Fundamental constants and electroweak phenomenology from the lattice

Lecture III: Chiral dynamics and light quark masses
III. Chiral dynamics and light quark masses

1. Chiral symmetry breaking and quark masses
   - GMOR relation
   - Chiral perturbation theory
   - Quark mass ratios

2. Lattice calculation of light quark masses
   - Basic strategy
   - Perturbative and non-perturbative matchings

3. Pion loop effects
   - Chiral log effects on chiral extrapolation
   - Pion form factor and general strategy
III. Chiral dynamics
1. Chiral symmetry breaking and quark masses
Chiral symmetry breaking

- In the QCD vacuum, chiral symmetry is broken.

- Flavor SU(3)_L × SU(3)_R → SU(3)_V

- Non-zero chiral condensate $\langle \bar{q}q \rangle$

- Nambu-Goldstone bosons (pion, kaon, $\eta$) nearly massless; in practice massive due to non-zero $m_q$.
  - Flavor-singlet axial U(1) is special, due to anomaly. $\eta'$ is substantially heavier.

- Other hadrons have a mass of $O(\Lambda_{QCD})$

- Low energy effective theory for pions (and K, $\eta$) can be constructed = chiral perturbation theory (ChPT, $\chi$PT).
PCAC relation

- Partially Conserved Axial Current (PCAC)
  - From the QCD Lagrangian,
    \[ A_\mu = \bar{u} \gamma_\mu \gamma_5 d, \]
    \[ \partial_\mu A^\mu = (m_u + m_d) \bar{u} \gamma_5 d \]
  - The axial current may annihilate pion to the vacuum; Lorentz invariance restricts its form.
    \[ \langle 0 | A_\mu (0) | \pi(p) \rangle = i f_\pi p_\mu, \]
    \[ \langle 0 | \partial_\mu A^\mu (0) | \pi(p) \rangle = f_\pi m_\pi^2; \]
    \[ \partial_\mu A^\mu (x) = f_\pi m_\pi^2 \phi_\pi (x) \quad \phi_\pi (x): \text{operator to create a pion}. \]
  - \( f_\pi \) is called the pion decay constant.
  - Can be measured from the leptonic decay \( \pi \to \mu \nu \).
    \[ f_\pi = 131 \text{ MeV} \]
  - Its analog for kaon is \( f_K \).
    \[ f_K = 160 \text{ MeV} \]
Consider two-point functions

\[ \Pi_{5}^{\mu\nu}(q) = i \int d^{4}x e^{iqx} \left\langle 0 \left| T \left( A^{\mu}(x) A^{\nu}(0)^{\dagger} \right) \right| 0 \right\rangle, \]

\[ \Psi_{5}(q) = i \int d^{4}x e^{iqx} \left\langle 0 \left| T \left( \partial_{\mu} A^{\mu}(x) \partial_{\nu} A^{\nu}(0)^{\dagger} \right) \right| 0 \right\rangle \]

Taking derivatives of T-products, we obtain

\[ q_{\mu} q_{\nu} \Pi_{5}^{\mu\nu}(q) = \Psi_{5}(q) - q^{\nu} \int d^{4}x e^{iqx} \delta(x^{0}) \left\langle 0 \left| \left[ A^{0}(x), A^{\nu}(0)^{\dagger} \right] \right| 0 \right\rangle \]

\[ + i \int d^{4}x e^{iqx} \delta(x^{0}) \left\langle 0 \left| \left[ \partial_{\mu} A^{\mu}(x), A^{0}(0)^{\dagger} \right] \right| 0 \right\rangle \]

In the limit of \( q^{\mu} \rightarrow 0 \), it leads to

\[ (m_{u} + m_{d}) \langle \bar{u}u + \bar{d}d \rangle = -if_{\pi}^{2} m_{\pi}^{4} \left\{ \frac{-i}{m_{\pi}^{2} - q^{2}} \right\}_{q \rightarrow 0} + \text{other resonances} \]

\[ = -f_{\pi}^{2} m_{\pi}^{2} \left\{ 1 + O(m_{\pi}^{2}) \right\} \]

