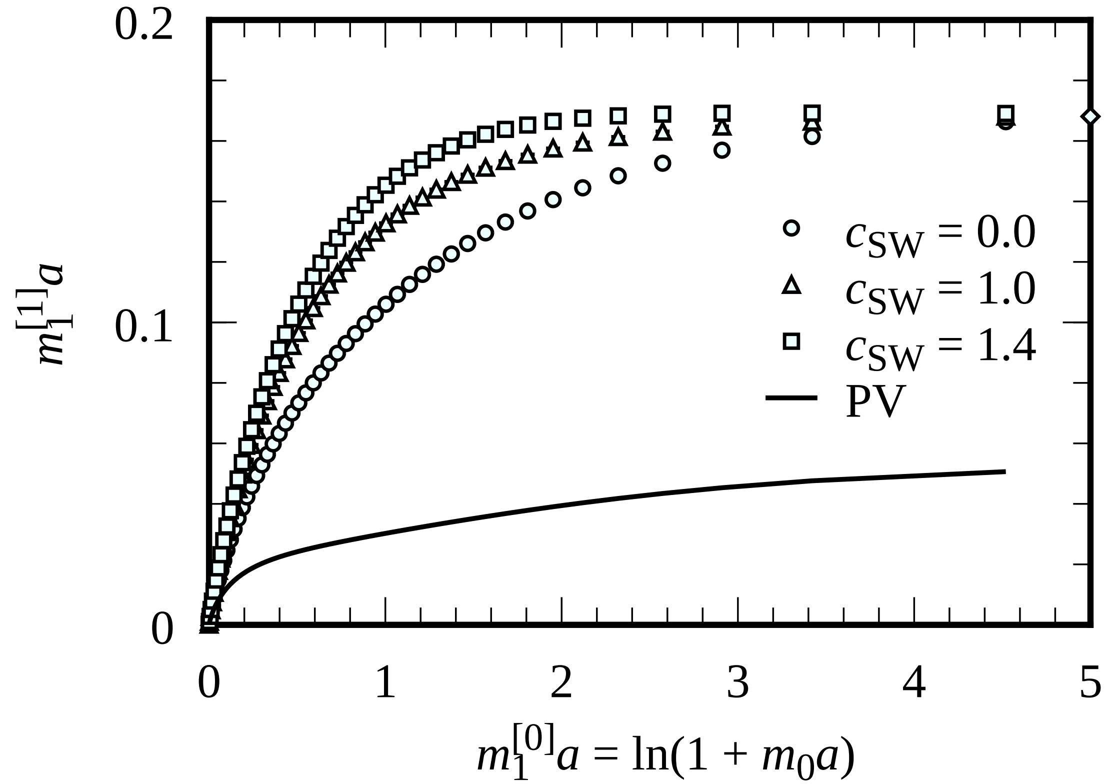
From examining the quark-gluon vertex one finds (tree level)

$$\frac{Z_{\sigma B}^{\mathrm{lat}}}{m_2 a} = \frac{2}{m_0 a(2 + m_0 a)} + \frac{c_{\mathrm{SW}}}{1 + m_0 a}, \tag{32}$$

where c_{SW} is the coefficient of the Sheikholeslami-Wohlert term.

