THEORETICAL ISSUES IN EXTRACTION $V_{ud}$ FROM NUCLEAR DECAYS

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- $V_{ud}$ from nuclear decays
  - radiative corrections
  - isospin-symmetry breaking corrections

- $V_{us}$ from kaon decays
  - SU(3)-symmetry breaking corrections

- Top row test of CKM unitarity
MASTER EQUATIONS

CVC: $\mathcal{F}t = ft(1 + \delta'_R)(1 - (\delta_C - \delta_{NS})) = \text{constant}$

$$V_{ud}^2 = \frac{K}{2G_F^2 \mathcal{F}t(1 + \Delta_R)} \quad \frac{K}{(\hbar c)^6} = \frac{2\pi^3 \hbar \ln 2}{(m_e c^2)^5}$$

where

$ft =$ experimental nuclear $ft$ values.

$\mathcal{F}t =$ average corrected $ft$ values (13 cases).

$G_F =$ weak interaction coupling constant

(from muon lifetime).

$$\Delta_R \quad \delta'_R \quad \delta_{NS}$$

$= \text{calculated radiative correction}.$

$\delta_C =$ calculated isospin symmetry breaking correction.
Note:

\[ V_{ud}^2 = \frac{\text{beta decay}}{\text{muon decay}} \]

Any radiative correction that is common to both beta decay and muon decay is called **universal**, cancels in ratio – not included in calculation.

**Example:**

![Diagrams showing beta decay and muon decay](image)

universal in limit: \[ \frac{m_h^2}{m_Z^2} \rightarrow 0 \]

\( m_h = \text{hadron mass} \).
RADIATIVE CORRECTION TO ORDER $\alpha$

bremsstrahlung  \hspace{1cm} W-box  \hspace{1cm} Z-box

$W$-box: \(\begin{cases} \text{long distance (low energies): sensitive to nucleon structure} \\ \text{short distance (high energies): only “see” quarks} \end{cases}\)

\[
\text{Brem} + W\text{-box}(V, \text{LD}) = \frac{\alpha}{4\pi} \bar{g}(E_m) \xrightarrow{\text{Large } E_m} \frac{\alpha}{4\pi} \left[ 3 \ln \left( \frac{m_p}{2E_m} \right) + \frac{81}{10} - \frac{4\pi^2}{3} \right]
\]

\[
W\text{-box}(A) = ?
\]

\[
W\text{-box}(V,\text{SD}) + Z\text{-box} = \frac{\alpha}{4\pi} \left[ 3 \ln \left( \frac{m_W}{m_p} \right) - 4 \ln \left( \frac{m_W}{m_Z} \right) \right]
\]
Break integration into short and long-distance regimes

a) **Short distance:** \( m_A^2 \leq Q^2 \leq \infty \)

\[
F(Q^2) \xrightarrow{Q^2 \to \infty} \frac{1}{Q^2} \left[ 1 - \frac{\alpha_s(Q^2)}{\pi} \right]
\]

\[
W-\text{Box}(A, \text{SD}) = \frac{\alpha}{4\pi} \left[ \ln \left( \frac{m_W}{m_A} \right) + A_g \right] \quad A_g = -0.34
\]
b) **Long distance:**  $0 \leq Q^2 \leq m_A^2$

\[ W\text{-Box}(A, \text{LD}) = \begin{array}{c}
\begin{array}{c}
\text{Born graphs}
\end{array}
\end{array} \]

\[ = \frac{\alpha}{2\pi} \left[ C_{\text{Born}} \right] = \frac{\alpha}{2\pi} [0.881] \]

Choose $m_A$?

Sirlin recommended:

\[ \frac{1}{2} m_{a_1} \leq m_A \leq 2 m_{a_1} \]

This range is **largest** contributor to error in radiative correction
For finite nuclei (but not neutron decay) there is a two-body contribution from the Born graphs:

Requires a shell-model calculation for its evaluation.

This is the ONLY piece of the radiative correction that depends on a nuclear-structure calculation and it is SMALL.