Spontaneous symmetry breaking \( Q_{5} \mid 0 \rangle \neq 0 \)
Gell-Mann-Oakes-Renner (GMOR) relation (1968)

\[(m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle = -f_\pi^2 m_\pi^2 \left\{1 + O(m_\pi^2)\right\}\]

- Chiral symmetry is broken = Non-zero chiral condensate \( \langle \bar{q}q \rangle \)
- Pion mass squared proportional to quark mass

\[m_\pi^2 = B_0 (m_u + m_d) + O(m_q^2)\]

\[= \frac{-2\langle \bar{q}q \rangle}{f_\pi^2} (m_u + m_d) + O(m_q^2)\]

- Also for kaons,

\[m_{K^+}^2 = B_0 (m_u + m_s) + O(m_q^2), \quad m_{K^0}^2 = B_0 (m_d + m_s) + O(m_q^2), \]

\[m_\eta^2 = \frac{1}{3} B_0 (m_u + m_d + 4m_s) + O(m_q^2), \]

- Quark mass ratios can be predicted up to \( O(m_q^2) \).
Chiral Lagrangian

- Low energy effective lagrangian is developed assuming
  - Spontaneous breaking of chiral symmetry
  - Pion (and kaon, eta) to be the Nambu-Goldston boson

- In the low energy regime, pions are only relevant dynamical degrees of freedom.

\[ L_2 = \frac{f^2}{4} \text{Tr} \left( D_\mu UD^\mu U^\dagger \right) + \frac{f^2}{4} \text{Tr} \left( \chi U^\dagger + U \chi^\dagger \right) , \]

\[ U = \exp \left( \frac{i\tau^a \pi^a}{f} \right) , \chi = 2B_0m \]

- Given by a non-linear sigma model.
- Provides a systematic expansion in terms of $m_\pi^2$, $p^2$; the leading order is given above.

For full details, see Bernard’s lectures
Expansion in the pion field gives

\[
L_2 = \frac{1}{2} \partial_\mu \pi^a \partial^\mu \pi^a - \frac{m_\pi^2}{2} \pi^a \pi^a + \frac{m_\pi^2}{24 f^2} (\pi^a \pi^a)^2 \\
+ \frac{1}{6 f^2} \left[ (\pi^a \partial_\mu \pi^a)(\pi^b \partial^\mu \pi^b) - (\pi^a \pi^a)(\partial_\mu \pi^b \partial^\mu \pi^b) \right] + \ldots
\]

- Pion mass is obtained as \( m_\pi^2 = 2B_0 m \)
- A chain of interaction terms: 4\( \pi \), 6\( \pi \), etc.

Loop corrections are calculable.

- Pick up a factor of \( (m_\pi/4\pi f)^2 \) or \( (p/4\pi f)^2 \)
- Counter terms must also be added at order \( (m_\pi/4\pi f)^2 \) or \( (p/4\pi f)^2 \)
  - introduce the low energy constants (LECs): \( L_1 \sim L_{10} \) at the one-loop level

For full details, see Bernard’s lectures
One-loop example

- Pion self-energy

\[
\frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} = \frac{1}{(4\pi)^2} \left[ \Lambda^2 + m^2 \ln \frac{m^2}{\Lambda^2 + m^2} \right]
\]

Cutoff regularization

\[
= \frac{m^2}{(4\pi)^2} \left( \frac{2}{\varepsilon} + \gamma - \ln 4\pi + \ln \frac{m^2}{\mu^2} - 1 \right)
\]