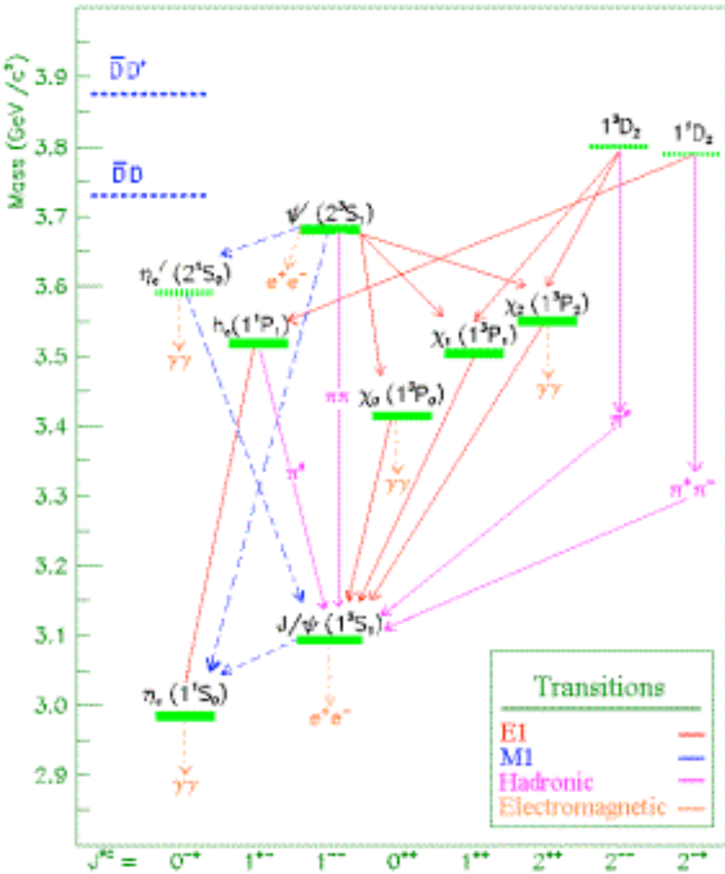
It is also possible to compute the (one-)loop corrections:

$$m_1 a = \ln(1 + m_0 a) + \sum_{\ell=1}^{\infty} g_0^{2\ell} m_1^{[\ell]} a \tag{33}$$



Applications: Lattice and Continuum

Charmonium Spectrum



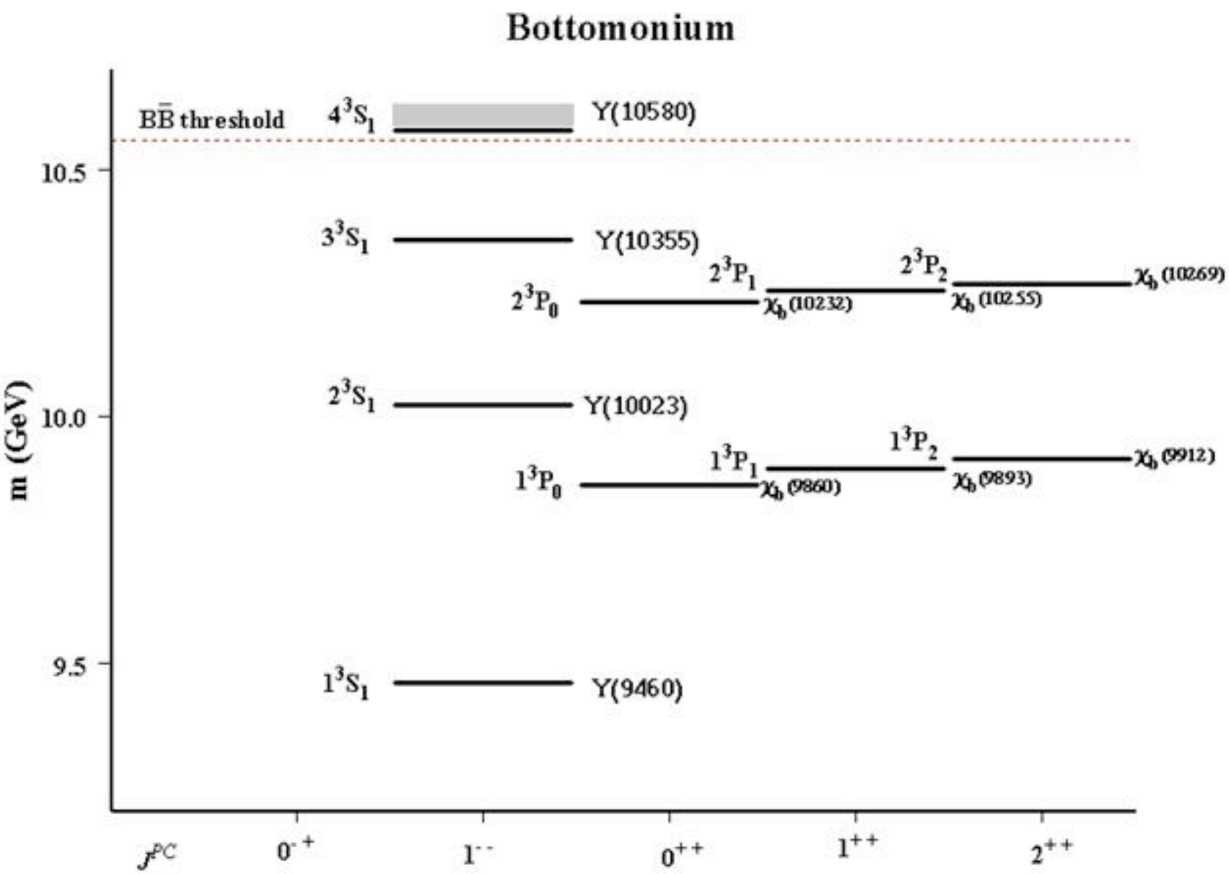
The gross structure, *e.g.*, $m_{\psi'} - m_{J/\psi}$, is dictated by kinetic term, with small corrections from the $(\boldsymbol{D}^2)^2$ term.

