Fundamental constants and electroweak phenomenology from the lattice

Lecture IV: CKM phenomenology: at tree level
IV. CKM phenomenology: at tree level

1. Quark flavor physics
   - Weak interaction; from W-exchange to four-fermion interactions
   - Quark mixings: the CKM matrix, unitarity triangle

2. $V_{us}$, the Cabibbo angle
   - Flavor SU(3) breaking: one-loop ChPT and higher order corrections; Lattice calculation

3. $V_{cb}$
   - Inclusive and exclusive semi-leptonic decays
   - Heavy quark symmetry; lattice calculation

4. $V_{ub}$
   - Continuum extraction from inclusive decays
   - Lattice calculation for exclusive processes
IV. CKM Phenomenology: at tree level
1. Quark flavor physics
Our goal:

- To understand this plot
- How lattice QCD may contribute to improve it.
Weak interaction

- Quarks may change their flavor through weak interaction.
  - Active only for left-handed quarks and right-handed anti-quarks.
    \[ \bar{q}^f \gamma_\mu (1 - \gamma_5) q^f W^\mu \]
  - Short distance (~1/M_W) interaction.

\[ G_F = \frac{1}{4\sqrt{2}} \frac{g^2}{M_W^2} = 1.17 \times 10^{-5} \text{ GeV}^{-2} \]

- Acts on (weak) isospin doublets.
  \[ \begin{pmatrix} u \\ d' \end{pmatrix}, \begin{pmatrix} c \\ s' \end{pmatrix}, \begin{pmatrix} t \\ b' \end{pmatrix}, \begin{pmatrix} v_e \\ u \end{pmatrix}, \begin{pmatrix} v_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} v_\tau \end{pmatrix} \]
Changing flavors

- Quark flavor may change:
  - Weak isospin is not identical with the real isospin.
    \[
    \begin{pmatrix}
    u \\
    d' \\
    s' \\
    b'
    \end{pmatrix}, \begin{pmatrix}
    c \\
    d \\
    s \\
    b
    \end{pmatrix}, \begin{pmatrix}
    t \\
    d' \\
    s' \\
    b'
    \end{pmatrix} \neq \begin{pmatrix}
    u \\
    d \\
    s \\
    b
    \end{pmatrix}, \begin{pmatrix}
    c \\
    d \\
    s \\
    b
    \end{pmatrix}, \begin{pmatrix}
    t \\
    d \\
    s \\
    b
    \end{pmatrix}
    \]
  - Related by a 3x3 unitary matrix.
    \[
    \begin{pmatrix}
    d' \\
    s' \\
    b'
    \end{pmatrix} = V_{CKM} \begin{pmatrix}
    d \\
    s \\
    b
    \end{pmatrix}, \quad V_{CKM} = \begin{pmatrix}
    V_{ud} & V_{us} & V_{ub} \\
    V_{cd} & V_{cs} & V_{cb} \\
    V_{td} & V_{ts} & V_{tb}
    \end{pmatrix}
    \]
  - Called Cabibbo-Kobayashi-Maskawa matrix.
    - Cabibbo (60s), Kobayashi-Maskawa (1973)
CKM matrix

- Degrees of freedom
  - $N \times N$ complex matrix has $2N^2$ real parameters.
  - Unitarity constraints $N$ (diagonal) + $N(N-1)$ (off-diagonal); thus $N^2$ parameters remain.
  - Quark phases are arbitrary $2N$ except for 1 (overall phase does not change $V_{CKM}$); thus $(N-1)^2$ remain.

- $N(N-1)/2$ are mixing angles.
- $(N-1)(N-2)/2$ are CP violating phases.

$$V_{CKM} = \begin{pmatrix}
  V_{ud} & V_{us} & V_{ub} \\
  V_{cd} & V_{cs} & V_{cb} \\
  V_{td} & V_{ts} & V_{tb}
\end{pmatrix}$$

$N=3 \Rightarrow$ 3 mixing angles + 1 CP phase
Mixing angles

- **Strength of the weak interaction is different among processes**
  
  \[ \mu^- \rightarrow \nu_\mu + e^- + \bar{\nu}_e \quad \Rightarrow \quad G_{\mu\nu}^2 = G_F^2 \]
  
  \[ n \rightarrow p + e^+ + \nu_e \quad \Rightarrow \quad G_{ud}^2 = 0.95 \times G_F^2 \]
  
  \[ K^- \rightarrow \pi^0 + e^- + \bar{\nu}_e \quad \Rightarrow \quad G_{us}^2 = 0.05 \times G_F^2 \]

- **\( V_{us} = \sin \theta_c \): the Cabibbo angle**
  
  \[ u = u \]
  
  \[ d' = d \cos \theta_c + s \sin \theta_c \]

  \[ \sin \theta_c \sim 0.22 \]

