Searching for a semiclassical structure in lattice QCD

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Work in collaboration with

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In this talk I shall

- explain some aspects of (multi-)caloron solutions
- report on search for similar structures in real Monte Carlo
- report on experience with an $SU(2)$ KvB caloron gas model

Overview

1. Introduction: What is meant by "semiclassical structure"?
2. $SU(N)$ (multi-)calorons with non-trivial holonomy
3. Semiclassical arguments applied near the phase transition
4. Have KvB calorons already been observed in Monte Carlo configurations?
5. An $SU(2)$ KvB caloron gas model and confinement
6. Summary and outlook
Recent papers:


5. E.-M. I., B. V. Martemyanov, M. Müller-Preussker, A. I. Veselov, "The monopole content of topological clusters: have KvB calorons been found?", Phys. Rev. D71, 034505 (2005)
6. E.-M. I., M. Müller-Preussker, D. Peschka,

7. F. Bruckmann and E.-M. I., "Laplacian modes probing gauge fields",

8. F. Bruckmann and E.-M. I., "Laplacian modes for calorons and as a filter",

9. F. Bruckmann and E.-M. I., "Laplacian modes as a filter",

10. E.-M. I., Ph. Gerhold, M. Müller-Preussker, B. V. Martemyanov, A. I. Veselov,
    "Topological clusters in SU(2) gluodynamics at finite temperature and the evidence

11. E.-M. I., B. V. Martemyanov, M. Müller-Preussker, A. I. Veselov,
    "Calorons and monopoles from smeared SU(2) lattice fields at non-zero temperature",
    e-Print Archive: hep-lat/0602002
Partly based on:

- Philipp Gerhold, Diploma thesis, Humboldt University (December 2005)

work in progress:

- An SU(2) KvB caloron gas model and confinement

  Ph. Gerhold, E.-M. I., M. Müller-Preussker
I. What means ”semiclassical structure”?

1. Instantons: prototype of ”semiclassical” structure

Idea: random superposition of classical solutions (instantons)

direct sampling: single-instanton parameter sampled according to the semiclassical path integral

't Hooft (for $T \neq 0$ Gross, Pisarski, Yaffe) amplitude $d(\rho, T)$
or just phenomenological density, size ($T$ dependent)

RILM: Random Instanton Liquid Model
E. Shuryak and J. Verbaarschot, good for $T = 0$!
random positions, hard core, random orientations, fixed size

IILM: Interacting Instanton Liquid Model
T. Schäfer and E. Shuryak: dealing with add. soft interactions
weak correlations, stronger at $T \to T_{\text{dec}}$ and with dynam. quarks!
Deficit of the model: confinement is not explained

2. Alternative: regular gauge instantons and/or merons

New: long range gauge fields, long range interactions

importance sampling: for all degrees of freedom

F. Lenz, J. Negele and M. Thies,
actually designed as a model for confinement

Virtue of the model: confinement is explained
by hidden long-range correlations between "pseudoparticles"
3. Evidence for instanton-like structure on the lattice

old review by J. Negele (plenary talk Lattice 1998):

lumps of action exist, but picture strongly dependent on procedure (mostly based on cooling or smearing)

- only slightly different sizes ($0.39 \ldots 0.54$ fm), size can be extrapolated to ”no smearing” limit
- strongly varying densities ($0.3 \ldots 50$ fm$^{-4}$ !!!) number of objects difficult to extrapolate to the ”no smearing” limit
4. Phenomenologically used forms of the $\rho$ distribution.

Figure 1: Left: The instanton $\rho$ distribution for quenched and full QCD, depending on the number of smearing steps (Brower et al. 1996); Right: Scaling of UKQCD instanton size distribution and the power-rise à la t’ Hooft at small $\rho$ (Smith and Teper 1998).
5. A new view at the topological vacuum structure using overlap or chirally improved fermions: 

I. Horvath et al. (2001 ff.), Ch. Gattringer et al. (2001 ff.), G. Schierholz et al. (QCDSF collaboration, 2005 ff.)

Again not a unique picture:

- **topological density without UV filter** (by overlap fermions) looks differently, forming multifractal objects:
  - no 4-dimensional sign-coherent clusters
  - negativity of the topological two-point function

- **topological density with UV filter** (eigenmode truncation) recovers clustering of topological charge:
  - remnant of negativity
  - (apparent) opposite-sign cluster attraction
6. Is there a relation between topology and confinement?