The splittings between states of $L = 0$ but different S, *e.g.*, $m_{J/\psi} - m_{\eta_c}$, come from the $i\boldsymbol{\sigma} \cdot \boldsymbol{B}$ term. Called “hyperfine.”

The splittings between states of $L = 1$ and $S = 1$ of different J, *e.g.*, among $m_{\chi_{cJ}}$, come from the $i\boldsymbol{\sigma} \cdot (\boldsymbol{D} \times \boldsymbol{E} - \boldsymbol{E} \times \boldsymbol{D})$ term.

Note that the last two are about the same size, smaller than the first.

Bottomonium Spectrum



Meson Masses

We now turn to some result from HQET for heavy-light hadrons.

The idea is to set up the interaction picture based on the leading Hamiltonian

$$H^{(0)}(t) = \int d^3x\, \mathcal{H}^{(0)}(x), \quad \mathcal{H}^{(0)} = \bar{h}^{(+)}(m_1 - iv\cdot A)h^{(+)}. \tag{34}$$

Recall that eigenstates of $H^{(0)}$ are independent of m_1 and eigenvalues depend trivially on m_1.

From the Gell-Mann–Low theorem

$$E = \frac{\langle B^{(0)}|H|B\rangle}{\langle B^{(0)}|B\rangle} = \frac{\langle B^{(0)}|H(0)U(0,-T)|B^{(0)}\rangle}{\langle B^{(0)}|U(0,-T)|B^{(0)}\rangle}, \tag{35}$$

where

$$U(t,t_0) = T\ \exp\int_{t_0}^{t} d^4x \sum_{s>0} \mathcal{L}^{(s)} \tag{36}$$

To leading order

$$M_1 = \langle B^{(0)}|[H_{\text{light}} + H^{(0)}]|B^{(0)}\rangle - \langle B^{(0)}|L^{(1)}|B^{(0)}\rangle, \tag{37}$$

$$= m_1 + \bar{\Lambda} + \frac{\mu_\pi^2}{2m_2} - Z_{\sigma B}^{(\text{LGT or QCD})} \frac{d_J}{3} \frac{\mu_G^2}{2m_2} \tag{38}$$

where $\mu_\pi^2 = -\lambda_1$ and $\mu_G^2 = 3\lambda_2$ are expectation values of the kinetic and chromomagnetic operators, in the state with spin J. $d_0 = 3$ for B meson, $d_1 = -1$ for B^*.

A similar formula holds for the Λ_b baryon, with a different $\bar{\Lambda}$ and kinetic matrix element, and no chromomagnetic contribution.