- **Other angles**
  
  - 2\( \leftrightarrow \)3: \( V_{cb} \)
  
  - 1\( \leftrightarrow \)3: \( V_{ub} \)

    - Much smaller in magnitude
CKM unitarity

- Unitary implies...
  - Normalization
    \[ |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \]
  - Orthogonality
    \[ V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0 \quad (b \rightarrow d) \]
    \[ V_{us}^* V_{ud} + V_{cs}^* V_{cd} + V_{ts}^* V_{td} = 0 \quad (s \rightarrow d) \]

Unitarity triangles
CKM hierarchy

- We don’t know why, but the CKM matrix has a hierarchical structure.
  \[
  V_{\text{CKM}} = \begin{pmatrix}
  0.97 & 0.23 & 0.004 \\
  0.23 & 0.96 & 0.04 \\
  0.007 & 0.04 & > 0.8
  \end{pmatrix}
  \]

- Wolfenstein parametrization: with \( \lambda = 0.225 \),
  \[
  V_{\text{CKM}} = \begin{pmatrix}
  1 - \lambda^2 / 2 & \lambda & A \lambda^3 (\rho - i \eta) \\
  -\lambda & 1 - \lambda^2 / 2 & A \lambda^2 \\
  A \lambda^3 (1 - \rho - i \eta) & -A \lambda^2 & 1
  \end{pmatrix}
  \]
  written explicitly by four parameters: \( \lambda, A, \rho, \eta \).
Unitarity triangle

- Most interesting unitarity condition.
  \[ V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0 \]

- Normalized by a better known side \( V_{cb}^* V_{cd} \).

- Apex is \((\rho, \eta)\).

- Defines three angles.
  \[ \phi_1(= \beta) = \arg\left(-\frac{V_{cd}^* V_{cb}}{V_{td} V_{tb}^*}\right), \quad \phi_2(= \alpha) = \arg\left(-\frac{V_{td}^* V_{tb}}{V_{cd}^* V_{ub}}\right), \quad \phi_3(= \gamma) = \arg\left(-\frac{V_{ud}^* V_{ub}}{V_{cd} V_{cb}^*}\right). \]
Several measurements (sides and angles) can be compared on a single plane of \((\rho, \eta)\).

- Tree processes (today)
- Loop processes (tomorrow)
Tree-level processes

<table>
<thead>
<tr>
<th>CKM element</th>
<th>generations</th>
<th>quark level process</th>
<th>exclusive processes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{us}$</td>
<td>2 $\rightarrow$ 1</td>
<td>$s \rightarrow ul\nu$</td>
<td>$K \rightarrow \pi l\nu, \Lambda \rightarrow pl\nu$</td>
</tr>
<tr>
<td>$V_{cb}$</td>
<td>3 $\rightarrow$ 2</td>
<td>$b \rightarrow cl\nu$</td>
<td>$B \rightarrow D(\ast) l\nu, \Lambda_b \rightarrow \Lambda_c l\nu$</td>
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</tbody>
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IV. CKM Phenomenology: at tree level
2. $V_{us}$, the Cabibbo angle
Cabibbo angle

- The best known mixing angle.
  - Primary information from $K \to \pi l \nu$ decay, that contains a quark level process $s \to u l \nu$. (Hyperon decay could also be used.)
  - Decay rate
    \[
    \Gamma_{Kl3} = \frac{G_F^2}{192 \pi^3} S_{EW} (1 + \delta_K) |V_{us}|^2 f_+^2 (0) I_K
    \]
  - $S_{EW}$: short distance EW radiative correction.
  - $\delta_K$: long distance EM radiative correction (sub %)
  - $f_+ (0)$: form factor = QCD soft physics
  - $I_K$: phase space integral = contains the info of the form factor shape.
Form factor

- **Matrix element**

\[
\langle \pi(p_\pi) | \overline{s} \gamma_\mu u | K(p_K) \rangle = f_+(t)(p_K + p_\pi)_\mu + f_-(t)(p_K - p_\pi)_\mu
\]

- Very similar to the pion form factor, but now contains \( f_- \) because (initial ↔ final) exchange symmetry is lost.
- Instead of \( f_- \), one can also use the scalar form factor

\[
f_0(t) = f_+(t) + \frac{t}{m_K^2 - m_\pi^2} f_-(t)
\]

which is the piece to survive the projection

\[
(p_K - p_\pi)^\mu \langle \pi(p_\pi) | \overline{s} \gamma_\mu u | K(p_K) \rangle = (m_K^2 - m_\pi^2) f_0(t)
\]
Before embarking on the hard (and costly) calculations on the lattice, analytically known facts should be used as much as possible.

