Examined toy QM model with SUSY.

Showed how naive discretization fails – lattice theory picks up new U.V contributions - fine tuning problem.

Avoid by correcting action with counterterm computed at 1-loop.

Generic way to deal with $D < 4$ theories (and maybe even $\mathcal{N} = 1$ in 4d with eg DWF).

Better. Find modified action which is invariant under linear combination of SUSYs.

New symmetry $Q^2 = 0$ and action Q-exact $S = Q\Lambda$.

Twisted formulation
Two dimensions

How can we generalize previous construction to field theory?

In QM needed 2 SUSYs. Also single scalar field and two real fermions. Try simplest 2D with these features: $\mathcal{N} = 2$ Wess-Zumino model ...

$$
S_{WZ} = \int d^2 x \partial_\mu \phi \partial_\mu \bar{\phi} + W'(\phi) W'(\bar{\phi}) + \bar{\psi} \gamma_\mu \partial_\mu \psi + \\
+ \bar{\psi} \left( \frac{1}{2} (1 + \gamma_5) W''(\phi) + \frac{1}{2} (1 - \gamma_5) W''(\bar{\phi}) \right) \psi
$$
Nicolai Map

Fermion operator can be rewritten (chiral basis):

\[ M_F = \begin{pmatrix} W''(\phi) & \partial_z \\ \partial_z & W''(\bar{\phi}) \end{pmatrix} \]

Associated Nicolai Map is

\[ N = \partial_z \bar{\phi} + W'(\phi) \]

Action \( N \bar{N} \) again differs by cross term from continuum (total derivative)
Lattice action

\[ S_L = \sum_x N \bar{N} + \bar{\omega} (\Delta^s_z \lambda + W''_L(\phi) \omega) + \bar{\lambda} (\Delta^s_z \omega + W''_L(\bar{\phi}) \lambda) \]

where \( \psi = \begin{pmatrix} \omega \\ \lambda \end{pmatrix} \), \( \bar{\psi} = \begin{pmatrix} \bar{\omega} \\ \bar{\lambda} \end{pmatrix} \).

Symmetric difference \( \Delta^s_z = \Delta^s_1 + i \Delta^s_2 \).

Doublers removed via:

\[ W'_L(\phi) = W'(\phi) + \frac{1}{2} D^+ \mu D^- \mu \phi \]

Equivalent in 1D
Supersymmetries

The single exact SUSY follows by analogy to QM:

\[ Q\phi = \omega \]
\[ Q\bar{\phi} = \lambda \]
\[ Q\lambda = 0 \]
\[ Q\omega = 0 \]
\[ Q\bar{\omega} = \bar{N} \]
\[ Q\bar{\lambda} = N \]

Note that \( Q^2 = 0 \) with E.O.M

Homework Problem. Check this invariance.
Additional SUSYs

In continuum this model has 3 additional SUSYs.
Go back to QM:
Continuum bosonic action same if $\mathcal{N}$ replaced by

$$\hat{N} = \Delta \phi - P'.$$

Equal to transpose of original fermion op.
Fermion action invariant if intercharge $\psi \rightarrow \bar{\psi}$ – Gives 2nd SUSY!
Similar for WZ model: 3 other ways of writing $\mathcal{N}$ –
corresponding to $\phi \rightarrow \bar{\phi}$ and flipping sign.
Three new fermion ops. Make fermion act invariant by
exchanging $\omega \rightarrow \lambda$ and $\bar{\omega} \rightarrow \bar{\lambda}$ etc.
Twisted/topological form

Not hard to verify that action can be rewritten (see QM) as

\[ S_L = Q \sum_x \bar{\omega} \left( \mathcal{N} + \frac{1}{2} B \right) + \bar{\lambda} \left( \bar{\mathcal{N}} + \frac{1}{2} \bar{B} \right) \]

with additional \( B, \bar{B} \)

\[
\begin{align*}
Q\bar{\omega} & = \bar{B} \\
Q\bar{\lambda} & = B \\
QB & = 0 \\
Q\bar{B} & = 0
\end{align*}
\]
Ward identities

Again, existence of exact SUSY leads to Ward identities:

\[ < S_B > = \frac{1}{2} N_{\text{d.o.f}} \]