Dimensional reg

- Log dependence \( m^2 \ln (m^2) \): called the chiral logarithm.
- Comes from the infrared end of the integral = long distance effect of (nearly massless) pion loop.
- Counter terms are necessary in order to renormalize the UV divergence.
- After subtracting the UV divergences

\[
m^2_\pi = 2B_0 m_q \left[ 1 + \frac{1}{2} \frac{m^2_\pi}{(4\pi f')^2} \ln \frac{m^2_\pi}{\mu^2} + (\text{const}) \times \frac{m^2_\pi}{(4\pi f')^2} + O(m^4_\pi) \right]
\]
Counter terms

- At the order \((m_\pi / 4\pi f)^2\) or \((p/4\pi f)^2\), there are 10 possible counter terms
  - 10 new parameters, \(L_1 \sim L_{10}\) = low energy constant at NLO
    c.f. 2 parameters at LO: \(\Sigma\) and \(f\).
  - Depends on how one renormalize the UV divergence, just as in the small coupling perturbation. \(L_1 \sim L_{10}\) depends on the renormalization scale \(\mu\).
  - Once these parameters are determined (e.g. from pion scattering data), one can predict other quantities.

- Lattice QCD may be used to calculate these parameters.
Quark mass ratio

- At NLO, the quark mass ratio is given as

\[
\frac{m_K^2}{m_\pi^2} = \frac{m_s + m_{ud}}{2m_{ud}} \left[ 1 + \frac{1}{2} \frac{m_\pi^2}{(4\pi f)^2} \ln \frac{m_\pi^2}{\mu^2} - \frac{1}{2} \frac{m_\eta^2}{(4\pi f)^2} \ln \frac{m_\eta^2}{\mu^2} - \frac{8(m_K^2 - m_\pi^2)}{f^2} (2L_8 - L_5) \right]
\]

- Assumes that the isospin breaking \( m_u \neq m_d \) is negligible.
- Requires the knowledge of the NLO LEC \( 2L_8 - L_5 \).
- Results in \( m_s/m_{ud} = 25 \sim 30 \) (PDG 2006); large uncertainty due to the unknown LEC.

- Comparison with the exp number gives LECs. But the predictive power is lost.
- Instead, lattice calculation can be used to fix LECs.
Isospin breaking

- In the real world, $\pi^\pm$ and $\pi^0$ have different masses; two sources
  - Small mass difference between up and down quarks.
  - Electromagnetic effect: $Q_u = +2/3$, $Q_d = -1/3$.

- Quark mass difference
  - When $m_u \neq m_d$, $\pi^0$ and $\eta$ can mix
    $$|\pi^0\rangle = |\pi^0\rangle_0 + \frac{\sqrt{3}}{4} \frac{m_d - m_u}{m_s - \hat{m}} |\eta\rangle_0 + O((m_d - m_u)^2)$$
    $$|\pi^0\rangle_0 = \frac{1}{\sqrt{2}} (u\bar{u} - d\bar{d})$$
    $$|\eta\rangle = |\eta\rangle_0 - \frac{\sqrt{3}}{4} \frac{m_d - m_u}{m_s - \hat{m}} |\pi^0\rangle_0 + O((m_d - m_u)^2)$$
    $$|\eta\rangle_0 = \frac{1}{\sqrt{6}} (u\bar{u} + d\bar{d} - 2s\bar{s})$$
  - Then, from
    $$M_\pi^2 = \langle \pi | \left[ m_u u\bar{u} + m_d d\bar{d} + m_s s\bar{s} \right] |\pi\rangle$$
    $$M_\pi^2 = B_0 (m_u + m_d) - \frac{B_0}{4} \frac{(m_d - m_u)^2}{(m_s - \hat{m})}$$
    $$\Delta M_\pi \sim 0.2 \text{ MeV}$$
    (c.f. $(\Delta M_\pi)^{(\text{phys})} = 4.6$ MeV)
Electromagnetic effect

