Initial particle production in high-energy nucleus-nucleus collisions

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Typical e+e- or pp collision
Why is QCD predictive there?

- Why is QCD predictive despite the fact that hadrons are non-perturbative bound states?

- **Factorization:**

  - When we add one extra loop to the hard part, all the divergences that remain after having summed over degenerate final states can be absorbed into the evolution of the parton distributions.

- **Universality:** the structure of these divergences is independent of the nature of the colliding hadrons and of the process under consideration; universal distributions.
Typical nucleus-nucleus collision

- Very high multiplicity (1000s of particles)
- 99% of the multiplicity below $p_\perp \sim 3$ GeV
Goals

- Describe nucleus-nucleus collisions within QCD

- Generalize the concept of “parton distribution”
  - Due to the high density of partons, we expect to need higher correlations (beyond the usual parton distributions, which are 2-point correlation functions)

- If divergences show up in loop corrections, one should be able to factor them out into the evolution of these distributions

- These distributions should be universal

- The non-perturbative information is relegated into the initial condition for this evolution

- Find which observables are “infrared safe”
Stages of a nucleus-nucleus collision

$z = -t$

$z = t$

$z$ (beam axis)
Stages of a nucleus-nucleus collision

- Stages of a nucleus-nucleus collision

- t

- z (beam axis)
Stages of a nucleus-nucleus collision

- Strong fields \( \rightarrow \) classical EOMs
- \( z \) (beam axis)
Stages of a nucleus-nucleus collision

- Gluons & quarks out of eq. → kinetic theory
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- gluons & quarks in eq. $\rightarrow$ hydrodynamics
- $z$ (beam axis)
Stages of a nucleus-nucleus collision

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- gluons & quarks in eq. $\rightarrow$ hydrodynamics

-hadrons in eq. $\rightarrow$ beam axis $z$

$\rightarrow t$
Stages of a nucleus-nucleus collision

- strong fields \( \rightarrow \) classical EOMs
- gluons & quarks out of eq. \( \rightarrow \) kinetic theory
- gluons & quarks in eq. \( \rightarrow \) hydrodynamics
- freeze out
- hadrons in eq.
- \( z \) (beam axis)
Stages of a nucleus-nucleus collision

- calculate the initial production of semi-hard particles
- prepare the stage for kinetic theory
Outline

Nucleons at high energy
- Basic principles and bookkeeping
- Inclusive gluon spectrum at leading order
- 1-loop corrections and resummations

- FG, Venugopalan, hep-ph/0601209, 0605246
- FG, Jeon, Venugopalan, in preparation
  + work in progress with Lappi, Venugopalan and Fukushima, McLerran
Nucleons at high energy
Nucleon at rest

- Very complicated non-perturbative object...
- Contains fluctuations at all space-time scales smaller than its own size
- Only the fluctuations that are longer lived than the external probe participate in the interaction process
- The only role of short lived fluctuations is to renormalize the masses and couplings
- Interactions are very complicated if the constituents of the nucleon have a non-trivial dynamics over time-scales comparable to those of the probe
Dilation of all internal time-scales of the nucleon

Interactions among constituents now take place over time-scales that are longer than the characteristic time-scale of the probe. The constituents behave as if they were free.

Many fluctuations live long enough to be seen by the probe. The nucleon appears denser at high energy.

Pre-existing fluctuations are totally frozen over the time-scale of the probe, and act as static sources of new partons.