Typical values:

\[
\begin{align*}
T_z = -1 : & \quad \delta_{NS}(^{10}C) = -0.36\% \\
T_z = 0 : & \quad \delta_{NS}(^{26}Al) = 0.01\% \\
& \quad \delta_{NS}(^{14}O) = -0.25\% \\
& \quad \delta_{NS}(^{46}V) = -0.04\% \\
& \quad \delta_{NS}(^{34}Ar) = -0.18\% \\
& \quad \delta_{NS}(^{74}Rb) = -0.06\%
\end{align*}
\]
Marciano-Sirlin (PL 96, 032002 (2006)) revision

Break integration into three regimes

a) Short distance:  \((1.5 \text{ GeV})^2 \leq Q^2 \leq \infty\)

\[
F(Q^2) = \frac{1}{Q^2} \left[ 1 - \frac{\alpha_s(Q^2)}{\pi} - C_2 \left( \frac{\alpha_s(Q^2)}{\pi} \right)^2 - C_3 \left( \frac{\alpha_s(Q^2)}{\pi} \right)^3 \right]
\]

QCD corrections to third order; \(C_2\) and \(C_3\) related to Bjorken sum rule for polarized electroproduction.

b) Intermediate distance:  \((0.823 \text{ GeV})^2 \leq Q^2 \leq (1.5 \text{ GeV})^2\)

\[
F(Q^2) = \frac{D_1}{Q^2 + m_\rho^2} + \frac{D_2}{Q^2 + m_A^2} + \frac{D_3}{Q^2 + m_{\rho'}^2}
\]

interpolation function parameterized by meson dominance

\(D_1, D_2, D_3\) fixed by matching and other constraints
c) Long distance: \[ 0 \leq Q^2 \leq (0.823 \text{ GeV})^2 \]

Born graphs: Change in integration range reduces value slightly

\[
C_{\text{Born}} : 0.881 \rightarrow 0.829
\]

Allow 10\% uncertainty in \( C_{\text{Born}} \); 100\% uncertainty in interpolator

**Result:**

a) factor of 2 reduction in error assigned to the radiative correction
b) little change in magnitude of radiative correction
BEYOND ORDER $\alpha$

1. QED Corrections

$\mathcal{O}(Z\alpha^2)$  $\mathcal{O}(Z^2\alpha^3)$

Sirlin, PR D35, 3423 (1987)
2. Leading Log corrections, $\alpha^n \ln^n (m_Z/m_p)$

Czarnecki, Marciano, Sirlin PR D70, 093006 (2004)

SD : \[ 1 + \frac{\alpha}{2\pi} \left[ 4 \ln \frac{m_Z}{m_p} \right] \rightarrow S(m_p, m_Z) = 1.02248 \]

LD : \[ 1 + \frac{\alpha}{2\pi} \left[ 3 \ln \frac{m_p}{2E_m} \right] \rightarrow L(2E_m, m_p) = 1.02673 \left[ 1 - \frac{2\alpha(m_e)}{3\pi} \ln \frac{2E_m}{m_e} \right]^{9/4} \]

where $S(m_p, m_Z)$ and $L(2E_m, m_p)$ are renormalization group summation of leading log.

\[
S(m_p, m_Z) = \left( \frac{\alpha(m_e)}{\alpha(m_p)} \right)^{3/4} \left( \frac{\alpha(m_e)}{\alpha(m_\tau)} \right)^{9/16} \left( \frac{\alpha(m_b)}{\alpha(m_e)} \right)^{9/19} \left( \frac{\alpha(m_W)}{\alpha(m_b)} \right)^{9/20} \left( \frac{\alpha(m_Z)}{\alpha(m_W)} \right)^{36/17}
\]

$\alpha^{-1}(0) = 137$, $\alpha^{-1}(m_e) = 137.089$, $\alpha^{-1}(m_p) \simeq 134$, $\alpha^{-1}(m_Z) \simeq 127.6$
SUMMARY

\[ 1 + RC = (1 + \delta_{NS}) \times (1 + \delta'_R) \times (1 + \Delta^V_R) \]

- Nuclear-structure dependent 2-body Born graphs

- Nucleus dependent trivially:
  \[ Z, \quad E_m \quad \text{Em} = \text{maximum electron energy} \]

- Nucleus independent

\[ \delta_{NS} \simeq -0.04\% \]

\[ \delta'_R \simeq 1.46\% \]

\[ \Delta^V_R \simeq 2.36\% \]

\[ 1 + \delta'_R = \left\{ 1 + \frac{\alpha}{2\pi} \left[ g(E_m) - 3\ln \frac{m_p}{2E_m} \right] \right\} \times \left\{ L(2E_m, m_p) + \frac{\alpha}{2\pi} [\delta_2 + \delta_3] \right\} \]

\[ 1 + \Delta^V_R = S(m_p, m_Z) + \frac{\alpha}{\pi} C_{\text{Born}} + \frac{\alpha(m_p)}{2\pi} \left[ \ln \frac{m_p}{m_A} + A_g \right] + \text{NLL} \]