- **Self-energy with a photon propagator**
  - Using the PCAC relation, related to a current two point function (VV-AA).
    \[
    \Pi_{\mu\nu}^J(q) = i \int d^4x \ e^{iqx} \left\langle 0 \left| T J^\mu(x) J^\nu(0) \right| 0 \right\rangle
    \]
  - Das-Guralnick-Low-Mathur-Young sum rule (1967), a close relative of the Weinberg sum rules (1967)
    \[
    \Delta M^2_\pi = \frac{3\alpha_{em}}{4\pi f^2} \int_0^\infty dQ^2 Q^2 \left[ \Pi_{VV}(Q^2) - \Pi_{AA}(Q^2) \right]
    \]
  - Sum rule estimate gives \( \Delta M_\pi \sim 5 \) MeV, comparative to the exp value 4.59 MeV. Lattice calculation is also possible.
  - At the leading order, the same effect for kaon (Dashen’s theorem)
    \[
    M^2_{K^+} - M^2_{K^0} = M^2_{\pi^+} - M^2_{\pi^0}
    \]
At the leading order, Weinberg (1977)

Using the GMOR relation and the EM correction,

\[
M_{\pi^\pm}^2 = B_0(m_u + m_d) + \Delta_{em}
\]

\[
M_{\pi^0}^2 = B_0(m_u + m_d)
\]

\[
M_{K^\pm}^2 = B_0(m_u + m_s) + \Delta_{em}
\]

\[
M_{K^0}^2 = B_0(m_d + m_s)
\]

Combine them to obtain

\[
\frac{m_u}{m_d} = \frac{M_{\pi^0}^2 + (M_{K^\pm}^2 - M_{K^0}^2) - (M_{\pi^\pm}^2 - M_{\pi^0}^2)}{M_{\pi^\pm}^2 - (M_{K^\pm}^2 - M_{K^0}^2)} = 0.55,
\]

\[
\frac{m_s}{m_d} = \frac{(M_{K^\pm}^2 + M_{K^0}^2) - M_{\pi^\pm}^2}{M_{\pi^\pm}^2 - (M_{K^\pm}^2 - M_{K^0}^2)} = 20.1
\]
Further estimate of $m_u/m_d$

- At NLO, those simple relations are lost.
  - NLO formula (Gasser-Leutwyler (1985))
    \[
    \frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{m_d + m_u} \left\{ 1 + \Delta_M + O(m^2) \right\}
    \]
    \[
    \frac{M_{K^0}^2 - M_{K^\pm}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}} \left\{ 1 + \Delta_M + O(m^2) \right\}
    \]
  - A double ratio is free from the NLO correction
    \[
    Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_\pi^2} \frac{M_{K^0}^2 - M_{K^\pm}^2}{M_K^2 - M_{K^\pm}^2} \left\{ 1 + O(m^2) \right\}
    \]
    which can be written in a simple form
    \[
    \left( \frac{m_u}{m_d} \right)^2 + \frac{1}{Q^2} \left( \frac{m_s}{m_d} \right)^2 = 1
    \]
  - A slight ambiguity comes from a violation of the Dashen’s theorem.

\[\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{f^2} (2L_8 - L_5) + \chi \text{logs} \]

\[(K^+ - K^0)_{\text{e.m.}} \] from Leutwyler, PLB378 (1996) 313.
NLO constraints

- The NLO formula makes an ellipse.
  \[
  \left( \frac{m_u}{m_d} \right)^2 + \frac{1}{Q^2} \left( \frac{m_s}{m_d} \right)^2 = 1
  \]

- Other constraints:
  - \( \Delta_M > 0 \), from large \( N_c \).
  - Another constraint on
    \[
    R = \frac{m_s - \hat{m}}{m_d - m_u}
    \]
    from a charmonium decay
    \[
    \frac{\Gamma(\psi' \to \psi\pi^0)}{\Gamma(\psi' \to \psi\eta^0)}
    \]
III. Chiral dynamics
2. Lattice calculation of light quark masses
Inputs

