Color Glass Condensate and its relation to statistical physics

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From RHIC to LHC: Achievements and Opportunities

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• Summary
Introduction
At very high energy, a fast moving proton looks as a dense gluon system!

Deep inelastic scattering (DIS) of electron off proton

→ Internal structure of a proton

Gluons are dominant at small-$x = high energies$

$\gamma^*$

$1/Q$

transverse

longitudinal

$1/xP^+$

$\leftrightarrow$ higher energies

$x \sim Q^2/(Q^2+W^2)$

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High-energy limit of QCD is the **Color Glass Condensate (CGC)**!!

**COLOR**: A matter made of gluons with colors.

**GLASS**: Almost “frozen” random color source creates gluon fields

**CONDENSATE**: High density. Occupation number $\sim O(1/\alpha_s)$

Many people contributed to establish the modern picture of the gluon saturation (Gribov, Levin, Ryskin, Mueller, Qiu, McLerran, Venugopalan, Jalilian-Marian, Iancu, Weigert, Leonidov, Kovner, Balitsky, Kovchegov)

A new “semi-hard” scale: “saturation scale” $Q_s > \Lambda_{QCD}$

= typical transverse size of gluons at saturation

Saturation scale = typical transverse momentum of gluons

$\rightarrow$ weak coupling $\alpha_s(Q_s) \ll 1$

Weakly interacting many body system of gluons

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Geometric scaling as an evidence

The $\gamma^*$-proton total cross section $\sigma(Q^2, x)$ becomes a function of only one variable $\xi = Q^2/Q_s^2(x)$ at small $x$ $\sigma(Q^2, x) = f(\xi)$, with $Q_s^2(x) \sim 1/x^\lambda$, $\lambda \sim 0.3$ determined by the fit.

→ Existence of saturation scale: $Q_s$ $x$-dependence of $Q_s$ is consistent with CGC
Multiple gluon production and evolution equation in QCD
Gluon cascade: linear evolution

BFKL evolution: multiple soft gluon production

Balitsky-Fadin-Kuraev-Lipatov

\[ n(x) \propto \sum_k \frac{1}{k!} \left( \alpha_s \ln \frac{1}{x} \right)^k \sim e^{\kappa \alpha_s Y} \]

Growth of gluon number per energy (rapidity) increase

\[ \frac{dn(Y)}{dY} \propto \kappa \ n(Y) \]

\( \kappa \): growth rate

BFKL equation:
linear evolution \( \rightarrow \) rapid growth \( \rightarrow \) Unitarity violation!!
Gluon cascade: nonlinear evolution

What is missing in the BFKL dynamics?

Rapid growth = “population explosion”
  ↵ feed back effect reduces the speed of growth

When the gluon density becomes high,
produced gluons start to interact with each other!

\[
\frac{dn(Y)}{dY} \propto \kappa n(Y) - \kappa n^2(Y)
\]

Logistic equation + transv. dep
→ Balitsky-Kovchegov eq.

* g→gg (increase) vs gg → g (recombination)*

Evolution becomes nonlinear → saturation

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The Balitsky-Kovchegov equation

- **Gluon number** \( \sim \) 2pt function of Wilson lines
  \[
  V_x^+ = P e^{i g \int A_+ dx^-}
  \]

  \[
  S_Y(x_\perp, y_\perp) = \frac{1}{N_c} \left\langle \text{tr}(V_x^+ V_y) \right\rangle_Y
  \]

- **Evolution equation for 2pt operator** contains 4pt function
  \[
  \left\langle \text{tr}(V_x^+ V_z) \cdot \text{tr}(V_z^+ V_y) \right\rangle_Y
  \]

  \( \rightarrow \) **Mean-field approx.**: necessary to obtain a closed equation
  \[
  \text{tr}(V_x^+ V_z) = \left\langle \text{tr}(V_x^+ V_z) \right\rangle_Y + "\text{fluctuation}"
  \]

- **The Balitsky-Kovchegov equation**
  \[
  \partial_Y \left\langle T_{xy} \right\rangle_Y = \frac{\bar{\alpha}}{2\pi} \int d^2z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ \left\langle T_{xz} \right\rangle + \left\langle T_{zy} \right\rangle - \left\langle T_{xy} \right\rangle - \left\langle T_{xz} \right\rangle \left\langle T_{zy} \right\rangle \right]
  \]