In a nucleus, soft gluons (long wavelength) belonging to different nucleons overlap in the longitudinal direction. Coherent effects (known as saturation)
Degrees of freedom and their interplay


- Soft modes have a large occupation number
  - they are described by a classical color field \( A^\mu \) that obeys Yang-Mills’s equation:

\[
[D_\nu, F^{\nu\mu}]_a = J_\mu^a
\]

- The source term \( J_\mu^a \) comes from the faster partons. The hard modes, slowed down by time dilation, are described as frozen color sources \( \rho_a \). Hence:

\[
J_\mu^a = \delta^{\mu+}(x^-) \rho_a(x_{\perp}) \quad (x^- \equiv (t - z)/\sqrt{2})
\]

- The color sources \( \rho_a \) are random, and described by a distribution functional \( W_Y[\rho] \), with \( Y \) the rapidity that separates “soft” and “hard”. Evolution equation (JIMWLK):

\[
\frac{\partial W_Y[\rho]}{\partial Y} = \mathcal{H}[\rho] \ W_Y[\rho]
\]
In order to study the collisions of two hadrons at leading order, solve the classical Yang-Mills equations in the presence of the following current:

\[ J^\mu \equiv \delta^{\mu+} \delta(x^-) \rho_1(\vec{x}_\perp) + \delta^{\mu-} \delta(x^+) \rho_2(\vec{x}_\perp) \]

Compute the observable \( O \) of interest in the background field created by a configuration of the sources \( \rho_1, \rho_2 \). Note: the sources are of order \( 1/g \) \( \triangleright \) very non-linear problem

Average over the sources \( \rho_1, \rho_2 \)

\[ \langle O \rangle_Y = \int \left[ D \rho_1 \right] \left[ D \rho_2 \right] W_{Y_{\text{beam}} - Y}[\rho_1] W_{Y + Y_{\text{beam}}}[\rho_2] \mathcal{O}[\rho_1, \rho_2] \]

Note: the boundary conditions depend on the observable
Basic principles
Main difficulties

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- **Dense regime**: non linearities are important
- Many gluons can be produced from the same diagram
- There can be many simultaneous disconnected diagrams
- Some of them may not produce anything (vacuum diagrams)
- All these diagrams can have loops (not at LO though)
In the saturated regime, the sources are of order $1/g$

The order of each disconnected diagram is given by:

$$\frac{1}{g^2} \cdot g \# \text{produced gluons} \cdot g^2(\# \text{loops})$$

The total order of a graph is the product of the orders of its disconnected subdiagrams ▷ quite messy...
Bookkeeping
Consider squared amplitudes (including interference terms) rather than the amplitudes themselves.
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Bookkeeping

- Consider **squared amplitudes** (including interference terms) rather than the amplitudes themselves
- See them as **cuts through vacuum diagrams**
- Consider **only the simply connected** ones, thanks to:

\[
\sum \left( \text{all the vacuum diagrams} \right) = \exp \left\{ \sum \left( \text{simply connected vacuum diagrams} \right) \right\}
\]

- Simpler power counting for connected vacuum diagrams:

\[
\frac{1}{g^2} g^2 (\# \text{ loops})
\]
There is an operator $D$ that acts on a pair of vacuum diagrams by removing two sources and attaching a cut propagator instead:
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![Diagram]

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- $\mathcal{D}$ can also act directly on single diagram, if it is already cut
- By repeated action of $\mathcal{D}$, one can generate all the diagrams with an arbitrary number of cuts
- Thanks to this operator, one can write:

$$ P_n = \frac{1}{n!} \mathcal{D}^n e^{iV} e^{-iV^*} , \quad iV = \sum \left( \text{connected uncut vacuum diagrams} \right) $$

$$ \sum \left( \text{all the cut vacuum diagrams} \right) = e^{\mathcal{D}} e^{iV} e^{-iV^*} $$
Inclusive gluon spectrum
First moment of the distribution

- It is easy to express the average multiplicity as:

\[
\overline{N} = \sum_n n \, P_n = \mathcal{D} \left\{ e^\mathcal{D} \, e^{iV} \, e^{-iV^*} \right\}
\]

- \(\overline{N}\) is obtained by the action of \(\mathcal{D}\) on the sum of all the cut vacuum diagrams. There are two kind of terms:
  - \(\mathcal{D}\) picks two sources in two distinct connected cut diagrams
  - \(\mathcal{D}\) picks two sources in the same connected cut diagram
Gluon multiplicity at LO