In general, lattice QCD simulation requires inputs for

- Lattice scale $1/a \Rightarrow$ determines $\alpha_s(1/a)$
  - Inputs discussed in Part I.
- Quark masses for each flavor
  - up and down quarks $m_{ud}$ (often assumed to be degenerate)
    - from pseudo-scalar meson mass $m_\pi$, good sensitivity because $m_\pi^2 \sim m_{ud}$.
  - Strange quark $m_s$
    - from $m_K$, for the same reason.
  - Charm quark $m_c$
    - either from D (heavy-light) or $J/\psi$ (heavy-heavy) mass
  - Bottom quark $m_b$
    - either from B (heavy-light) or $\Upsilon$ (heavy-heavy) mass
**Chiral extrapolation**

- Lattice simulation is harder for lighter sea quarks.
  - Computational cost grows as $m_q^{-n}$ (n~2).
  - Finite volume effect becomes more important $\sim \exp(-m_\pi L)$

- Practical calculation involves the *chiral extrapolation*. At the leading order, it is very simple:
  1. Fit the pseudo-scalar mass with $m_\pi^2 = B_0 (m_u + m_d) + O(m_q^2)$
  2. Input the physical pion mass $m_{\pi0} = 135$ MeV to get $m_{ud} = (m_u + m_d)/2$. (Forget about the isospin breaking for the moment.)
  3. Renormalize it to the continuum scheme.

- Including higher orders is non-trivial…
Chiral expansion

\[ m^2 = 2B_0 m_q \left[ 1 + \frac{1}{2} \frac{m^2}{(4\pi f^2)} \ln \frac{m^2}{\mu^2} + c_3 \frac{m^2}{(4\pi f^2)^2} + \text{NNLO} \right] \]

- LO (linearity) looks very good, but if you look more carefully NLO is visible.
- \( m^2_{\pi}/m_q \) not constant.
- Chiral log term has a definite coefficient = curvature fixed.
- Analytic term has an unknown constant, to be fitted with lattice data = linear slope.
Strange quark

- Must consider 2+1-flavor theory.
  - If your simulation contains only 2-flavors (up and down quarks), then a possible choice is to use the Partially Quenched ChPT. Although it is not the correct theory after all, it will provide a consistent description of the lattice data.

- NLO effect is less pronounced for strange.

\[
M_\pi^2 = 2\hat{m}B_0 \left[ 1 + \mu_\pi - \frac{1}{3} \mu_\eta + 2\hat{m}K_3 + (2\hat{m} + m_s)K_4 \right], \quad \mu_\pi = \frac{1}{2} \frac{m_\pi^2}{(4\pi f)^2} \ln \frac{m_\pi^2}{\mu^2}
\]

\[
M_K^2 = (\hat{m} + m_s)B_0 \left[ 1 + \frac{2}{3} \mu_\eta + (\hat{m} + m_s)K_3 + (2\hat{m} + m_s)K_4 \right], \quad \mu_\eta = \frac{1}{2} \frac{m_\eta^2}{(4\pi f)^2} \ln \frac{m_\eta^2}{\mu^2}
\]

- No singularity in the chiral limit. (This may not be the case for other quantities like \(f_K\).)
- Numerical analysis will be more stable.
A case study: MILC+HPQCD 2+1

MILC+HPQCD, PRD70, 031504(R) (2004),
MILC, PRD70, 114501 (2004).

- Bare quark masses taken from the MILC 2+1 asqtad simulations.
  - Two lattice spacings “coarse” ($a=0.125$ fm) and “fine” ($a=0.090$ fm).
  - Complicated fit including the taste-breaking effects of staggered fermion, vanishing at $a=0$.
  - NNLO analytic terms are included. Non-analytic (chiral log) terms are discarded.
Once the bare quark mass is fixed on the lattice, it must be converted to the continuum definition, because the pole mass is not adequate.