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The Balitsky-Kovchegov equation

\[ \partial_Y \langle T_{xy} \rangle = \frac{\bar{\alpha}}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ \langle T_{xz} \rangle + \langle T_{zy} \rangle - \langle T_{xy} \rangle - \langle T_{xz} \rangle \langle T_{zy} \rangle \right] \]

Consequences

- \( \langle T_{xy} \rangle_Y \) : scattering amplitude of a color dipole \( \sim \) gluon number
- Derived from QCD in leading log accuracy \( (\alpha_s \ln 1/x) \) in the mean-field approximation
- BFKL + non-linear term
  \[ \rightarrow \langle T_{xy} \rangle_Y \text{ saturates (unitarizes) at fixed } b = (x+y)/2 : \langle T_{xy} \rangle_Y \leq 1 \]
- Saturation scale \( Q_s(Y) \) increases with rapidity \( Y \) : \( Q_s^2(Y) \sim e^{\lambda Y} \)
- Geometric Scaling \( \rightarrow \) amplitude \( \langle T_{xy} \rangle_Y \) is a function of \( (x-y)Q_s(Y) \)
  [Levin, Tuchin, Motyka, etc, etc]
- Approximate scaling persists even outside of the CGC regime
  [Iancu, KI, McLerran]

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“Phase diagram” of a proton in DIS

Non-perturbative (Regge)

Extended scaling regime

Parton gas

\( Q_s^2(x) \sim 1/x^\lambda \): grows as \( x \to 0 \)

\( Q_s^4(x)/\Lambda_{QCD}^2 \)

Higher energies

1/x in log scale

\( \Lambda_{QCD}^2 \)

Fine transverse resolution

\( Q^2 \) in log scale

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The Reaction-Diffusion dynamics


Within a reasonable approximation, the BK equation in momentum space is rewritten as the F-KPP equation (Fisher, Kolmogorov, Petrovsky, Piscounov)

\[ \partial_t u = \partial_x^2 u + u - u^2 \]

where \( t \sim Y, \ x \sim \ln k_t^2 \) and \( u(t, x) \sim 1 - \langle T(k) \rangle_Y \).

More precisely,…

The BK equation in momentum space \( L = \log(k^2/k_0^2) \)

\[ \partial_Y N = \bar{\alpha} \chi (-\partial_L) N - \bar{\alpha} N^2 \]

where \( \bar{\alpha} = \alpha_s N_c / \pi, \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \)

expand \( \chi \) around its saddle point \( \gamma=1/2 \) to second order

\[ \bar{\chi} (-\partial_L) = \chi(\frac{1}{2}) + \frac{\chi''(\frac{1}{2})}{2} \left( \partial_L + \frac{1}{2} \right)^2 \]

\[ \omega = \chi(\frac{1}{2}), \ D = \chi''(\frac{1}{2}), \text{ and } \bar{\gamma} = 1 - \frac{1}{2} \sqrt{1 + 8\omega/D}, \]

\[ t = \frac{\bar{\alpha} D}{2} (1-\bar{\gamma})^2 Y, \quad x = (1-\bar{\gamma}) \left( L + \frac{\bar{\alpha} D}{2} Y \right), \]

\[ u(t, x) = \frac{2}{D(1-\bar{\gamma})^2} N \left( \frac{2t}{\bar{\alpha} D (1-\bar{\gamma})^2}, \frac{x}{1-\bar{\gamma}} - \frac{t}{(1-\bar{\gamma})^2} \right) \]
The Reaction-Diffusion dynamics

FKPP eq. \[ \partial_t u = \partial_x^2 u + u - u^2 \] = “reaction” + “diffusion”

Famous equation in non-equilibrium statistical physics with broad range of application: pattern formulation, expansion of epidemic, chemical reaction, etc

**Reaction** : logistic growth (g \rightarrow gg, vs gg \rightarrow g)
**Diffusion** : expansion of stable region

→ Traveling wave solution

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For a “traveling wave” solution, one can define the position of a “wave front” \( x(t) = v(t)t \).

\[ x(t) \sim \ln Q_s^2(Y) \text{ Saturation scale!} \]

- “Boundary” btw dilute and saturated regimes
- Precise form of \( Q_s(Y) \) determined
  \[ Q_s^2(x) \propto \left(\frac{1}{x}\right)^\lambda = e^{2Y} \]
  