- At LO, only tree diagrams contribute the second type of topologies can be neglected (it starts at 1-loop)

- In each blob, we must sum over all the tree diagrams, and over all the possible cuts:

\[ \overline{N}_{LO} = \sum_{\text{trees}} \sum_{\text{cuts}} \]

- A major simplification comes from the following property:

\[ \sim \sim \sim + \sim \times \sim = \text{retarded propagator} \]

- The sum of all the tree diagrams constructed with retarded propagators is the retarded solution of Yang-Mills equations:

\[ [D_{\mu}, F^{\mu\nu}] = J^\nu \quad \text{with} \quad A^\mu (x_0 = -\infty) = 0 \]
Gluon multiplicity at LO


\[ \frac{dN_{LO}}{dY d^2p_\perp} = \frac{1}{16\pi^3} \int_{x,y} e^{ip \cdot (x-y)} \mathcal{D}_{x y} \sum_{\lambda} \epsilon_{x \lambda}^{\mu} \epsilon_{y \lambda}^{\nu} A_{\mu}(x) A_{\nu}(y) \]

At LO: Yang-Mills equations with null retarded boundary conditions
Gluon multiplicity at LO


\[
\frac{d\bar{N}_{LO}}{dY d^2\vec{p}_\perp} = \frac{1}{16\pi^3} \int_{x,y} e^{i\vec{p} \cdot (x-y)} \Box_x \Box_y \sum_{\lambda} \epsilon_\lambda^\mu \epsilon_\lambda^\nu A_\mu(x) A_\nu(y)
\]

- At LO: Yang-Mills equations with null retarded boundary conditions
  - initial value problem on the light-cone
Gluon multiplicity at LO

- Lattice artefacts at large momentum (they do not affect much the overall number of gluons)
- Important softening at small $k_\perp$ compared to pQCD (saturation)
Initial conditions and boost invariance

- **Gauge condition**: \( x^+ A^- + x^- A^+ = 0 \)

\[
\begin{align*}
A^i(x) &= \alpha^i(\tau, \eta, \vec{x}_\perp) \\
A^\pm(x) &= \pm x^\pm \beta(\tau, \eta, \vec{x}_\perp)
\end{align*}
\]

- Initial values at \( \tau = 0^+ \): \( \alpha^i(0^+, \eta, \vec{x}_\perp) \) and \( \beta(0^+, \eta, \vec{x}_\perp) \) do not depend on the rapidity \( \eta \)

- \( \alpha^i \) and \( \beta \) remain independent of \( \eta \) at all times (invariance under boosts in the \( z \) direction)

- numerical resolution performed in \( 1 + 2 \) dimensions
Local anisotropy

- After some time, the gluons have a longitudinal velocity tied to their space-time rapidity by \( v_z = \tanh(\eta) \):
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\[
v_z = \tanh(\eta)
\]

▷ at late times: if particles fly freely, only one longitudinal velocity can exist at a given \( \eta \): 
\[
v_z = \tanh(\eta)
\]
1-loop corrections and summation of leading divergences
1-loop contributions to $N$

- 1-loop diagrams for $\overline{N}$
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1-loop contributions to $N$

The 1-loop corrections to $\overline{N}$ can be written in terms of the generator $T(\vec{x})$ of translations of the initial condition at the point $\vec{x}$ of the light-cone:

\[
\delta \overline{N}_1 = \int_{\vec{x} \in \text{light cone}} \left[ \delta A(\vec{x}) T(\vec{x}) \right] \overline{N}_{LO}
\]

\[
\delta \overline{N}_2 = \int \frac{1}{2} \left[ \int \zeta(\vec{x}) \zeta^*(\vec{y}) T(\vec{x}) T(\vec{y}) \right] \overline{N}_{LO}
\]
If taken at face value, the 1-loop corrections are plagued by two kinds of divergences:

- The two coefficients $\delta A(\vec{x})$ and $\int \zeta_\lambda(\vec{x})\zeta^*_\lambda(\vec{y})$ are infinite, because of an unbounded integration over a rapidity variable.