- \( f_+(t) \) reduces to the pion form factor \( G_\pi(t) \) in the limit of degenerate \( \pi \) and \( K \).
- In this limit, \( f_+(0) = 1 \).
- Away from this limit, there is a correction starting from second order (no \( O(m_s-m_u) \) = Ademollo-Gatto theorem, 1964).
  - Simple explanation: \( f_+ \) is a symmetric piece under the exchange \( \pi \leftrightarrow K \). So, \( m_s-m_u \) cannot appear.
    \[
    f_+(0) = 1 + f_2 + f_4 = 1 - 0.023 - 0.016(8) = 0.961(8)
    \]
For small $q^2=t$ region, ChPT provides a reliable framework to calculate the form factor.

At one-loop, non-analytic dependence on quark mass is predicted (chiral log). Gasser-Leutwyler (1985)

$$f_+(0) = 1 + \frac{3}{2} H_{K\pi}(0) + \frac{3}{2} H_{K\eta}(0),$$

$$H_{PQ}(0) = -\frac{1}{128\pi^2 f^2} \left( m_P^2 + m_Q^2 \right) h_0 \left( \frac{m_P^2}{m_Q^2} \right),$$

$$h_0(x) = 1 + \frac{2x}{1 - x^2} \ln x$$

gives $f_2 = -0.023$, now two-loop is known (Bijinens et al.)
What lattice can do

- Main target is $f_4$: only a few % contribution, but that is the accuracy one wants to achieve.
  - $f_2$ should also be calculated. Good consistency check with ChPT.
- Many other predictions/cross-checks possible. Mainly the form factor shape,
  \[ f_\pm(q^2) = f_\pm(0)[1 + \lambda_\pm(q^2/m^2_\pi)] \]
  - The slope parameter $\lambda_\pm$ can be compared with the experimental data.
Lattice calculation

- Very similar to the pion form factor calculation.
  - But need to separate $f_+$ from $f_0$.
    \[
    \langle \pi(p_\pi) | \bar{s} \gamma_\mu u | K(p_K) \rangle = f_+(t)(p_K + p_\pi)_\mu + f_-(t)(p_K - p_\pi)_\mu
    \]
  - Possible by looking at different $\mu$ directions and solving linear equations, but…

- When both $\pi$ and $K$ are at rest, only $f_0$ can be obtained.

- Statistical error is larger with finite momentum insertion.
**Statistical noise**

- Larger the momentum, larger the noise.

  Possible to understand as follows (Lepage, 1990)

  \[
  N^2(t) \sim \left\langle \left( \text{Tr} \left[ \Gamma S_q(x,0)\Gamma' S_q(0,x) \right] \right)^2 \right\rangle - \left\langle \text{Tr} \left[ \Gamma S_q(x,0)\Gamma' S_q(0,x) \right] \right\rangle^2
  \]

  \[
  = \exp\left[ -E_\pi (\mathbf{p} \pm \mathbf{p})t \right] - \exp\left[ -2E_\pi (\mathbf{p}) \right]
  \]

  \[
  N(t) / S(t) \sim \exp\left[ (E_\pi (\mathbf{p}) - E_\pi (0) / 2) t \right]
  \]

  \[
  \sim \exp\left[ (E_\pi (\mathbf{p}) - E_\pi (0)) t \right], \quad E_\pi (\mathbf{p}) = \sqrt{m_\pi^2 + \mathbf{p}^2}
  \]
A clever method

- Precision is the key for this quantity. Consider ratios in which the bulk of stat fluctuation cancel.

\[ \frac{C^{\pi V_4 K}(t)C^{K V_4 \pi}(t)}{C^{\pi V_4 \pi}(t)C^{K V_4 K}(t)} \xrightarrow{\text{1st ratio}} \frac{\langle \pi(0) | V_4 | K(0) \rangle \langle K(0) | V_4 | \pi(0) \rangle}{\langle \pi(0) | V_4 | \pi(0) \rangle \langle K(0) | V_4 | K(0) \rangle} = \frac{(m_K + m_{\pi})^2}{4m_K m_{\pi}} f_0(q_{\text{max}}^2) \]

- Precisely calculated.
- Renormalization factors cancel.

- Can be obtained only at
  \[ q_{\text{max}}^2 = (m_K - m_{\pi})^2 \]

- Need to extrapolate back to \( q^2 = 0 \).