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WZ Ward identity

\[ \langle x_2^2(0) N_2(t) \rangle \]

\[ -G_{22}(t) \]

Graph showing the behavior of \( \langle x_2^2(0) N_2(t) \rangle \) and \( -G_{22}(t) \) as a function of time \( t \).
Twisting in 2D

Process of exposing this nilpotent supercharge can be made systematic
Fields transform as spinors under $SO(2)_R$ and $SO(2)_{rot}$
Choose to decompose fields under twisted symmetry

$$SO(2)' = \text{Diag} \left( SO(2)_R \times SO(2)_{rot} \right)$$

Regard spinors/supercharges as matrices
Decompose on products of gamma matrices

$$q = QI + Q_\mu \gamma_\mu + Q_{12} \gamma_1 \gamma_2$$

Scalars, vectors, tensors
Note $\text{Tr}(\Gamma^i \Gamma^j) = 2\delta^{ij}$ where $\Gamma^i = (I, \gamma_\mu, \gamma_5)$
Original supersymmetry algebra \( \{q, q\} = \gamma_\mu p_\mu \)

\[
\begin{align*}
\{Q, Q\} &= \{Q_{12}, Q_{12}\} = \{Q, Q_{12}\} = \{Q_\mu, Q_\nu\} = 0 \\
\{Q, Q_\mu\} &= p_\mu \quad \{Q_{12}, Q_\mu\} = \epsilon_{\mu\nu} p_\nu
\end{align*}
\]

Note that \( p_\mu \) is \( Q \)-variation of something. Makes it plausible that entire \( T_{\mu\nu} \) is \( Q \)-exact.

(why?: \( p_\mu p_\nu = Q \Lambda_\mu Q \Lambda_\nu = Q (\Lambda_\mu Q_\nu) \))

Hence twisted theories have \( Q \)-exact actions! (remember \( T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}} \))
Kähler-Dirac fermions

Typical Dirac action can be rewritten as:

\[ S_F = \text{Tr} \left[ \bar{\Psi} \gamma_\mu \partial_\mu \Psi \right] \]

where

\[ \Psi = \frac{\eta}{2} I + \psi_\mu \gamma_\mu + \chi_{12} \gamma_5 \]

Components \((\eta/2, \psi_\mu, \chi_{12})\) constitute a Kähler-Dirac field. In real (why real?) components:

\[ S_F = \frac{1}{2} \eta \partial_\mu \psi_\mu + \chi_{12} (\partial_1 \psi_2 - \partial_2 \psi_1) \]


KD treatment equivalent to twisting
Discretize KD equations

Can discretize such geometrical actions without introducing doubles (Rabin, Joos)

- Replace $\partial_\mu \rightarrow \Delta^+_\mu$ in curl
- Replace $\partial_\mu \rightarrow \Delta^-_\mu$ in div.

Prohibits doubles!

$$S_F = \begin{pmatrix} \eta/2 & \chi_{12} \end{pmatrix} \begin{pmatrix} \Delta^-_1 & \Delta^-_2 \\ -\Delta^+_2 & \Delta^+_1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$
Relation to staggered fermions

Natural to place scalars on sites, vectors on links and tensors on diagonal links.
Introduce a lattice with half spacing – all fields now site fields.
Free KD action reduces to staggered action – fermion phases now arise from alternate use of $\Delta^-, \Delta^+$. Usual U(1) symmetry: rotate $\psi_{1,2} \rightarrow e^{i\alpha} \psi_{1,2}$ and $\eta, \chi_{12} \rightarrow e^{-i\alpha} \eta, \chi_{12}$.
Homework problem 4. Check this.
Wess-Zumino using KD fermions

If we choose

\[ \omega = \frac{1}{2} \eta + i \chi_{12} \]

\[ \lambda = \psi_1 + i \psi_2 \]

Find KD action

\[ \omega^\dagger D_z \lambda + \text{h.c} \]

Technically uses self-dual parts of KD field

Duality \( f_p \rightarrow f_{d-p} \) with

\[ * f_{\mu_1 \ldots \mu_d} = \epsilon_{\mu_1 \ldots \mu_d} f_{\mu_{d-p+1} \ldots \mu_d} \]

\[ P^+ = \frac{1}{2} (I + i*) \]