- At late times, $T(\vec{x})A(\tau, \vec{y})$ diverges exponentially,

$$T(\vec{x})A(\tau, \vec{y}) \sim e^{\sqrt{\mu \tau}}$$

because of an instability of the classical solution of Yang-Mills equations against non-boost invariant perturbations (Romatschke, Venugopalan (2005)).

We can decompose the 1-loop correction $\delta \overline{N}$ as

$$\delta \overline{N} = \begin{bmatrix} \delta \overline{N} \end{bmatrix}_{\text{finite}} + \begin{bmatrix} \delta \overline{N} \end{bmatrix}_{\text{divergent coefficients}} + \begin{bmatrix} \delta \overline{N} \end{bmatrix}_{\text{unstable modes}}$$
Initial state factorization

- Anatomy of the full calculation:

\[
\begin{align*}
W_{\text{beam}-Y}[\rho_1] & \quad N[A_{\text{in}}(\rho_1, \rho_2)] & \quad W_{\text{beam}+Y}[\rho_2]
\end{align*}
\]
Initial state factorization

Anatomy of the full calculation:

\[
\begin{align*}
W_{Y\text{beam}}^{-Y}[\rho_1] \\
N[A_{\text{in}}(\rho_1, \rho_2)] + \delta N \\
W_{Y\text{beam}}^{+Y}[\rho_2]
\end{align*}
\]

When the observable \( N[A_{\text{in}}(\rho_1, \rho_2)] \) is corrected by an extra gluon, one gets divergences of the form \( \alpha_s \int dY \) in \( \delta N \). One would like to be able to absorb these divergences into the \( Y \) dependence of the source densities \( W_Y[\rho_{1,2}] \).
Initial state factorization

- Anatomy of the full calculation:

\[
\begin{align*}
W_{Y_{\text{beam}} - Y_0}[\rho_1] & \\
N[A_{\text{in}}(\rho_1, \rho_2)] + \delta N & \\
W_{Y_{\text{beam}} + Y_0'}[\rho_2]
\end{align*}
\]

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- Equivalently, if one puts some arbitrary frontier \( Y_0 \) between the “observable” and the “source distributions”, the dependence on \( Y_0 \) should cancel between the various factors.
Initial state factorization

- The two kind of divergences don’t mix, because the divergent part of the coefficients is boost invariant. Given their structure, the divergent coefficients seem related to the evolution of the sources in the initial state.

- In order to prove the factorization of these divergences in the initial state distributions of sources, one needs to establish:

\[
\left[ \delta \mathcal{N} \right]_{\text{divergent coefficients}} = \left( Y_0 - Y \right) \mathcal{H}^\dagger [\rho_1] + \left( Y - Y'_0 \right) \mathcal{H}^\dagger [\rho_2] \quad \mathcal{N}_{LO}
\]

where \( \mathcal{H}[\rho] \) is the Hamiltonian that governs the rapidity dependence of the source distribution \( W_Y[\rho] \):

\[
\frac{\partial W_Y[\rho]}{\partial Y} = \mathcal{H}[\rho] \ W_Y[\rho]
\]

FG, Lappi, Venugopalan (work in progress)
Initial state factorization

Why is it plausible?