JLQCD, 2005
Clever ratios

- **Extrapolate back to** $q^2=0$
  - **2$^{nd}$ ratio with finite momentum.**
    \[
    \frac{\langle \pi(p)|V_4|K(0) \rangle}{\langle \pi(0)|V_4|K(0) \rangle} = \frac{m_K + E_\pi}{m_K + m_\pi} \frac{f_+(q^2)}{f_+(q_{max}^2)} \left[ 1 + \xi(q^2) \frac{m_K - E_\pi}{m_K + E_\pi} \right] \]
  - **3$^{rd}$ ratio with different $\mu$**
    \[
    \frac{\langle \pi(p)|V_4|K(0) \rangle}{\langle \pi(0)|V_4|K(0) \rangle} = 1 - \xi(q^2) \]
    \[
    \frac{\langle \pi(p)|\bar{V}_4^*|\pi(0) \rangle}{\langle \pi(p)|\bar{V}_4^*|\pi(0) \rangle} = \frac{m_K + E_K}{m_\pi + E_\pi} \frac{m_K - E_K}{m_K + E_\pi} \]

- Subtract $f_-$ to get $f_+$

$\xi(q^2) = \frac{f_-(q^2)}{f_+(q^2)}$
\( f_+(0) \)

- Combine the 3 ratios.
- \( q^2 \) conversion is dominant.

- Analysis with one-loop \( \chi \)PT plus an analytic term \((m_K^2 - m_\pi^2)^2\).

\[
f^+(0) = 1 + \frac{3}{2} H_{K} (0) + \frac{3}{2} H_{\eta} (0),
\]

\[
H_{PQ} (0) = -\frac{1}{128 \pi^2 F^2} (m_p^2 + m_Q^2) h_0 \left( \frac{m_p^2}{m_Q^2} \right),
\]

\[
h_0 (x) = 1 + \frac{2x}{1 - x^2} \ln x.
\]

**JLQCD (2005):** \( f_+(0) = 0.954(9) \)

- Note: there are several newer calculations…
Other recent results

- Recent results compiled by Juettner at Lattice 2007.
- Now, several groups are interested in this quantity.
- Results including 2+1-flavors of dynamical quarks.
- Light enough sea quarks.
- Results compatible with the original estimate by Leutwyler-Roos (1984).
Points to be checked

- Presenting just a final number is not good enough. Form factor shapes (charge radii) contain lots of info.
- Chiral extrapolation: consistency with $\chi$PT.
- Analyticity: consistency with the known $K^*$ pole.
- Consistency with the experimental measurements for both $f_+$ and $\xi$.

Ex). Charge radius:

$$f_{K\pi}^+(t) = f_{K\pi}^+(0) \left[ 1 + \frac{1}{6} \left< r^2 \right>_{K\pi} t + \cdots \right],$$

$$f_{K\pi}^0(t) = f_{K\pi}^0(0) \left[ 1 + \frac{1}{6} \left< r^2 \right>_{S} t + \cdots \right].$$

Not satisfactory, so far.
IV. CKM Phenomenology: at tree level
3. $V_{cb}$
Heavy-to-heavy

- Second well-known parameter: A
  - Inclusive:
    - do not specify the final state (except that it contains a charm).
    - Heavy decays can be well controlled by perturbation theory.
  - Exclusive:
    - treats a definite final state (e.g. D, or D*).
    - Heavy quark symmetry constrains the form factors.

\[ V_{\text{CKM}} = \begin{pmatrix}
1 - \lambda^2 / 2 & \lambda & A\lambda^3 (\rho - i\eta) \\
-\lambda & 1 - \lambda^2 / 2 & A\lambda^2 \\
A\lambda^3 (1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix} \]
Inclusive decay

- **Perturbation theory**
  - Valid when energy scale is large. In this case, provided by the mass difference \( m_b - m_c \approx 3 \) GeV.
  - Valid when smeared over final state. Thus, consider inclusive. In this case, the sum is over \( D, D^*, D\pi, D^*\pi, \) etc.

- **Decay rate**
  - At the quark level,
    \[
    \Gamma_{sl}(b \to c) = \frac{G_F^2 m_b^5(\mu)}{192\pi^3} |V_{cb}|^2 \left(1 + A_{ew}\right) A^{pert} \left(\frac{m_c^2}{m_b^2}, \mu\right)
    \]
  - Contains \( m_b^5 \): precise knowledge of \( m_b \) is crucial, or to be fitted with exp data.
Heavy quark expansion

- Initial state is a B meson, not a b quark.
- Correction can be calculated by the Operator Product Expansion (OPE); in this case, called the Heavy Quark Expansion.

\[
\Gamma_{sl}(b \rightarrow c) = \frac{G_F^2 m_b^5(\mu)}{192\pi^3} |V_{cb}|^2 \left( 1 + A_{\text{ew}} \right) A_{\text{pert}}(r, \mu) \\
\times \left[ z_0(r) + z_2(r) \left( \frac{\mu_2^2}{m_b^2}, \frac{\mu_G^2}{m_b^2} \right) + z_3(r) \left( \frac{\rho_D^3}{m_b^3}, \frac{\rho_{LS}^3}{m_b^3} \right) + \ldots \right]
\]