Czarnecki, Marciano and Sirlin, PR D70, 093006 (2005).
**ISOSPIN-SYMMETRY BREAKING CORRECTION**

Beta decay in nuclei described by one-body operator

$$F = \sum_{\alpha, \beta} \langle \alpha | \tau_+ | \beta \rangle \hat{a}_\alpha \dagger \hat{a}_\beta$$

Matrix element in many-body system

$$\langle M_F \rangle = \sum_{\alpha, \beta} \langle f | \hat{a}_\alpha \dagger \hat{a}_\beta | i \rangle \langle \alpha | \tau_+ | \beta \rangle$$

- shell-model one-body density
- matrix elements evaluated in many-body states
- single-particle matrix elements

$$\Omega_\alpha = \delta_{\alpha, \beta} \int_0^\infty R_{n_{\alpha \alpha}}^{\text{proton}} R_{n_{\beta \beta}}^{\text{neutron}} r^2 \, dr$$

Define:

$$\langle M_F \rangle^2 = 2 \left( 1 - \delta_C \right) ; \quad \delta_C = \delta_{C1} + \delta_{C2}$$

- Isospin Mixing ~0.1%
- Radial Overlap ~0.4%
Radial Overlap: contribution constrained by:

asymptotic radial function for proton matched to proton separation energy, $S_p$, in decaying nucleus

$$R(r) \sim e^{-\alpha r}$$

$$\alpha^2 = \frac{2mS}{\hbar^2}$$

ditto neutron, matched to neutron separation energy, $S_n$, in daughter nucleus

Towner-Hardy: used Saxon-Woods functions PR C66, 035501 (2002)

Ormand-Brown: used Hartree-Fock functions PR C52, 2455 (1995)

$$\Omega_\alpha = \delta_{\alpha,\beta} \int_0^\infty R_{n}\alpha l_{\alpha} R_{n}\beta l_{\beta} r^2 \, dr$$

Radial integral departs from the value of unity because proton and neutron radial functions are matched to different separation energies. Further these separation energies depend on the parentage expansions.
Radial Overlap (continued)

\[ \langle M_F \rangle^2 = 2 \left( 1 - \delta_{C_2} \right) \]

\[
\langle M_F \rangle = \sum_{\alpha, \beta} \langle f | \hat{a}^\dagger_{\alpha} \hat{a}_{\beta} | i \rangle \langle \alpha | \tau_+ | \beta \rangle \\
= \sum_{\alpha, \pi} \langle f | \hat{a}^\dagger_{\alpha} | \pi \rangle \langle \pi | \hat{a}_{\alpha} | i \rangle \Omega_{\alpha}^\pi \\
= \delta_{\alpha, \beta} \Omega_{\alpha}
\]

Use shell model to calculate these parentage coefficients. Consider:

\[ | j^n_{\alpha}; J = 0, T = 1 \rangle \quad | j^{n-1}_{\alpha}; J = j_{\alpha}, T = 1/2 \text{ or } 3/2 \rangle \]

\[ \delta_{C_2} \simeq \frac{n + 4}{3} \left( 1 - \Omega_{\alpha}^< \right) - \frac{n - 1}{3} \left( 1 - \Omega_{\alpha}^> \right) \]

\[
\text{for } T = 1/2 \\
\text{for } T = 3/2
\]

Also contribution from core orbitals:

\[ | j^n_{\alpha} j^{-1}_{c}; J = j_{c}, T = 1/2 \text{ or } 3/2 \rangle \]

\[ \delta_{C_2} \simeq \sum_{c} \frac{4j_{c} + 2}{3} \left[ (1 - \Omega_{c}^<) - (1 - \Omega_{c}^>) \right] \]

Core contribution \( \rightarrow 0 \) as \( \Omega_{c}^< \rightarrow \Omega_{c}^> \rightarrow 1 \) as separation energies increase.
Isospin Mixing:

Introduce charge-dependent terms in shell-model Hamiltonian: **Constrain** the calculation to reproduce coefficients of IMME equation

\[ M(A, T, T_z) = a + bT_z + cT_z^2 \]

Require calculation to fit experimental \( b \) and \( c \) coefficients

Then compute: \( \langle M_F \rangle \rightarrow \delta_{C_1} \)

With isospin symmetry:
- Parent state can **only** decay to its isospin analogue state

With isospin-symmetry breaking:
- Parent state can **now** decay (weakly) to non-analogue states.
  - Calculation, besides yielding \( \delta_{C_1} \), predicts these weak branches.