- Just like the conversion of the coupling constant.
- May use the perturbation theory (Use the renormalized coupling!). But, in most cases, known only at the one-loop level. (Exceptions are HQET, Asqtad, stochastic PT(?).)

Ex). O(a)-improved Wilson fermion

\[ m^{MS}(\mu = a^{-1}) = [1 + 2.05 \alpha_s + \ldots] m^{lat}(a^{-1}) \]

- Non-perturbative renormalization is desirable.
Conversion is done perturbatively.

Calculated to two-loop. HPQCD (Mason et al.), PRD73, 114501 (2006).

\[ Z_m(\mu a = 1) = 1 + 0.119 \alpha_V(q^*) + (2.22 - 0.02 n_f) \alpha_V^2 \]

with \( q^* = 1.88/a \).

With \( \alpha_V(q^*) = 0.27 \), this series is

\[ Z_m = 1 + 0.03 + 0.16 + \ldots \]

The uncertainty from higher orders is estimated as \( 2 \alpha_V^3(q^*) \sim 5\% \).

After the continuum extrapolation with \( \alpha_V a^2 \), they quote

\[ m_s^{\overline{MS}} (2 \text{ GeV}) = 87(0)(4)(4)(0) \text{ MeV} \]
Another work by QCDSF-UKQCD with the O(a)-improved Wilson fermion, PRD73, 054508 (2006).

One may doubt about the convergence of the perturbative expansion. Non-perturbative renormalization is desirable if possible.

Non-perturbative renormalization is done using the RI/MOM scheme. It has its own subtlety: the renormalization constant is not really constant, due to $S\chi$SB.

NP results are about 20% larger than the one-loop calculation.

For more details, see the lectures by S. Sint.
Lattice data fit to the one-loop PQ$\chi$PT formula (but the magnitude of the $\chi\log$ is a free parameter).

Conversion to MSbar is (partially) non-perturbative.

Continuum extrapolation is carried out with 4 data points. Substantial rise in the continuum limit.

$$m_s^{\text{MS}}(2\text{ GeV}) = 111(6)(4)(6)\text{ MeV}$$
Present status

  - Non-perturbative renormalization yields higher $m_s$?
    or
  - $N_f=3$ gives lower $m_s$?

- Including other determinations (Plot from Davier, Hocker, Zhang, Rev. Mod. Phys.78,1043 (2006)).
  - Consistent within large errors.
Up and down quarks

- **Ratio to strange**
  - Basically obtained by the ChPT formula.
  - At NLO, equivalent to the calculation of the LEC.

\[
\frac{m_K^2}{m_\pi^2} = \frac{m_s + m_{ud}}{2m_{ud}} \left[ 1 + \frac{1}{2} \left( \frac{m_\pi}{\mu} \right)^2 \ln \frac{m_\pi^2}{\mu^2} - \frac{1}{2} \left( \frac{m_\eta}{\mu} \right)^2 \ln \frac{m_\eta^2}{\mu^2} - \frac{8(m_K^2 - m_\pi^2)}{f^2} (2L_8 - L_5) \right]
\]

- MILC obtained \(m_s/m_{ud}=27.4(1)(4)(1)\) from the \(S\chi PT\) fit. Much more delicate.
- \(m_u/m_d\) can also be obtained (up to the EM uncertainty) from the relation

\[
\left( \frac{m_u}{m_d} \right)^2 + \frac{1}{Q^2} \left( \frac{m_s}{m_d} \right)^2 = 1
\]
III. Chiral dynamics

3. Pion loop effects
Chiral log effects

- We learned that the chiral extrapolation is non-trivial.
  - Especially so, if pions are involved as external states
    - pion mass, decay constant
    - form factors, $\pi\pi$-scattering...
  - Even when pion does not appear as external states, it could be there in the loop. May lead to the chiral log.
    - Kaon decay constant
    - Nucleon (masses, matrix elements)
    - Heavy-light meson (masses, decay constants, form factors)