- B meson matrix elements represent the bound state effects.

\[
\mu_\pi^2 = - \langle B | \bar{b} (iD_\perp)^2 b | B \rangle,
\mu_G^2 = \langle B | \bar{b} (iD_\perp^\mu)(iD_\perp^\nu)\sigma_{\mu\nu} b | B \rangle,
\]

- Can be fitted with exp data.
Several moments

Theoretical calculation also possible for differential decay rate. To avoid duality errors, one must use moments, instead.

Several moments $\langle M_X^n \rangle$ and $\langle E_l^n \rangle$ compared with exp data.

May also combine with the photon energy spectrum in $B \rightarrow X_s \gamma$, which is governed by the same matrix elements.
$V_{cb}$ inclusive

- $|V_{cb}|$ obtained to 1-2%.
  - $|V_{cb}| = 0.0417(7)$ (PDG 2006)
  - Other parameters, such as $m_b$, $m_c$, $\mu_\pi^2$, etc., can be obtained at the same time. Very strong method!
- Duality issue?
  - $B \rightarrow X_c \nu$ is dominated by $D$ and $D^*$ (80%).
Exclusive decays

- Use the exclusive decays $B \to D^{(*)} l \nu$ to determine $|V_{cb}|$.
  - Analogous to the $|V_{us}|$ determination through $K \to \pi l \nu$.
  - Need a precise calculation of the form factors = non-perturbative physics.

- Very different systematic effect from the inclusive decays, thus a good cross-check.
- Heavy quark symmetry plays an important role, like the chiral symmetry (or flavor SU(3)) in $K \to \pi l \nu$. 
Heavy quark symmetry

- Heavy quarks look similar...
  - In the heavy-light meson (or heavy-light-light baryon), the heavy quark hardly moves, looks as if a static color source.
  - Therefore, no difference between b and c in the heavy quark limit ($m_Q \rightarrow \infty$).

- Also, there is no difference between spin-up and spin-down heavy quarks, because the spin-(chromo-)magnetic interaction is at $\mathcal{O}(1/m_Q)$.

$$H_{sm} = \psi^\dagger \frac{\sigma \cdot B}{2m_Q} \psi$$
Heavy quark symmetry

- Symmetry relations among form factors
  - Interchange of $b \leftrightarrow c$
  - Interchange of $\uparrow \leftrightarrow \downarrow$

ex) Isgur-Wise function

- $B \rightarrow D$ and $B \rightarrow D^*$ are governed by the same form factor $\xi(w)$, called the Isgur-Wise function.

\[
\left\langle D(v') \left| \bar{c}_v \gamma_\mu b_v \right| B(v) \right\rangle = \xi(w) \left( v_\mu + v'_\mu \right),
\]

\[
\left\langle D^*(v', \epsilon) \left| \bar{c}_v \gamma_\mu \gamma_5 b_v \right| B(v) \right\rangle = -i \xi(w) \left[ (1 + w) \epsilon^* \cdot (\epsilon^* \cdot v) v'_\mu \right]
\]

- A function of $w = v \cdot v'$, see below.
Scale separation

- Write the momentum of heavy quark as
  \[ p = m_Q v + k \]
  - \( v \): four-velocity of the heavy quark.
  - \( k \): residual momentum

- Heavy quark mass limit:
  - propagator
    \[
    i \frac{p + m_Q}{p^2 - m_Q^2 + i\varepsilon} = i \frac{m_Q v + m_Q + k}{2m_Q v \cdot k + k^2 + i\varepsilon} \rightarrow i \frac{1 + v}{2} \frac{1}{v \cdot k + i\varepsilon}
    \]

- Lagrangian
  \[
  L_Q = \overline{Q}_v (iv \cdot D) Q_v; \quad Q(x) = e^{-im_Q v \cdot x} Q_v(x)
  \]
  Georgi (1990), Eichten-Hill (1990)

- States are distinguished by the heavy quark velocity.
Heavy meson states

- Usual normalization
  \[ \langle H(p')|H(p)\rangle = 2E_p(2\pi)^3\delta^3(p-p') \]
  \[ |H(p)\rangle = \sqrt{m_H}[H(v) + O(1/m_Q)] \]

- Decay constant
  \[ \langle 0|\bar{q}\gamma_\mu\gamma_5Q(0)|P(p)\rangle = if_p\gamma_\mu \]