  NLO BFKL: \( \lambda \sim 0.3 \quad x = 10^{-2} - 10^{-4} \)

At late time, the shape of a traveling wave is preserved, and the solution is only a function of \( x - vt \).

\[ x - v(t)t \sim \ln k^2/Q_s^2(Y) \text{ Geometric scaling!!} \]

- Observed in HERA DIS at small \( x \) [Stasto,Golec-Biernat,Kwiecinski]
- \( Q_s(Y) \) from the data consistent with theoretical results.
The Reaction-Diffusion dynamics
Dynamics in 1 dimension

Splitting $A \rightarrow AA$ and merging $AA \rightarrow A$ occur at each site.

**Diffusion**: Hopping to the right and left

$N$ : allowed number of particles at each site

Equation for $n(i)$: the number of particles at site $i$

mean-field approximation $\rightarrow$ FKPP equation
The Reaction-Diffusion dynamics (2/9)

**Exact implementation: Master equation**

\[ n_i(t) = \text{number of particles at site } i, \quad \{n_i\} = (n_0, n_1, n_2, \ldots) \text{ configuration} \]

\[ P(\{n_i\}; t) : \text{probability to have a particular configuration } \{n_i\} \text{ at time } t \]

Change of \( P(\{n_i\}; t) \)

1) **Diffusion** (hopping to the right \( i \rightarrow i+1 \), and to the left \( i \rightarrow i-1 \)) at rate \( D/h^2 \)

\[
\frac{D}{h^2} \sum_{<i,j>} \left[ (n_i + 1)P(...n_{i+1}, n_{j-1},...; t) + (n_j + 1)P(...n_{i-1}, n_{j+1},...; t) \right] - \frac{D}{h^2} \sum_{<i,j>} \left[ n_i P(\{n\}; t) + n_j P(\{n\}; t) \right]
\]

\[ \text{gain} \quad \leftarrow \text{loss} \]

2) **Splitting** (A \( \rightarrow \) AA) at each site \( i \) at rate \( \lambda_s \)

\[ \lambda_s \sum_i \left[ (n_i - 1)P(...n_{i-1},...; t) - n_i P(\{n\}; t) \right] \]

\[ \text{gain} \quad \text{loss} \]

3) **Merging** (AA \( \rightarrow \) A) at each site \( i \) at rate \( \lambda_m \)

\[ \lambda_m \sum_i \left[ \frac{n_i(n_i + 1)}{2} P(...n_{i+1},...; t) - \frac{n_i(n_i - 1)}{2} P(\{n\}; t) \right] \]

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Second quantization method  

Doi ’76, Peliti ’85, Cardy & Tauber ’98

a useful technique for a system with creation and annihilation of particles

Introduce bosonic operators \( a_i^+ \) creates a particle at site \( i \).

\[
[a_i, a_j^+] = \delta_{ij} \quad [a_i, a_j] = [a_i^+, a_j^+] = 0
\]

(Non-hermite) “Hamiltonian” is defined for a “probability vector”

\[
-\partial_t |\Phi(t)\rangle = H |\Phi(t)\rangle, \quad |\Phi(t)\rangle = \sum_{\{n\}} P(\{n\};t) |\{n\}\rangle
\]

For \( A \rightarrow AA \) and \( AA \rightarrow A \), the master equation is exactly reproduced by

\[
H = \frac{D}{\hbar^2} \sum_{<i,j>} (a_i^+ - a_j^+) (a_i - a_j) - \lambda_s \sum_i \left( (a_i^+)^2 a_i - a_i^+ a_i \right) - \frac{\lambda_m}{2} \sum_i \left( a_i^+ a_i^2 - (a_i^+)^2 a_i^2 \right)
\]

Can construct a “field theory” by using coherent state path-integral

\[
S = \int dx dt \left[ \bar{\phi} \left( \partial_t - D \nabla^2 \right) \phi - g_s (\bar{\phi} \phi + \bar{\phi}^2 \phi) + g_m (\bar{\phi}^2 \phi^2 + \bar{\phi}^2 \phi^2) \right]
\]

where \( \phi \) is a coherent state eigen-value of annihilation operator, \( a_i |\phi_i\rangle = \phi_i |\phi_i\rangle \)

\( g_s = \lambda_s, \quad g_m = \lambda_m \hbar/2 \),  particle number (density) : \( n_i = \langle a_i^+ a_i \rangle \sim \phi_i \)
The Reaction-Diffusion dynamics (4/9)