- Reminder:

\[
\left[ \delta N \right]_{\text{divergent coefficients}} = \left\{ \int_{\tilde{x}} \left[ \delta A(\tilde{x}) \right]_{\text{div}} T(\tilde{x}) + \frac{1}{2} \int_{\tilde{x}, \tilde{y}} \left[ \int_{\lambda} \zeta_\lambda(\tilde{x}) \zeta_{\lambda}^*(\tilde{y}) \right]_{\text{div}} T(\tilde{x}) T(\tilde{y}) \right\} \overline{N}_{LO}
\]

- Compare with the evolution Hamiltonian:

\[
H[\rho] = \int_{\tilde{x}_\perp} \sigma(\tilde{x}_\perp) \frac{\delta}{\delta \rho(\tilde{x}_\perp)} + \frac{1}{2} \int_{\tilde{x}_\perp, \tilde{y}_\perp} \chi(\tilde{x}_\perp, \tilde{y}_\perp) \frac{\delta^2}{\delta \rho(\tilde{x}_\perp) \delta \rho(\tilde{y}_\perp)}
\]

- The coefficients $\sigma$ and $\chi$ in the Hamiltonian are well known. There is a well defined calculation that will tell us if it works...
Unstable modes

- The coefficient $\delta A(\vec{x})$ is boost invariant.
  Hence, the divergences due to the unstable modes all come from the quadratic term in $\delta \overline{N}$:

$$\left[ \delta \overline{N} \right]_{\text{unstable modes}} = \left\{ \frac{1}{2} \int_{\vec{x}, \vec{y}} \Sigma(\vec{x}, \vec{y}) \; T(\vec{x})T(\vec{y}) \right\} \overline{N}_{\text{LO}} [A_{\text{in}}(\rho_1, \rho_2)]$$

  with

$$\Sigma(\vec{x}, \vec{y}) = \int_{\lambda} \zeta_{\lambda}(\vec{x}) \zeta_{\lambda}^*(\vec{y})$$

- When summed to all orders, these divergences exponentiate:

$$\left[ \delta \overline{N} \right]_{\text{unstable modes}} = e^{\frac{1}{2} \int_{\vec{x}, \vec{y}} \Sigma(\vec{x}, \vec{y}) \; T(\vec{x})T(\vec{y})} \; \overline{N}_{\text{LO}} [A_{\text{in}}(\rho_1, \rho_2)]$$
Unstable modes

- This can be arranged in a more intuitive way:

\[
\left[ \delta \bar{N} \right]_{\text{unstable modes}} = \int [D\xi] e^{\frac{i}{2} \int \bar{\alpha}, \bar{\eta} \frac{\xi(\bar{\alpha}) \xi(\bar{\eta})}{\Sigma(\bar{\alpha}, \bar{\eta})}} e^{i \int \bar{\alpha} \xi(\bar{\alpha}) T(\bar{\alpha})} \bar{N}_{LO} [A_{\text{in}}(\rho_1, \rho_2)]
\]

\[
\bar{Z}_{\text{fluct}}[\xi] = \int [D\xi] \bar{Z}_{\text{fluct}}[\xi] \bar{N}_{LO} [A_{\text{in}}(\rho_1, \rho_2) + \xi]
\]

- summing these divergences simply requires to add Gaussian fluctuations to the initial condition for the classical problem

- Interpretation:
  Despite the fact that the fields are coupled to strong sources, the classical approximation alone is not good enough, because the classical solution has unstable modes that can be triggered by the quantum fluctuations
Unstable modes
Unstable modes
Unstable modes

Combining everything, one should write

\[
\frac{dN}{dY \, d^2 \vec{p}_\perp} = \int [D\rho_1] [D\rho_2] \ W_{Y_{\text{beam}} - Y}[^1\rho_1] \ W_{Y_{\text{beam}} + Y}[^1\rho_2] \\
\times \int [D\xi] \ Z_{\text{fluct}}[^1\xi] \ \frac{dN[^1\mathcal{A}_{\text{in}}(\rho_1, \rho_2) + \xi]}{dY \, d^2 \vec{p}_\perp}
\]

▷ This formula resums the divergences that occur at Leading Order
Conclusions
Conclusions

- When the parton densities in the projectiles are large, the study of particle production becomes rather involved
  - Many disconnected diagrams contribute
  - Vacuum diagrams play a crucial role