Matrix Elements of Currents

Currents can be described in HQET in a way analogous to the Lagrangian. For example, the heavy$\rightarrow$light current

$$\mathcal{V}^\mu = \bar{q}\gamma^\mu b \doteq C_{V_\parallel} v^\mu \bar{q} h_v + C_{V_\perp} \bar{q} i\gamma_\perp^\mu h_v - \sum_{i=1}^{6} B_{Vi} Q_{Vi}^\mu + \cdots, \tag{39}$$

$$\mathcal{A}^\mu = \bar{q}\gamma^\mu\gamma_5 b \doteq C_{A_\perp} \bar{q} i\gamma_\perp^\mu \gamma_5 h_v - C_{A_\parallel} v^\mu \bar{q}\gamma_5 h_v - \sum_{i=1}^{6} B_{Ai} Q_{Ai}^\mu + \cdots, \tag{40}$$

where at the one loop level

$$C_{J_\parallel} = 1 + \frac{g^2 C_F}{16\pi^2}\left(\gamma_h \ln(m^2/\mu^2) - 2\right), \tag{41}$$

$$C_{J_\perp} = 1 + \frac{g^2 C_F}{16\pi^2}\left(\gamma_h \ln(m^2/\mu^2) - 4\right), \tag{42}$$

and the “hybrid anomalous dimension” $\gamma_h = \frac{3}{2}$.

In lattice gauge theory we can similarly describe

$$V^{\mu}_{\text{lat}} \doteq C^{\text{lat}}_{V_{\parallel}} v^{\mu} \bar{q} h_v + C^{\text{lat}}_{V_{\perp}} \bar{q} i\gamma^{\mu}_{\perp} h_v - \sum_{i=1}^{6} B^{\text{lat}}_{Vi} Q^{\mu}_{Vi} + \cdots, \tag{43}$$

$$A^{\mu}_{\text{lat}} \doteq C^{\text{lat}}_{A_{\perp}} \bar{q} i\gamma^{\mu}_{\perp} \gamma_5 h_v - C^{\text{lat}}_{A_{\parallel}} v^{\mu} \bar{q} \gamma_5 h_v - \sum_{i=1}^{6} B^{\text{lat}}_{Ai} Q^{\mu}_{Ai} + \cdots, \tag{44}$$

where the short-distance coefficients depend on the details of the discretization. For example, $V^{\mu}_{\text{lat}} = \bar{\psi}_q \gamma^{\mu} \psi_b$, $A^{\mu}_{\text{lat}} = \bar{\psi}_q \gamma^{\mu} \gamma_5 \psi_b$, where ψ and $\bar{\psi}$ are lattice (not continuum) fermion fields.

They may have some different structure: for example, the logarithm may be missing, but we shall see what happens.

One can derive a formula similar to that for the mass:

$$
\begin{aligned}
\langle L|v\cdot\mathcal{V}|B\rangle = &-C_{V_\parallel}^{\mathrm{QCD}}\langle L|\bar{q}h_v|B_v^{(0)}\rangle \\
&- B_{V1}^{\mathrm{QCD}}\langle L|v\cdot Q_{V1}|B_v^{(0)}\rangle - B_{V4}^{\mathrm{QCD}}\langle L|v\cdot Q_{V4}|B_v^{(0)}\rangle \\
&- \mathcal{C}_2^{\mathrm{QCD}}C_{V_\parallel}^{\mathrm{QCD}}\int d^4x\langle L|T\,\mathcal{O}_2(x)\bar{q}h_v|B_v^{(0)}\rangle^\star \\
&- \mathcal{C}_{\mathcal{B}}^{\mathrm{QCD}}C_{V_\parallel}^{\mathrm{QCD}}\int d^4x\langle L|T\,\mathcal{O}_{\mathcal{B}}(x)\bar{q}h_v|B_v^{(0)}\rangle^\star + O(\Lambda^2/m^2),
\end{aligned}
$$