Deterministic & stochastic F-KPP equations

\[ S = \int dxdt \left[ \overline{\phi} (\hat{\partial}_t - D \nabla^2) \phi - \overline{\phi} (g_s \phi - g_m \phi^2) - \overline{\phi^2} (g_s \phi - g_m \phi^2) \right] \]

Introduce a Gaussian noise \( \eta(x,t) \) as an auxiliary field to resolve the last “fluctuation” term

\[ e^{\int \overline{\phi^2} (\phi - \kappa \phi^2)} = \int D\eta \, e^{-\frac{1}{2} \eta^2 + \eta \phi \sqrt{2(\phi - \kappa \phi^2)}} \]

EOM \( \rightarrow \) Stochastic F-KPP equation (after rescaling)

\( (\hat{\partial}_\tau - \nabla^2)\phi - (\phi - \phi^2) - \sqrt{2(\phi - \phi^2)} / N \cdot \eta(x,t) = 0 \)

\[ \langle \tilde{\eta}(\tau,\xi) \tilde{\eta}(\tau,\xi) \rangle = \zeta(\tau - \tau, \xi - \xi) \quad \text{Gaussian noise} \]

In the limit \( N = g_s / g_m \rightarrow \) infinity, reduces to the FKPP equation
Two major modifications to deterministic FKPP

1. Front velocity becomes slower.

2. The shape of the traveling wave does not change a lot, but the position of the front becomes stochastic (Gaussian).

Enberg, Golec-Biernat, Munier PRD72 (05)
**Discreteness** becomes important when the number of particles are few.

→ At the tail of a traveling wave: \( \varphi(x,t) \sim 1/N \ll 1 \)

→ large effect: *Diffusion* controls the propagation because linear growth does not work without “seeds”.

→ The velocity of a traveling wave is reduced.

Brunet, Derrida, PRE ’97

FKPP with a cutoff

\[
(\partial_t - \nabla_x^2)\varphi - (\varphi - \varphi^2)\Theta(\varphi - 1/N) = 0.
\]

\[
v = v_0 - \Delta = v_0 - \frac{1}{2}v''(\gamma_0)\frac{\pi^2\gamma_0^2}{\ln^2 N}.
\]
Stochastic front position

1. Treat fluctuation as *perturbation* to the *cutoff* FKPP equation

\[
F[\varphi] \equiv \nabla_x^2 \varphi + (\varphi - \varphi^2) \Theta(\varphi - 1/N),
\]

\[
\Delta F[\varphi] \equiv \epsilon \sqrt{2(\varphi - \varphi^2)/N \cdot \eta},
\]

\(\varphi_0(z=x-v\tau)\) : unperturbed solution, scaled \(\leftarrow \) cutoff FKPP

\(\delta \varphi(z, \tau): \) perturbed, not scaled, to be determined

**Linearized equation**

\[
\frac{\partial}{\partial \tau} \delta \phi(z, \tau) = v \frac{\partial}{\partial z} \delta \phi + \frac{\delta F}{\delta \varphi}_{\varphi=\varphi_0} \delta \phi + \Delta F[\varphi_0](z, \tau).
\]

2. Equation for \(\delta \varphi\) can be solved by introducing Green's function

\[
\delta \phi(z, \tau) = \int d\tau' \int dz' G(z, \tau|z', \tau') \Delta F[\varphi_0](z', \tau')
\]

where

\[
\frac{\partial}{\partial \tau} G(z, \tau|z', \tau') + \mathcal{H}[\varphi_0] G(z, \tau|z', \tau') = \delta(z-z') \delta(\tau-\tau'),
\]

Green function is associated with *fluctuation of the (cutoff) FKPP equation.*

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3. **Stability analysis** of the (cutoff) FKPP equation

\[
G(z, \tau | z', \tau') = e^{-\frac{\lambda}{2}(z-z')^2} \sum_{n=0}^{\infty} e^{-\omega_n(\tau-\tau')} \xi_n(z) \xi_n^*(z') \theta(\tau - \tau')
\]

Spectrum of the fluctuation around the FKPP solution

\[
h \xi_n = \omega_n \xi_n, \quad h \equiv -\frac{\partial^2}{\partial z^2} + \frac{v^2}{4} - \frac{\delta}{\delta \varphi} \left\{ (\varphi - \varphi^2) \Theta(\varphi - \frac{1}{N}) \right\} \bigg|_{\varphi = \varphi_0}
\]

Dominant fluctuation around \( \varphi_0 \rightarrow \text{zero mode} \)

\[
\xi_0(z) = C e^{\frac{vz}{2}} \phi_0'(z)
\]

Zero mode is due to the translational invariance.

induces the shift of the solution!!