**Subject to experimental test.**
Test of calculation of $\delta_{C_1}$

Branches to non-analogue $0^+$ states

$$\langle M_0 \rangle^2 = 2 (1 - \delta_{C_1})$$

$$\langle M_1 \rangle^2 = 2\delta_{C_1}^1$$

$$\langle M_2 \rangle^2 = 2\delta_{C_1}^2$$

$$\vdots$$

$$\sum_{n=1}^{\infty} \delta_{C_1}^n = \delta_{C_1} + \text{corrections due to mixing with } T = 0, 2, 3 \ldots$$

$$\text{BR} = \frac{f_1}{f_0} \delta_{C_1}^1$$
Experimental non-analogue branching ratios

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<tr>
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<th>Theory</th>
<th>Experiment</th>
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<tbody>
<tr>
<td></td>
<td>$\delta_{C_1}^1$ (%)</td>
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<tr>
<td>$^{38}$K</td>
<td>0.090(30)</td>
<td>$&lt; 0.280$</td>
</tr>
<tr>
<td>$^{42}$Sc</td>
<td>0.020(20)</td>
<td>0.040(9)</td>
</tr>
<tr>
<td>$^{46}$V</td>
<td>0.035(15)</td>
<td>0.053(5)</td>
</tr>
<tr>
<td>$^{50}$Mn</td>
<td>0.045(20)</td>
<td>$&lt; 0.016$</td>
</tr>
<tr>
<td>$^{54}$Co</td>
<td>0.040(20)</td>
<td>0.035(5)</td>
</tr>
<tr>
<td>$^{62}$Ga</td>
<td>0.085(20)</td>
<td>$\leq 0.040(15)$</td>
</tr>
<tr>
<td>$^{74}$Rb</td>
<td>0.050(30)</td>
<td>$&lt; 0.070$</td>
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Theory is within a factor of two of these small experimental quantities.
**SUMMARY:**

Isospin-symmetry breaking

\[ \delta_C = \delta_{C1} + \delta_{C2} \]

- isospin mixing
- radial overlaps

**Typical values (in percent)**

- Avg (A = 10 --> 54): 0.05 0.40
- Avg (A = 62 --> 74): 0.25 1.10

**Two calculations (constrained by separation energies and fits to IMME coefficients):**

Results in a systematic uncertainty of 0.9s in Ft of 3070s.

Both calculations produce similar nucleus to nucleus variations.
CVC Test: \( \mathcal{F} t = \text{constant} \)

\[
\mathcal{F} t \equiv ft(1 + \delta'_R)(1 - (\delta_C - \delta_{NS})) = \frac{K}{2G^2 F V^2_{ud}(1 + \Delta_R)}
\]

Average \( \mathcal{F} t = 3073.9 \pm 0.8 \pm 0.9 \text{ s} \)

\( \chi^2/\nu = 0.9 \)

\( \delta_C \)
Alternative Strategy: Take the CVC test as a given, and use it to probe the nucleus-to-nucleus variations in the corrections.
CKM Unitarity Test

From: \[ \text{Average } \overline{Ft} = 3073.9 \pm 0.8 \pm 0.9 \text{ s} \]

And: \[ \Delta_R = (2.361 \pm 0.038)\% \]

Yields: \[ V_{ud} = 0.97378(27) \]

V\text{ud} also obtained from neutron and pion decay

And:

\[ V_{ud}^2 + V_{us}^2 + V_{ub}^2 = 0.9992(11) \]

Within the estimated errors: CKM unitarity fully satisfied
Summary on $V_{ud}$

- $0^+ \rightarrow 0^+$ provides best determination
- In 3-5 years $n-\beta$ will catch up: stay tuned!
NEUTRON BETA DECAY

Serious problem with lifetime:

\[ \tau_n(s) = \begin{cases} 
878.5 \pm 0.7_{\text{stat}} \pm 0.3_{\text{syst}} & \text{Serebrov et.al. PL B605, 72} \\
885.7 \pm 0.8 & \text{PDG06}
\end{cases} \]

Differs from World Average by 6.5\(\sigma\).
V_{us} \text{ from } K_{\ell 3} \text{ rates}