- HQET normalization
  \[ \langle H(v',k')|H(v,k)\rangle = 2v^0\delta_{v,v'}(2\pi)^3\delta^3(k-k') \]
  \[ \langle 0|\bar{q}\gamma_\mu\gamma_5Q_v|P(v)\rangle = i(f_p\sqrt{m_p})\gamma_\mu \]

- Heavy quark scaling
  \[ f_p \sim \frac{1}{\sqrt{m_p}}[1 + O(1/m_p)] \]

- Form factors: \( B \to Dl\nu \) as an example
  \[ \langle D(p')|V_\mu|B(p)\rangle = f_+(q^2)(p+p')_\mu + f_-(q^2)(p-p')_\mu = \sqrt{m_Bm_D}[h_+(w)(v+v')_\mu + h_-(w)(v-v')_\mu] \]
  \[ \xi(w) \]
  \[ 0 \]
  given by a function of \( w = v \cdot v' \)
Isgur-Wise function

- **Form factors:**
  \[
  \langle D(v')|V^\mu|B(v)\rangle = h_+(w)(v+v')^\mu + h_-(w)(v-v')^\mu,
  \]
  \[
  \langle D^*(v',\varepsilon)|V^\mu|B(v)\rangle = h_v(w)\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\nu'}\varepsilon_{\alpha'}\varepsilon_{\beta'}
  \]
  \[
  \langle D^*(v',\varepsilon)|A^\mu|B(v)\rangle = -ih_{A_1}(w)(1+w)\varepsilon_{\mu}^\ast + ih_{A_2}(w)(\varepsilon\cdot v')v^\mu + ih_{A_3}(w)(\varepsilon\cdot v)v'^\mu
  \]

- **Heavy quark limit:** \(m_b, m_c \to \infty\)
  \[
  h_+(w) = h_v(w) = h_{A_1}(w) = h_{A_3}(w) = \xi(w) \quad \text{Isgur-Wise function}
  \]
  \[
  h_-(w) = h_{A_2}(w) = 0
  \]

- **Zero recoil limit**
  - \(\xi(w=1)=1\) because of the vector current conservation (number of heavy quark).
  - A strong constraint, like the \(f_+(0)=1\) of pion/kaon form factor.
1/$m_Q$ corrections

- **Luke's theorem**
  - The leading correction of $O(1/m_Q)$ vanishes in the zero recoil limit $w=1$.
    
    $$h_x(1) = \eta_v \left[ 1 - l_p \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 + O(1/m_Q^3) \right],$$
    
    $$h_{A1}(1) = \eta_A \left[ 1 - \frac{l_v}{(2m_c)^2} + \frac{2l_A}{(2m_c)(2m_b)} - \frac{l_p}{(2m_b)^2} + O(1/m_Q^3) \right]$$
  
  - An analog of the Ademollo-Gatto theorem; comes from the symmetry $\langle D|\leftrightarrow|B\rangle$.
  
  - Extraction of $|V_{cb}|$ is most precise in the zero recoil limit.
Lattice calculation: IW function shape

- Lattice calculation is possible with a similar method as used for $K \to \pi$ form factor
  - Except that the heavy quark is heavy: treated by HQET on the lattice (for example).
    - Putting velocity is non-trivial.
  - Because of a S/N issue, NRQCD is better (or the conventional lattice formulation).
    - Sometimes called Moving NRQCD.

For detailed discussion of heavy quark formulations, see Kronfeld’s lecture.
Lattice calculation: zero recoil limit

- In the zero recoil limit, lattice can calculate the $O(1/m_Q^2)$ (or higher) deviation from the heavy quark limit.

- Clever ratios (again!)

  \[
  R_+ = \frac{\langle D|\bar{c}\gamma_4 b|\bar{B}\rangle \langle \bar{B}|\bar{b}\gamma_4 c|D\rangle}{\langle D|\bar{c}\gamma_4 c|D\rangle \langle \bar{B}|\bar{b}\gamma_4 b|\bar{B}\rangle} = |h_+(1)|^2 = \eta_v \left[ 1 - l_p \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 + O(1/m_Q^2) \right]
  \]

  \[
  R_\tau = \frac{\langle D^*|\bar{c}\gamma_4 b|\bar{B}^*\rangle \langle \bar{B}^*|\bar{b}\gamma_4 c|D^*\rangle}{\langle D^*|\bar{c}\gamma_4 c|D^*\rangle \langle \bar{B}^*|\bar{b}\gamma_4 b|\bar{B}^*\rangle} = |h_\tau(1)|^2 = \eta_v \left[ 1 - l_v \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 + O(1/m_Q^2) \right]
  \]

  \[
  R_{A1} = \frac{\langle D^*|\bar{c}\gamma_4 b|\bar{B}^*|\bar{b}\gamma_4 c|D^*\rangle \langle \bar{B}^*|\bar{b}\gamma_4 b|\bar{B}^*\rangle}{\langle D^*|\bar{c}\gamma_4 c|D^*\rangle \langle \bar{B}^*|\bar{b}\gamma_4 b|\bar{B}^*\rangle} = |h_{A1}(1)|^2 = \eta_A \left[ 1 - l_A \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 + O(1/m_Q^2) \right]
  \]