- At Leading Order, the inclusive gluon spectrum can be calculated from a classical solution with retarded boundary conditions

- However, the 1-loop corrections bring divergences:
  - Large logs of $1/x$
  - Unstable modes of the classical solution

- Resummation of the leading divergences to all orders:
  - Evolution with $Y$ of the distribution of sources
  - Quantum fluctuations on top of initial condition for the classical solution in the forward light-cone

- The devil is in the details, and many need to be checked...
Extra bits

- Parton saturation
- Generating function
- Quark production
- Exclusive processes
- Link with Kinetic Theory
assume that the projectile is big, e.g. a nucleus, and has many valence quarks (only two are represented)

- on the contrary, consider a small probe, with few partons

- at low energy, only valence quarks are present in the hadron wave function
Parton evolution

▷ when energy increases, new partons are emitted

▷ the emission probability is $\alpha_s \int \frac{dx}{x} \sim \alpha_s \ln\left(\frac{1}{x}\right)$, with $x$ the longitudinal momentum fraction of the gluon

▷ at small-$x$ (i.e. high energy), these logs need to be resummed
as long as the density of constituents remains small, the evolution is **linear**: the number of partons produced at a given step is proportional to the number of partons at the previous step (BFKL)
eventually, the partons start overlapping in phase-space

parton recombination becomes favorable

after this point, the evolution is non-linear:
the number of partons created at a given step depends non-linearly on the number of partons present previously
Saturation criterion

Gribov, Levin, Ryskin (1983)

- Number of gluons per unit area:
  \[ \rho \sim \frac{xG_A(x, Q^2)}{\pi R_A^2} \]

- Recombination cross-section:
  \[ \sigma_{gg \rightarrow g} \sim \frac{\alpha_s}{Q^2} \]

- Recombination happens if \( \rho \sigma_{gg \rightarrow g} \gtrsim 1 \), i.e. \( Q^2 \lesssim Q_s^2 \), with:
  \[ Q_s^2 \sim \frac{\alpha_s xG_A(x, Q_s^2)}{\pi R_A^2} \sim A^{1/3} \frac{1}{x^{0.3}} \]

- At saturation, the phase-space density is:
  \[ \frac{dN_g}{d^2\vec{x}_\perp d^2\vec{p}_\perp} \sim \frac{\rho}{Q^2} \sim \frac{1}{\alpha_s} \]
Saturation domain

\[ \log(x^{-1}) \quad \log(Q^2) \]

\[ \Lambda_{QCD} \]
Definition

- One can encode the information about all the probabilities $P_n$ in a generating function defined as:

$$F(z) \equiv \sum_{n=0}^{\infty} P_n z^n$$

- From the expression of $P_n$ in terms of the operator $D$, we can write:

$$F(z) = e^{zD} e^{iV} e^{-iV^*}$$

- Reminder:
  - $e^{zD} e^{iV} e^{-iV^*}$ is the sum of all the cut vacuum diagrams
  - The cuts are produced by the action of $D$

- Therefore, $F(z)$ is the sum of all the cut vacuum diagrams in which each cut line is weighted by a factor $z$
What would it be good for ?

Let us pretend that we know the generating function $F(z)$. We could get the probability distribution as follows:

$$P_n = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-in\theta} F(e^{i\theta})$$

Note: this is trivial to evaluate numerically by a FFT:
F(z) at Leading Order

- We have: \( F'(z) = D \{ e^{zD} e^{iV} e^{-iV^*} \} \)

- By the same arguments as in the case of \( \overline{N} \), we get:

\[
\frac{F'(z)}{F(z)} = \frac{F(z)}{F(z)} + \frac{F(z)}{F(z)}
\]

- The major difference is that the sum of cut graphs that must be evaluated have a factor \( z \) attached to each cut line.

