Our Limited Understanding of

Plasma Instabilities at Large Anisotropy

Guy Moore: work with Peter Arnold

1. What plasma instabilities are about
2. Why to expect large anisotropy (at weak coupling)
3. Physical picture of instability at large anisotropy
4. Handwaves about linear regime at large anisotropy
5. Our first numerical studies
6. Conclusions and Issues Still to be Addressed
Plasma begins as a “pancake” in position space.
Momentum selection: how that turns to momentum space anisotropy
Why momentum anisotropy implies Weibel instability

Consider the effects of a seed magnetic field $\hat{B} \cdot \hat{p} = 0$ and $\hat{k} \cdot \hat{p} = 0$

How do the particles deflect?
Positive charges:

No net $\rho$. Net current is induced as indicated.
Negative charges:

Induced $B$ adds to seed $B$. Exponential Weibel instability

Linearized analysis: $B$ grows until bending angles become large.
Scales in the problem

\( p \): typical momentum of a “hard” particle \( Q_s \)

\( n \): number density of particles \( Q_s^3 / g^2 (Q_s \tau) \)

\( n \propto 1/\tau \) in absence of splitting

\( m^2 = g^2 n/p \): plasma scale \( Q_s^2 / (Q_s \tau) \)

\( \theta \): angular spread of hard particles To determine

Gauge fields: Vlasov works at scale \( k \) provided \( p > k/\theta \).
We will need \( p > m/\theta^2 \).
Growth limits: two possibilities

1. Abelianization: gauge fields grow until particle deflection angles become large, as occurs in abelian plasmas.

\[ \frac{B^2}{g^2} \sim np: \text{O}(1) \text{ of energy in soft fields.} \]

2. Growth to nonabelian scale. Nonabelian effects cut off field growth at a lower scale.

\[ \frac{B^2}{g^2} \sim m^4 \theta^\# \text{ power to be determined} \]

Abelianization: fast isotropization. Disfavored.

Nonabelian: anisotropy grows with time
Why $B^2/g^2 \sim m^4 \theta^\# \Rightarrow \theta$ shrinks with time

$$\frac{dp_\perp^2}{dt} \sim \frac{B^2}{\tau_{\text{dyn}}} \sim \frac{B^2}{m} \sim m^3 \theta^\# \sim \tau^{-3/2} \theta^\#$$

but to keep $p_\perp^2$ fixed during expansion, need

$$\tau \frac{p_\perp^2}{dt} \sim \theta^2 p^2 \Rightarrow m^3 \theta^\# \sim \tau \tau^{-3/2} \theta^\#$$

so $\theta$ must shrink with time, unless $p$ or $m$ change.

$p$ change: splitting. Doesn’t occur early but will eventually, see Peter Arnold’s talk
What does high-anisotropic Weibel look like?

$m$ is $t^{-1}$ for back-reaction to become important.

\[ \delta \theta_p \sim \frac{Bt}{p} \quad \delta z \sim \frac{Bt^2}{p} \quad j = g^2n\delta zk \sim \frac{g^2n}{p} t^2k = m^2t^2k \]

Compare to the $D \times B$ term in Ampere's Law:

\[ D \times B \sim kB \sim j \sim m^2t^2k \quad \Rightarrow \quad t \sim 1/m \]
Which modes are unstable?

Last argument: growth timescale $\sim 1/m$ ($\gamma \sim m$)

Particles must be coherently in same-sign $B$ for $1/m$ time.

Gauge invariant— you should include Wilson line along particle trajectory

This can fail when

- $B$ varies in plane: $k_\perp > m$

- Range of $\theta$ too big: $k_z > m/\theta \equiv k_{\text{max}}$
Picture during pertubative growth: $B$ pancakes

$B$ varies in $z$ with $\sim 1/k_{\text{max}}$ but in $x, y$ with $\sim 1/m$ coherence length.
What cuts off the growth?

Assume all unstable modes present, comparable amplitude.