$$
\begin{aligned}
\langle L|v\cdot V_{\mathrm{lat}}|B\rangle = &-C_{V_\parallel}^{\mathrm{lat}}\langle L|\bar{q}h_v|B_v^{(0)}\rangle \\
&- B_{V1}^{\mathrm{lat}}\langle L|v\cdot Q_{V1}|B_v^{(0)}\rangle - B_{V4}^{\mathrm{lat}}\langle L|v\cdot Q_{V4}|B_v^{(0)}\rangle \\
&- \mathcal{C}_2^{\mathrm{lat}}C_{V_\parallel}^{\mathrm{lat}}\int d^4x\langle L|T\,\mathcal{O}_2(x)\bar{q}h_v|B_v^{(0)}\rangle^\star \\
&- \mathcal{C}_{\mathcal{B}}^{\mathrm{lat}}C_{V_\parallel}^{\mathrm{lat}}\int d^4x\langle L|T\,\mathcal{O}_{\mathcal{B}}(x)\bar{q}h_v|B_v^{(0)}\rangle^\star \\
&- K_{\sigma\cdot F}C_{V_\parallel}^{\mathrm{lat}}\int d^4x\langle L|T\,\bar{q}i\sigma F q(x)\bar{q}h_v|B_v^{(0)}\rangle^\star + O(\Lambda^2a^2b(ma)).
\end{aligned}
$$

If one multiplies the equations for the lattice matrix elements with

$$Z_{J_\parallel} = \frac{C_{J_\parallel}}{C^{\text{lat}}_{J_\parallel}}, \tag{45}$$

$$Z_{J_\perp} = \frac{C_{J_\perp}}{C^{\text{lat}}_{J_\perp}}, \tag{46}$$

and subtracts the result from the continuum equations, one finds that the difference can be traced solely to the mismatch of the short-distance coefficients, or

$$\delta\mathcal{C}_i = \mathcal{C}^{\text{lat}}_i - \mathcal{C}_i, \tag{47}$$

$$\delta B_{Ji} = Z_{Ji} B^{\text{lat}}_{Ji} - B_{Ji}, \tag{48}$$

where the normalization factors Z_{Ji} are $Z_{J_\parallel}$ for $i = 1$, 4, and $Z_{J_\perp}$ for $i = 2$, 3, 5, 6.

This is, then, a living, breathing example of how the discretization effects of heavy quarks can be isolated into short-distance coefficients of the HQET.

Summary

It is important to observe that the procedure for renormalizing and matching via HQET is non-perturbative.

The framework is established in perturbative QCD, but should hold non-perturbatively.

Hence, it is an "implementation detail" to compute the Zs and other coefficients non-perturbatively.

In the same sense this procedure is analogous to renormalizing and matching via the Symanzik EFT.

Next time we will survey methods to compute both the matrix elements, and the matching factors.

Bibliography for Lectures 1 & 2

HQET: the textbook *Heavy Quark Physics*, by A. Manohar and M. Wise (Cambridge U., Cambridge, UK, 2004), contains a wealth of information. They may overemphasize the need of the phase factor $e^{-imv\cdot x}$ for the emergence HQS, but they cover the most important phenomenological applications pedagogically.

NRQCD: for power counting I follow the original paper "Improved nonrelativistic QCD for heavy quark physics," by G.P. Lepage, L. Magnea, C. Nakhleh, U. Magnea, and K. Hornbostel, *Phys. Rev. D* **46**, 4052 (1992) [arXiv:hep-lat/9205007].

For a comprehensive survey of quarkonium physics see "Heavy quarkonium physics," by the Quarkonium Working Group (N. Brambilla *et al.*), arXiv:hep-ph/0412158.

The HQ theory of cutoff effects was developed in a series of papers titled "Application of heavy-quark effective theory to lattice QCD," with subtitles:

"I: Power corrections," A.S. Kronfeld, *Phys. Rev. D* **62**, 014505 (2000) [arXiv:hep-lat/0002008];

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