- At tree level (LO), we can write \( F'(z)/F(z) \) in terms of solutions of the classical Yang-Mills equations, but these solutions are not retarded anymore, because:

\[
\frac{F(z)}{F(z)} + z \frac{F(z)}{F(z)} \neq \text{retarded propagator}
\]
The derivative $F'/F$ has an expression which is formally identical to that of $\langle n_{\text{gluons}} \rangle$,

$$
\frac{F'(z)}{F(z)} = \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_p} \int_{x,y} e^{i\vec{p} \cdot (x-y)} \delta(x-y) \sum_{\lambda} \epsilon_\lambda^\mu \epsilon_\lambda^\nu A_\mu^{(+)}(x) A_\nu^{(-)}(y),
$$

with $A_\mu^{(\pm)}(x)$ two solutions of the Yang-Mills equations.

If one decomposes these fields into plane-waves,

$$
A^{(\varepsilon)}_\mu(x) \equiv \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_p} \left\{ f^{(+)}(x^0, \vec{p}) e^{-i\vec{p} \cdot x} + f^{(-)}(x^0, \vec{p}) e^{i\vec{p} \cdot x} \right\}
$$

the boundary conditions are:

$$
f^{(+)}_-(\infty, \vec{p}) = f^{(-)}_+(\infty, \vec{p}) = 0
$$

$$
f^{(+)}_+(\infty, \vec{p}) = z f^{(-)}_+(\infty, \vec{p}), \quad f^{(-)}_+(\infty, \vec{p}) = z f^{(+)}_-(\infty, \vec{p})
$$

There are boundary conditions both at $x_0 = -\infty$ and $x_0 = +\infty$ ☢ not an initial value problem ☢ hard+++...
Remarks on factorization

- As we’ve seen before, the fact that the calculation of the first moment $\overline{N}$ can be formulated as an initial value problem is probably going to be crucial for proving factorization.

- If this is indeed the case, then **factorization does not hold for the generating function** $F(z)$, or equivalently for the individual probabilities $P_n$.

- Preliminary studies seem to indicate that the computation of higher moments of the distribution can also be reformulated as initial value problems.
  - the moments would be factorizable as well.
Quark production


\[ E_p \frac{d\langle n_{\text{quarks}} \rangle}{d^3 \vec{p}} = \frac{1}{16\pi^3} \int_{x,y} e^{i\vec{p} \cdot (x-y)} \hat{\phi}_x \hat{\phi}_y \langle \bar{\psi}(x)\psi(y) \rangle \]

- Dirac equation in the classical color field:
Quark production


\[
E_p \frac{d\langle n_{\text{quarks}} \rangle}{d^3 \vec{p}} = \frac{1}{16\pi^3} \int_{x,y} e^{i \vec{p} \cdot (x-y)} \hat{\phi}_x \hat{\phi}_y \langle \bar{\psi}(x) \psi(y) \rangle
\]

- **Dirac equation** in the classical color field:
Spectra for various quark masses

- Parton saturation
- Generating function
- Quark production
- Exclusive processes
- Link with Kinetic Theory
Exclusive processes

- So far, we have considered only **inclusive quantities** – i.e. the \( P_n \) are defined as probabilities of producing particles anywhere in phase-space.

- What about events where a part of the phase-space remains unoccupied? e.g. **rapidity gaps**
Main issues

1. How do we calculate the probabilities $P_{excl}^n$ with an excluded region $\Omega$ in the phase-space? 
   Can one calculate the total gap probability $P_{gap} = \sum_n P_{excl}^n$?