Growth can stop due to

- Color randomization: Particle color changes in length $< 1/m$, making $J^a$ wrong component to grow $B^a$

- Nonabelian interaction: $D_\perp = \partial_\perp - iA_\perp$ and large $A_\perp$ changes $B$ evolution to take on nonvanishing $k_\perp \sim m$ components

- Nonabelian interaction: color rotation $A_\perp$ terms in YM equation can make $A$ color precess so it’s out of alignment with what $J$ is doing

Which is which? Probably gauge dependent.
How large does $B$ get?

Color randomization: Wilson line $U = \exp(-iA \cdot dl)$ far from 1 on length $1/m$: $A_\perp/m \sim 1$ or $A_\perp \sim m$

Other nonabelian effects also give $A_\perp \sim m$.

Magnetic energy

$$\frac{1}{g^2}B^2 = \frac{1}{g^2}(D \times A)^2 \sim \frac{1}{g^2}k_{\text{max}}^2 m^2 = \frac{m^4}{g^2\theta^2}$$
Aside: Nielsen-Olsen Instability

What if fields are very “clean” with 1 mode + tiny fluctuations?

$B$ field splits states into Landau levels. Split by $\vec{s} \cdot \vec{B}$.

One spin-1 mode unstable. But at nonzero $k_z$ it takes

$$E^2 = k_z^2 + (s_z + \frac{1}{2})B < 0 \rightarrow B > k_{\text{max}}^2, B^2 / g^2 > m^4 / g^2 \theta^4$$
Aside: scattering

When unstable fields reach nonabelian scale:
How would they scatter if particles were absent?

$$\frac{1}{g^2} B^2 = \epsilon \sim \frac{k_{\text{max}}^2 m^2}{g^2} \sim \int d^3 k \ k f \sim k_{\text{max}}^2 m^2 f \implies f \sim \frac{1}{g^2}$$

Such particles give an

$$m_{\infty}^2 \sim g^2 n/k_{\text{max}} \sim g^2 (k_{\text{max}} m^2 / g^2) / k_{\text{max}} \sim m^2$$

Scattering rate

$$\Gamma_{\text{Coulomb}} \sim \sigma n \sim \frac{g^4 f n}{Q^2} \sim \frac{k_{\text{max}} m^2}{Q^2}$$

$$\sim k_{\text{max}} \text{ for } Q \sim m \text{ but } \sim m^2 / k_{\text{max}} \text{ for } Q \sim k_{\text{max}}$$
Trying to think about end of instability

Nonabelian interaction $k_\perp$ broadens modes out of unstable region?

Right behavior as small-angle physics dominates in gauge theory
Possible obstacle: frequency mismatch

Perturbatively unstable fields evolve, $\omega \sim m$.

But other $k \sim k_{\text{max}}$ modes typically have $\omega \sim k_{\text{max}} \gg m$.

Little spectral weight near $\omega \sim m$.

Does this make “pumping” of other modes by soft ones inefficient?
Relevant modes probably those with $k_\perp \sim \text{(few)} \times m$

\[ G_R^{-1} \sim k^2 P_T^{ij} + \Pi_T^{ij} \]

\[ \Pi_T^{ij} = O(m^2) + \int d^3p \frac{g^2 f(p)}{p} v^i_p v^j_p \frac{k^2 - \omega^2}{(v_p \cdot k - \omega - i\epsilon)^2} \]

Landau cut, $\omega \in [-k_\perp, k_\perp]$ and plasmon pole, $\omega \sim k$

Spectral weight at $\omega = \text{(few)} m$, but real-time behavior expected to be damped

Energy can get here, but should then be re-absorbed by hard modes
Prediction

Perturbative unstable modes should gain energy

\[ \frac{1}{g^2} B^2 \sim \frac{k_{\text{max}}^2 m^2}{g^2} \sim \frac{m^4}{g^2 \theta^2} \]

in time scale \( \sim 1/m \). IF it goes in cascade,

\[ \frac{d\epsilon_{\text{soft}}}{dt} \sim \frac{k_{\text{max}}^2 m^3}{g^2} \sim \frac{m^5}{g^2 \theta^2} \]

BUT most energy should be Landau-damped back into hard modes and real growth rate should be smaller.