4. **This zero mode** couples to the external noise term

The front position \( \delta X(\tau) \) due to the noise is proportional to the noise

\[
\delta X(t) = C^2 \int' dt' \int dz' e^{\frac{vz'}{2}} \frac{d\varphi_0}{dz}(z') \sqrt{2(\varphi_0 - \varphi_0^2) / N} \eta(z', t')
\]

can easily compute the diffusion coefficient \( < \delta X(\tau)^2> = D \tau \)
Stochastic front position

Comments...
1. Already done some years ago… ex) Rocco, et al. PRE65(2001)012102
   Exactly the same formula for the displacement $\delta X$ !!

2. Not the same as the diffusion coefficient obtained by the other method
   [Brunet, Derrida, Mueller, Munier, PRE73(2006)056126]

   $D = \pi^2 \gamma_0^3 v''(\gamma_0) \frac{\pi^2/3}{\gamma_0^2 \ln^3 N}$

   ! Our result shows different asymptotic behavior $D \sim 1/\ln^6 N$
   $\leftarrow$ because of different definition of the front?? [Panja ’03]

3. Possible to treat the front position as a collective coordinate
   in the field theory representation [work in progress]
   (cf: Zero-mode fluctuations of a soliton are treated as collective coordinates)
   $\rightarrow$ can go beyond the linear response theory??

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Translate everything into CGC language
Effects of fluctuation on the BK eq.

"Mean-field BK equation = Deterministic FKPP equation"

Equation representing “full” reaction-diffusion dynamics with $A \rightarrow AA$ and $AA \rightarrow A$ is not the FKPP equation, but the stochastic FKPP equation.

Need to find an equation representing the full dynamics of both the Pomeron merging and splitting. (BK contains the effects of splitting only partially)

1. Effects of fluctuation is significant when gluon (dipole) number is small (high transverse momentum).
   \[ \langle T(r) \rangle_y \approx \alpha_s^2 n(r, Y) \ll \alpha_s^2 \]

2. Add 3 Pomeron vertex (Pomeron splitting) which becomes important in the dilute regime (in scattering of 2 dipoles)

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Effects of fluctuation on the BK eq.

2. Construct the evolution equation so that the BK + noise term can correctly reproduce the evolution equation of the 2 dipoles → Stochastic BK eq. [Iancu-Triantafyllopoulos]

$$\frac{\partial T_Y(x, y)}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int_z \left[ M_{xyz} \otimes T_Y(x, y) - M(x, y, z) T_Y(x, z) T_Y(z, y) \right]$$

$$+ \frac{\alpha_s}{2\pi} \sqrt{\frac{\bar{\alpha}_s}{2\pi}} \int_{u,v,z} A_0(x, y|u, z) \frac{|u-v|}{(u-z)^2} \sqrt{\nabla_u^2 \nabla_v^2 T_Y(u, v)} \nu(u, v, z, Y)$$

3. Stochastic BK reduces to Stochastic FKPP in the diffusive approximation.

4. Saturation scale becomes slowly increasing due to diffusion at the edge, stochastic variable due to fluctuation term in sFKPP eq. [Iancu,Mueller,Munier]

5. The stochastic saturation scale is induced by the zero mode fluctuation of the BK equation, which is related to the scale transformation in 2 dimensional transverse momentum.

6. The “field theory” representation of the reaction-diffusion dynamics is very similar to the Reggeon field theory.

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Summary

• At very high energy, a proton (in fact, any hadrons) looks as the Color Glass Condensate, a densely saturated gluonic system. This is a weakly interacting many body state.

• Its dynamics is essentially equivalent to the reaction-diffusion dynamics. The BK equation ~ the FKPP equation. Rich information from the statistical physics is available.

• In particular, the effects of fluctuation beyond the mean-field BK picture have been recognized to be significant in dilute regime (at high transverse momentum)

• Slowly-growing and stochastic saturation scale is obtained.