- These determine the expansion coefficients $l_p$, $l_v$, $l_A$ to reconstruct $h_{A1}(1)$.

  \[
  h_{A1}(1) = \eta_A \left[ 1 - \frac{l_v}{(2m_c)^2} + \frac{2l_A}{(2m_c)(2m_b)} - \frac{l_p}{(2m_b)^2} + O(1/m_Q^2) \right]
  \]
More recent work

Laiho et al. (at Lattice 2007)

- Including dynamical fermions
  - Asqtad improved staggered (2+1 flavors)
- With a single ratio
  \[
  \frac{\langle D^* | \bar{c} \gamma_j \gamma_5 b | B \rangle \langle B | \bar{b} \gamma_j \gamma_5 c | D^* \rangle}{\langle D^* | \bar{c} \gamma_4 c | D^* \rangle \langle B | \bar{b} \gamma_4 b | B \rangle} = |h_{A_1}(1)|^2
  \]
- Not equal to one in the heavy quark limit, but the error is still controllable.
- Three lattice spacings
- Result
  \[ h_{A_1}(1) = 0.924(11)(19). \]
  - Error is competitive with inclusive.
“Exclusive” summary

- **Experiment**
  - $F(1)|V_{cb}|$ is now measured to 2%.
  - Slope of the form factor has not been well measured, but now converging.

- **Theory**
  - Most recent lattice calculation has got 2%.
  - Theoretical calculation of $F(1)$ can become better than 1%?

- **Combined**
  - $|V_{cb}| = 0.0402(7)(8)$, compared to $0.0417(7)$ from inclusive.
IV. CKM Phenomenology: at tree level
4. $V_{ub}$
Less known parameter:
$(\rho^2 + \eta^2)^{1/2}$

- Inclusive:
  - do not specify the final state (except that it contains a charm).
  - Heavy decays can be well controlled by perturbation theory.

- Exclusive:
  - treats a definite final state (e.g. $\pi$, $\rho$, $\omega$, ...).
  - Heavy quark symmetry does not help a lot...

\[
V_{\text{CKM}} = \begin{pmatrix}
1 - \lambda^2 / 2 & \lambda & A\lambda^3 (\rho - i\eta) \\
-\lambda & 1 - \lambda^2 / 2 & A\lambda^2 \\
A\lambda^3 (1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix}
\]
Inclusive $b \to u$

- **Perturbation theory**
  - Valid when energy scale is large. In this case, provided by the mass difference $m_b \sim 5$ GeV; better than $b \to c$
  - Valid when smeared over final state. In this case, the sum is over many final states, thus much safer.

- **Experimentally harder**
  - Must distinguish $b \to u$ from $b \to c$ background, which is 100x larger.
  - Needs cut to enhance the signal.
B→X_uν kinematics

- 3-body decay characterized by:
  - $E_l$: charged lepton energy
  - $q^2$: $l\nu$ invariant mass
  - $m_X$: hadron invariant mass

- Several cuts to enhance $b\rightarrow u$
  - $E_l$ cut
  - $m_X$ cut
  - $q^2$ cut

- Light-cone parameter $P_+ = E_X - |P_X|$
Shape function

- Non-perturbative physics enters as the shape function
- An analog of the B meson matrix element in HQE
- In this case, the distribution in the light-cone variable

\[ f(k_+) = \frac{1}{2m_B} \langle \bar{B} | \bar{b}_v \delta(i n \cdot D + k_+) b_v | B \rangle \]

also observable from \( B \rightarrow X_s \gamma \).
- Possible to calculate on the lattice??
Now, $|V_{ub}|$ is reasonably precise $\sim 8\%$.

$|V_{ub}| = 0.0440(20)(27)$ (PDG 2006)

Sets the challenge for lattice QCD.
Exclusive decays

- Use the exclusive decays $B \rightarrow \pi l \nu$, $\rho l \nu$, $\omega l \nu$, etc. to determine $|V_{ub}|$.

- Need a precise calculation of the form factors = non-perturbative physics.

- Very different systematic effect from the inclusive decays, thus a good cross-check.