[[NEEDS TO BE QUANTIFIED]]
Prediction 2

We should measure hard particle scattering:

\[ \hat{q} \equiv \frac{dP^2}{dt} = \int dt' (E + v \times B)_a [tv^\mu] U_{ab} (tv^\mu, 0) (E + v \times B)_b [0] \]

Expect: \( \sim B^2 \tau \) where \( \tau \sim m \).

Should be dominated by would-be unstable modes

Definite angular pattern: mostly planar \( B \) fields

Decompose in \( dP^2_L(\theta) \), \( dP^2_\theta(\theta) \), \( dP^2_\phi(\theta) \). Expect \( dP^2_L \ll dP^2_\theta, dP^2_\phi \); largest near \( \theta = 0 \).
Test using highly anisotropic distribution $\Omega(x = \cos \theta)$

- Positivity: $\Omega(x) > 0$ all $x \in [-1, 1]$
- $Y_{\ell m}$ expansion truncates at some $\ell = 2n$
- Maximally peaked near $x = 0$ given $\ell$ cutoff

Our choice:

$n$ odd: $\Omega = N(1 - x^2)(a_1 - x^2)(a_2 - x^2)^2 \ldots (a_{\left\lfloor \frac{n}{2} \right\rfloor} - x^2)^2$

$1 > a_1 > a_2 > \ldots > a_{\left\lfloor \frac{n}{2} \right\rfloor}$

$n$ even: $\Omega = N(a_1 - x^2)^2(a_2 - x^2)^2 \ldots (a_{n/2} - x^2)^2$

$1 = a_1 > a_2 > \ldots > a_{n/2}$

Determine $a_i$ to minimize $\int x^2\Omega(x)$ with $\int \Omega(x) = 1$. 
What our functions look like

\[ \sqrt{\langle x^2 \rangle} \approx \frac{6}{4n+9} \]
\[ k_{\text{max}} \approx \frac{n+1}{2} m_\infty \]
Initial conditions

Real world: $\theta$ shrinks with time: $\theta_{\text{init}} \sim 1$ or $\sim g$

$\theta(\tau)$ shrinks slower than $1/Q_s \tau$ but $m$ scales as $m \propto \tau^{-1/2}$.

Many e-folds occur before $\theta$ gets really small

Should start with large IR fields.

We choose thermal, $T \sim k_{\text{max}}/g^2$, smeared on a scale $\sim k_{\text{max}}$. $\epsilon \sim k_{\text{max}}^4/g^2$ initially.
Energy growth with time

\[ \frac{d\epsilon_{\text{soft}}}{dt} \sim \frac{k_{\text{max}}^3 m^2}{g^2} \]

Why?

Bottom to top: n=4,...,15

\[ g^2[B_x^2 + B_y^2]/2k_{\text{max}}^3 m_\infty \]
Particle momentum diffusion

As mentioned, $dP^2/dt$ of hard modes is really 3 functions:

$dP_L^2/dt$ energy change—feels only $E$ fields.

$dP_\theta^2/dt$ polar angle change: $E \times E$, $E \times B$, $B \times B$

$dP_\phi^2/dt$ azimuthal angle change: also $E$ and $B$

$dP_\theta^2 \neq dP_\phi^2$ would be interesting for jet observables

Expect $\theta$ dependence with maximum value in-plane ($B$ field coherence)
Measured momentum diffusion, $n = 10$
Conclusions

- Large angular anisotropy is interesting
- Large initial conditions give linear growth in $\epsilon$
- Cascade seems to work,
  $$\frac{d\epsilon}{dt} \sim \frac{k_{\text{max}}^3 m^2}{g^2} \sim \frac{m^5}{g^2 \theta^3}, \quad \text{not} \quad \frac{m^5}{g^2 \theta^2}$$
- Much to understand: my naive pictures give wrong answers
- We need more theoretical thinking!