Not within the instanton liquid model
(at least with a size distribution as above):
no asymptotic string tension, no approximate Casimir scaling

For a new attempt: F. Lenz, J. Negele and M. Thies

7. What other d.o.f. might be important?

both instantons and holonomies must be extracted from
lattice configs in order to reconstruct hadronic correlators

8. Could fractional instantons (instanton constituents) help?

- instantons ↔ merons in YM (Callan, Dashen, Gross),
- in model theories (\(\sigma\)-model, \(CP^{N-1}\)-model)
- in \(\mathcal{N} = 1\) Susy YM:
  - gluino condensation/confinement by fractional instantons
  (Davies et al. [1999]; Diakonov and Petrov [2003])

9. Are fractionally charged instanton solutions known?

- only with twisted b.c. on the \((T = 0)\) torus
- or as constituents of finite \(T\) instantons (calorons)

Finite \(T\) calorons with non-trivial holonomy
generically composed of \(|Q| \times N_{\text{color}}\) approximate
BPS monopoles (dyons) each of size \(\sim 1/(\pi T)\)

KvBLL solutions:
T. C. Kraan and P. van Baal, K. Lee and C. Lu [1998]
Attractive feature:

Holonomy (Polyakov loop, the order parameter of deconfinement/confinement) coupled to the distribution of action (topological charge) in space-time.

Non-trivial holonomy
(defined at distance $|\vec{x}| \to \infty$ from lumps of action):

$$\mathcal{P}(\vec{x}) = \text{P exp} \left( i \int_0^{1/T} A_4(\vec{x}, t)dt \right) \Rightarrow \mathcal{P}_\infty \notin \mathbb{Z}(N_{\text{color}})$$
10. Sketch of the phase transition in terms of calorons

- **High temperature phase** to be described, as before, by Harrington-Shepard (HS) calorons ($T$ dependent solutions) or KvB calorons with almost trivial holonomy!

- **Confinement** due to calorons dissociating into dyons! Is the semiclassical approximation reliable there?

- Does a phenomenological model in terms of independent dyons or KvB calorons with internal d.o.f. describe confinement at $T \lesssim T_{\text{dec}}$ and at lower temperature?

Such kind of model is presently under examination
II. SU(N) (multi-)calorons with general holonomy

1. Generalities on $SU(N)$ calorons

Simple formula for a charge-one $SU(N)$ corlon action density ($= \pm$ topological density):

$$\text{Tr} \ F_{\alpha \beta}^2(x) = \partial^2_{\alpha} \partial^2_{\beta} \log \psi(x), \quad \psi(x) = \frac{1}{2} \text{tr}(A_N \cdots A_1) - \cos(2\pi t)$$

with $m = 1, \ldots, N$ numbering the constituents

$$A_m = \frac{1}{r_m} \begin{pmatrix} r_m & |\vec{\rho}_{m+1}| \\ 0 & r_{m+1} \end{pmatrix} \begin{pmatrix} \cosh(2\pi \nu_m r_m) & \sinh(2\pi \nu_m r_m) \\ \sinh(2\pi \nu_m r_m) & \cosh(2\pi \nu_m r_m) \end{pmatrix}$$

with $b = 1/T = 1$ and distances from/between centers $\vec{y}_i$

$$r_m = |\vec{x} - \vec{y}_m|, \quad |\vec{\rho}_m| = \vec{y}_m - \vec{y}_{m-1}$$
Parameters:

positions (approximately those of constituent monopoles):

\[ \vec{y}_1, \vec{y}_2, \ldots, \vec{y}_N \]

asymptotic holonomy:

\[ \mathcal{P}_\infty = \text{diag} \left( e^{2\pi i \mu_m} \right), \quad \sum_{m=1}^{N} \mu_m = 0 \]

eigenvalues, ordered as

\[ \mu_1 < \mu_2 < \ldots < \mu_N < \mu_{N+1} \equiv 1 + \mu_1 \]

action (topological charge) distributed among lumps:

\[ S_m = \nu_m \ S_{\text{inst}} \quad \text{with} \quad \nu_m = \mu_m - \mu_{m-1} \]
Fermion zero-mode (present according to Atiyah-Singer index theorem), endowed with periodicity to be varied at will:

\[ \Psi_z(t + 1/T, \vec{x}) = e^{-2\pi i z} \Psi_z(t, \vec{x}) \]