2. What is the appropriate distribution of sources $W_{Y}^{excl}[\rho]$ to describe a projectile that has not broken up?

3. How does it evolve with rapidity?
   See: Hentschinski, Weigert, Schafer (2005)

4. Are there some factorization theorems, and for which quantities do they hold?
Exclusive probabilities

The probabilities $P_n^{\text{excl}}[\Omega]$, for producing $n$ particles – but none in the region $\Omega$ – can also be constructed from the vacuum diagrams, as follows:

$$P_n^{\text{excl}}[\Omega] = \frac{1}{n!} D^n_{\Omega} e^{iV} e^{-iV^*}$$

where $D_{\Omega}$ is an operator that removes two sources and links the corresponding points by a cut (on-shell) line, for which the integration is performed only in the region $\Omega$.

One can define a generating function,

$$F_{\Omega}(z) = \sum_n P_n^{\text{excl}}[\Omega] z^n,$$

whose derivative is given by the same diagram topologies as the derivative of the generating function for inclusive probabilities.
Exclusive probabilities

- Differences with the inclusive case:
  - In the diagrams that contribute to $\frac{F'_\Omega(z)}{F_\Omega(z)}$, the cut propagators are restricted to the region $\Omega$ of the phase-space.
    - ▶ at leading order, this only affects the boundary conditions for the classical fields in terms of which one can write $\frac{F'_\Omega(z)}{F_\Omega(z)}$.
    - ◀ not more difficult than the inclusive case.
  - Contrary to the inclusive case – where we know that $F'(1) = 0$ – the integration constant needed to go from $\frac{F'_\Omega(z)}{F_\Omega(z)}$ to $F_\Omega(z)$ is non-trivial. This is due to the fact that the sum of all the exclusive probabilities is smaller than unity.
    - ▶ $F_\Omega(1)$ is in fact the probability of not having particles in the region $\Omega$ – i.e. the gap probability.
Survival probability

- We can write:

\[
F_{\Omega}(z) = F_{\Omega}(1) \exp \left\{ \int_1^z d\tau \frac{F'_{\Omega}(\tau)}{F_{\Omega}(\tau)} \right\}
\]

- The prefactor \(F_{\Omega}(1)\) will appear in all the exclusive probabilities.

- This prefactor is nothing but the famous “survival probability” for a rapidity gap.

- One can in principle calculate it by the general techniques developed for calculating inclusive probabilities:

\[
F_{\Omega}(1) = F_{1-\Omega}^{\text{incl}}(0)
\]

- Note: it is incorrect to say that a certain process with a gap can be calculated by multiplying the probability of this process without the gap by the survival probability.
Factorization?

- Except for the case of Deep Inelastic Scattering, nothing is known regarding factorization for exclusive processes in a high density environment.

- For the overall framework to be consistent, one should have factorization between the gap probability, $F_\Omega (1)$, and the source density studied in Hentschinski, Weigert, Schafer (2005) (and the ordinary $W_Y [\rho]$ on the other side).

- The total gap probability is the “most inclusive” among the exclusive quantities one may think of. For what quantities – if any – does factorization work?
Link with Kinetic Theory

- The distribution $n(\tau_0, \vec{x}, \vec{p})$ of gluons is evaluated from the previous formula at a finite time $\tau_0$.

- After the time $\tau_0$, it evolves according to a Boltzmann equation:

$$\nu \cdot \partial_x n + g \partial_x A \cdot \partial_p n = S + C[n]$$

  - The source term $S$ comes from the fact that there are still time-dependent fields in the system, that keep producing particles.

- This procedure makes sense only if the final result is insensitive to the value of $\tau_0$, within a reasonable range. This independence comes from the fact that, for field amplitudes $1 \ll A \ll g^{-1}$, the system can be described in a dual way:
  - as fields governed by classical equations of motion
  - as on-shell particles governed by a kinetic equation.
Importance of fluctuations

- With a background field, the collision term of the Boltzmann equation allows $1 \rightarrow 2$ splittings.

- In the classical calculation, the average over the fluctuations of the initial condition also amounts to including gluon splittings.
  
  ▶ these fluctuations may be crucial for the independence with respect to the transition time $\tau_0$ (so that the two descriptions work equally well between $\tau_0$ and $\tau'_0$).