\Gamma(K_{\ell 3}) = \frac{C_{K}^{2} G_{F}^{2} m_{K}^{5}}{192\pi^{3}} S_{EW} |f_{+}(0)|^{2} I(\lambda)(1 + 2\Delta_{K}^{SU(2)} + 2\Delta_{K}^{EM})

C_{K}^{2} = 1/2 \text{ for } K^{+}, = 1 \text{ for } K^{0}

S_{EW} = \text{universal short-distance radiative correction}

\text{Inputs from theory:}

- Hadronic matrix element (form factor) at zero momentum transfer \((t = 0)\)
- Form-factor correction for \(SU(2)\) breaking
- Form-factor correction for long-distance EM effects

\text{Inputs from experiment:}

- Rates with well-determined treatment of radiative decays:
  - Branching ratios
  - Kaon lifetimes
- Integral of form-factor over phase space: \(\lambda\)s parameterize evolution in \(t\)
  - \(K_{e3}\): Only \(\lambda_{+}\) (or \(\lambda_{+}^{'}, \lambda_{+}^{''}\))
  - \(K_{\mu3}\): Need \(\lambda_{+}\) and \(\lambda_{0}\)
New fit to Kaon branching ratios

$$\text{BR}(K^\pm \to p^0 \nu)$$

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$$\text{BR}(K^\pm \to p^0 \mu \nu)$$

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Preliminary results from KLOE, ISTR+ and NA48/2

$$\text{BR}(K_S \to p \nu) = 7.046(91) \times 10^{-4}$$

KLOE

$$\text{BR}(K_L \to p e\nu)$$

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$$\text{BR}(K_L \to p \mu\nu)$$

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KTeV KLOE

NA48 preliminary
$|V_{us}|f_+(0)$ from $K_{l3}$ data

| $|V_{us}|f_+(0)$ | % err | Approx. contrib. to % err from: |
|-----------------|-------|-------------------------------|
| $K_L e^3$       | 0.21646(59) | 0.27  | 0.09  | 0.19  | 0.10  | 0.11 |
| $K_L \mu^3$     | 0.21665(71) | 0.33  | 0.12  | 0.19  | 0.15  | 0.18 |
| $K_S e^3$       | 0.21555(143) | 0.66  | 0.65  | 0.02  | 0.10  | 0.11 |
| $K^\pm e^3$     | 0.21875(104) | 0.47  | 0.37  | 0.07  | 0.27  | 0.11 |
| $K^\pm \mu^3$   | 0.21817(125) | 0.57  | 0.30  | 0.07  | 0.45  | 0.18 |

Average: $|V_{us}|f_+(0) = 0.21686(49)$ \quad $\chi^2$/ndf = 5.0/4 (29.0%)
Evaluations of $f_+(0)$

\[ f_+(0) = 1 + f_2 + f_4 \]

\[ = 1 - 0.023 \pm ? \]

- LR 84 quark model
- BT 03 ChPT + LR 84
- Cir 05 ChPT + disp
- JOP 04 ChPT + disp
- C$^+$ 05 ChPT + $1/N_c$
- SPQcdR 05 $N_f = 0$
- FNAL/MILC/HPQCD 04 $N_f = 2_{stag} + 1$
- JLQCD 05 $N_f = 2$
- RBC 06 $N_f = 2_{DW}$
- UKQCD/RBC 06 (revised) $N_f = (2+1)_{DW}$

Leutwyler & Roos estimate (LR 84) still widely used: $f_+(0) = 0.961(8)$

Lattice evaluations generally agree well with this value
CONCLUSIONS

- Superallowed $\beta$ decay currently yields most precise value of $V_{ud}$, limited by theory uncertainties.
  
  \[ V_{ud} = 0.97378(27) \]

- Value of $V_{ud}$ proving to be very robust.

- Neutron and pion decays yield $V_{ud}$ consistent with nuclear result, but with larger experimental errors. This will change in 3 – 5 years.

- Much activity in nuclear physics focussed on reducing errors still further via tests of structure-dependent corrections.
CONCLUSIONS (continued)

- Experimental uncertainty in $f_+(0)|V_{us}|$ currently at 0.2% with good consistency.

- Dominant uncertainty in $V_{us}$ is still from estimate of $SU(3)$-symmetry breaking: $f_+(0)$.

- With benchmark Leutwyler-Roos value: $V_{us} = 0.2257(20)$

- CKM Unitarity now verified to 0.1% – dominant error from $V_{us}$.