- Thus, could always be a delicate problem when one aims at good precision.
An example

- Pion decay constant
  - At NNLO, it has the form
    \[
    (f_\pi)_{\text{NNLO}} = f \left[ 1 - 2 \xi \ln \xi + 5(\xi \ln \xi)^2 + \frac{3}{2} \left( \frac{L_{\text{phys}}}{L} + \frac{53}{2} \right) \xi^2 \ln \xi \right] + L_4(\xi - 10\xi^2 \ln \xi) + \alpha_2 \xi^2 + O(\xi^3),
    \]
    with \( \xi = m_\pi^2 / (4\pi f_\pi)^2 \).
  - Leading-log terms have known coefficients.
  - Free parameters in the analytic term (NLO, NNLO) and NLL term (NNLO).
  - Turns out that the chiral log effect is not substantial, but non-negligible.

JLQCD (2007)
dynamical overlap (N_f=2)
(talk by Noaki at lat07)
The simplest form factor
\[ \langle \pi(p')|V_\mu|\pi(p)\rangle = i(p_\mu + p_\mu')F_V(q^2), \quad q_\mu \equiv p_\mu' - p_\mu \]

Momentum transfer \( q_\mu \) by a virtual photon. Space-like \( q^2<0 \) in the \( \pi e \rightarrow \pi e \) process.

Vector form factor \( F_V(q^2) \) normalized as \( F_V(0)=1 \), because the vector current is conserved.

\[ F_V(q^2) = 1 + \frac{1}{6} \left\langle r^2 \right\rangle^\pi \langle p^2 \rangle + O(q^4), \]

Vector (or EM) charge radius \( \left\langle r^2 \right\rangle^\pi \) is defined through the slope at \( q^2=0 \).
Lattice calculation of 3pt function

- $\pi(p) \rightarrow \pi(p')$
  - An interpolating operator for the initial state $\pi(p)$ at $t=t_0$
  - Another interpolating operator for the final state $\pi(p')$ at $t=t_1$
  - Current insertion $V_{\mu}$ in the middle $t$.
  - Spatial momentum inserted at two operators.

*(sequential) source method*

- Calculate a quark propagator starting from a previous quark propagator at $t$.

$$ (\mathcal{D} + m)S_2(x) = e^{i\mathbf{q} \cdot \mathbf{x}} \Gamma S_1(x) \delta(x_0 - t) $$
Working on the Euclidean lattice

- On-shell particle will never appear (except for the massless pion in the chiral limit).
- Instead, one calculates two-point function

\[ C^{(2)}(t_1, t_0) \sim Z e^{-m(t_1-t_0)} \]

- This is a Fourier transform of the two-point function in the space-like regime.

\[ C^{(2)}(t) \sim \int_{-\pi/a}^{\pi/a} dq_0 \frac{dq_0}{2\pi} \Pi(q^2) e^{iq_0 t} = \int_{-\pi/a}^{\pi/a} dq_0 \frac{dq_0}{2\pi} \frac{e^{iq_0 t}}{m^2 + q_0^2 + q^2} \]

- All the information encoded in the space-like two-point function \( \Pi(q^2) \).
Lattice calculation of 3pt function

- At large enough time separations $\Delta t = t - t_0, \Delta t' = t_1 - t$, the ground state pions dominate.

$$C_{4, \text{smr,smr}}(\Delta t, \Delta t'; \mathbf{p}, \mathbf{p}') \rightarrow \frac{\sqrt{Z_{4, \text{smr}}(|\mathbf{p}|)} Z_{\pi, \text{smr}}(|\mathbf{p}'|)}{4E(p)E(p')Z_V} e^{-E(p)\Delta t} e^{-E(p')\Delta t'} \langle \pi(p') | V_4 | \pi(p) \rangle$$

- Extra factors can be taken off with 2pt functions.