- Heavy quark symmetry does not help a lot. Only the heavy quark scaling is useful.
Kinematics

- \( B \rightarrow \pi l \nu \)
  - \( q^2: l \nu \) invariant mass; \( 0 \leq q^2 \leq (m_B - m_\pi)^2 \)
  - small \( q^2 \) ⇒ large recoil of \( \pi \)
  - large \( q^2 \) ⇒ small recoil of \( \pi \)

- Differential decay rate
  \[
  \frac{d\Gamma}{dq^2} = \frac{G_F^2 |V_{ub}|^2}{24\pi^3} |p_\pi|^3 |f_+(q^2)|^2
  \]
  - Depends only on \( f_+ \); \( f_0 \) term is suppressed by small \( m_l \).

- Lattice calculation
  - Possible only when both \( B \) and \( \pi \) have small spatial momenta.
  \( \Rightarrow \) large \( q^2 \) region
Lattice calculation

- **Calculation of 3pt function**
  - Use the sequential source method with momentum insertion.
  - The clever ratios not so much useful: numerator and denominator are not similar.

- **Most difficult among other semi-leptonic decays. Several checks to be done**
  - Operator matching
  - Heavy quark scaling
  - Chiral extrapolation
  - Dispersion relation
Lattice operators have to be matched to the continuum operator.

- Usually done using perturbation theory at one-loop. Neglected higher orders could be sizable.
- For light-light currents the non-perturbative matching is available in many cases.
- In the double ratios (K→π, B→D) the matching factors largely cancel.
- Cancellation is less precise for heavy-to-light, thus larger systematic error.
Heavy quark scaling

- HQET normalization

\[
\langle \pi(k_\pi)|\bar{q}\gamma^\mu b|B(v)\rangle = 2 \left[ f_1(v \cdot k_\pi)v^\mu + f_2(v \cdot k_\pi)\frac{k_\pi^\mu}{v \cdot k_\pi} \right]
\]

- Related to the conventional form factors

\[
f^+(q^2) = \sqrt{m_B} \left\{ \frac{f_2(v \cdot k_\pi)}{v \cdot k_\pi} + \frac{f_1(v \cdot k_\pi)}{m_B} \right\},
\]

\[
f^0(q^2) = \frac{2}{\sqrt{m_B}} \frac{m_B^2}{m_B^2 - m_\pi^2} \left\{ [f_1(v \cdot k_\pi) + f_2(v \cdot k_\pi)] - \frac{v \cdot k_\pi}{m_B} \left[ f_1(v \cdot k_\pi) + \frac{m_\pi^2}{(v \cdot k_\pi)^2} f_2(v \cdot k_\pi) \right] \right\}
\]

- HQET scaling is manifest

\[
f^+(q^2) \sim \sqrt{m_B},
\]

\[
f^0(q^2) \sim \frac{1}{\sqrt{m_B}},
\]

Possible to check the consistency among different heavy quark formulations.
Chiral extrapolation

- **Soft pion theorem**
  \[ f_0(q_{max}^2) = \frac{f_B}{f_\pi} \]
  - Valid in the chiral limit.
  - Chiral extrapolation is not trivial because \( q^2 \) changes as \( m_q \) changes.

- **ChPT predicts the chiral log**
  - Calculation exists (Becirevic-Prelovsek-Zupan, 2002), not fully tested so far.
Dispersion relation

- Near $q^2_{\text{max}}$ the $B^*$ pole dominates the dispersion relation.

\[ F(q^2) = \frac{1}{2\pi i} \int dt \frac{F(t)}{t - q^2} = \frac{1}{\pi} \int_{t_0}^\infty dt \frac{\text{Im} F(t)}{t - q^2} \]

- Using the $B^*B\pi$ coupling,

\[ \lim_{q^2 \to m^2_B} f^+(q^2) = \frac{f_{B^*}}{f_\pi} \frac{g}{1 - q^2/m^2_{B^*}} \]

or

\[ \lim_{v \cdot k_\pi \to 0} f_2(v \cdot k_\pi) = \frac{f_{B^*} \sqrt{m_{B^*}}}{2f_\pi} \frac{v \cdot k_\pi}{v \cdot k_\pi + \Delta_B} \]

which implies constant $f_2$. 

JLQCD (2001)
Most recent results

- Lattice data available only in the large q² region; take exp data only in that region to extract |V_{ub}|.
  - |V_{ub}| = 0.00384(+67-49), to be compared with 0.00440(20)(27) from inclusive.
  - Error is x(2-3) larger, mostly theoretical.

HPQCD 2006
Semi-leptonic decays...

- Complicated!
  - But good, because there are many different ways to check lattice calculations.
  - All come from symmetries (chiral, heavy quark) and analyticity.
  - Lattice calculation must pass these stringent *theoretical* tests in order to make reliable predictions.
  - Heavy-to-light is the greatest challenge. Need <5% accuracy to be competitive.