$$C_{\phi, \phi'}(\Delta t; \mathbf{p}) = \frac{\sqrt{Z_{\pi, \phi}(|\mathbf{p}|)} Z_{\pi, \phi'}(|\mathbf{p}'|)}{2E(p)} e^{-E(p)\Delta t}, \quad \sqrt{Z_{\pi, \phi}(|\mathbf{p}|)} = \langle \pi(p) | O_{\pi, \phi}(\mathbf{p}) | \rangle$$

$$R(\Delta t, \Delta t'; \mathbf{p}, \mathbf{p}') = \frac{C_{\mu, \text{smr,smr}}^\pi(\Delta t, \Delta t'; \mathbf{p}, \mathbf{p}')}{C_{\text{smr,icl}}^\pi(\Delta t; \mathbf{p}) C_{\text{icl,smr}}^\pi(\Delta t'; \mathbf{p}')}$$

$$F_\pi(q^2) = \frac{2M_\pi}{E(p) + E(p')} \frac{R_{\mu, \phi, \phi'}(\Delta t, \Delta t'; 0, 0)}{R_{4, \phi, \phi'}(\Delta t, \Delta t'; 0, 0)}$$
A recent calculation

- **Lattice signal**
  - Look for a plateau, where the ground state pion dominates.
  - Noisier for larger pion momentum.

- **Note:**
  - The actual data were obtained using the all-to-all technique, so that the data points at different $t_0, t, t_1$ and different momentum combinations can all be averaged.

JLQCD (2007) dynamical overlap (Nf=2) (talk by Kaneko at lat07)
A recent calculation

- Many points corresponds to many momentum combinations \((p, p')\).
  - \((1,0,0) \rightarrow (0,1,0), \ldots\) etc.
  - in units of \(2\pi/L\).
  - Too large \(p\)'s are contaminated by discretization effects \((ap)^2\).

- \(q^2\) dependence well approximated by a vector meson pole

\[
F_{\pi}(q^2) = \frac{1}{1 - q^2 / m_v^2} + c_1 q^2 + \ldots
\]

- with the independently calculated \(m_V\) at the same quark mass.
Vector meson dominance is understood using analyticity.

\[ F(q^2) = \frac{1}{2\pi i} \oint dt \frac{F(t)}{t-q^2} = \frac{1}{\pi} \int_{i0}^{\infty} dt \frac{\text{Im} F(t)}{t-q^2} \]

written in terms of the form factor in the time-like region \( t > 0 \).

\[ \langle \pi(p)\pi(p')|V_\mu|0 \rangle \]

In the heavier quark mass region, \( \rho \) meson is a nearest isolated pole. \( \pi\pi \) is subleading.

For the physical quark mass, \( \pi\pi \) is nearest. \( \rho \) is a part of \( \pi\pi \) (broad resonance).
Charge radius has a chiral log contribution.

\[ \langle r^2 \rangle^\pi_V = \frac{1}{(4\pi f_\pi)^2} \left[ \ln \frac{m_\pi^2}{\mu^2} + 12(4\pi)^2 L_9 + O(m_\pi^2) \right] \]

- Must diverge in the chiral limit: pion cloud gets larger.
- Valid only in the region where \( 2m_\pi < m_\rho \).
- Lattice data actually increases towards the chiral limit. Chiral log further enhance its value.

\[ \langle r^2 \rangle^\pi_V = 0.388(9)(12) \text{ fm}^2 \]
General problem

- Chiral extrapolation is a serious issue in current lattice QCD studies. Questions arise...
  - How closely one must approach the chiral limit?
  - One-loop enough? Two-loop needed?
  - Finite volume effect might become significant.

- How big is the effect of chiral symmetry violation of Wilson, twisted-mass, staggered and domain-wall fermions?
- Modified $\chi$PTs for these lattice actions contain many parameters. Possible to determine all of them to necessary precision?

- Answer depends on the process, action, ...