Jumping and delocalization (breathing) of zero-mode:

for \( z \in [\mu_m, \mu_{m+1}] \) \( \Rightarrow \) \( |\Psi_z(x)|^2 \) localized at \( \vec{y}_m \)

iff \( z \to \mu_m \) \( \Rightarrow \) \( |\Psi_z(x)|^2 \) becomes delocalized

To see this, for classical configurations the simple massless clover-improved Dirac operator with \( \kappa = \frac{1}{8} \) and \( c_{sw} = 1 \) is sufficient!
2. Explicit $Q = \pm 1$ solution for $SU(2)$:

\[
A^{KV_B}_\mu = \frac{1}{2} \bar{\eta}^3_{\mu\nu} \tau_3 \partial_\nu \log \phi \\
+ \frac{1}{2} \phi \Re \left( (\bar{\eta}^1_{\mu\nu} - i\bar{\eta}^2_{\mu\nu})(\tau_1 + i\tau_2)(\partial_\nu + 4\pi i\omega \delta_{\nu,4})\tilde{\chi} \right) \\
+ \delta_{\mu,4} 2\pi \omega \tau_3,
\]

depends on

- temperature: $b = 1/T$ (put $= 1$)
- scale-size $\rho$: $\pi \rho^2 T = |\vec{x}_1 - \vec{x}_2| = d$
- asymptotic holonomy: $P_\infty = \frac{1}{2} \tr P_\infty = \cos(2\pi\omega)$
- two complementary holonomy parameter: $\omega$ and $\bar{\omega} = 1/2 - \omega$

describing the action/size of two complementary constituents
First $SU(2)$ cooling results have been reported in: “On the topological content of $SU(2)$ gauge field below $T_c$”, E.-M. I., B. V. Martemyanov, M. Müller-Preussker, S. Shcheredin, A. I. Veselov, Phys. Rev. D 66, 074503 (2002)

A $SU(2)$ caloron from cooling $16^3 \times 4$ lattice ($\beta = 2.2$):

![Figure 2](image)

**Figure 2:** Various portraits of a selfdual $LM$ pair: the figures show 2D cuts of the topological charge density (a) and of the Polyakov loop (b).
Remarks:

Cooled configurations on finite $T$ torus satisfactorily fitted by the analytical solutions!

Higher plateaus: configurations with many dyons, are marked by jumping zero-modes

Cooling cascades consist of two types of events:

- instantons collapse: $\Delta S = S_{\text{inst}}, \Delta Q = \pm 1$,
- dyon pairs annihilate: $\Delta S = S_{\text{inst}}, \Delta Q = 0$, seen at $T \lesssim T_{\text{dec}}$


Finally three limiting cases seen:

- $\omega \to 0$ or $\bar{\omega} \to 0 \Rightarrow ” \text{old caloron}”$ (close to trivial holonomy)
- $\omega \neq 0$ and $\omega \neq \frac{1}{2}$ and $|\vec{x}_1 - \vec{x}_2|$ large $\Rightarrow$ dissociating into two static BPS monopoles ($L$ and $M$) with action (topological charge) shared in proportion $\sim \bar{\omega}/\omega$
- $\omega \neq 0$ and $\omega \neq \frac{1}{2}$ and $|\vec{x}_1 - \vec{x}_2|$ small $\Rightarrow$ coalescing into a single caloron
Figure 3: The figures show the lowest eigenvalues of the simple Wilson-Dirac operator (a,b) and the $2D$ cut of the corresponding real-mode fermion densities (c,d), for time-periodic (a,c) and time-antiperiodic (b,d) boundary conditions.
3. First $SU(3)$ cooling results reported in:

"Calorons in $SU(3)$ lattice gauge theory",

Monte Carlo generated $12^3 \times 4$ configurations,
also $20^3 \times 4$ (for bigger volume) and $12^3 \times 6$ (for lower $T$) in confined phase, Wilson action with $\beta = 5.65$

cooling with Wilson action (and improved action)
down to the first "plateau" yielding configurations with

$$|Q| = 1, |Q| = 2, |Q| = 3 \text{ and } |Q| = 4 \text{ (as well as } Q = 0)$$

⇒ Various ensembles of several 1000 configurations

Allow to explore the space of solutions in a finite volume,
dependence on the aspect ratio $L_t/L_s$. 
Nahm’s ”no-go theorem”

Non-existence of $|Q| = 1$ exactly selfdual calorons on the torus

Figure 4: Distribution of (anti-)selfduality violation for $|Q| > 1$ and $|Q| = 1$. 
Figure 5: Non-trivial $Q=1$ caloron: Spectral flow (upper left), IPR of zero-mode (breathing, upper right), action density (bottom left) and zero-mode density (bottom right), for three $z$’s corresponding to maximal localization.
4. **Specific for $SU(3)$:**

the complex landscape of the Polyakov loop

Holonomies at the constituents’ positions $\vec{z}_m$ (for $SU(3)$)

For well-separated constituents: all $\vec{z}_m \rightarrow \vec{y}_m$.

$$
\mathcal{P}(\vec{z}_1) = \text{diag} \left( e^{-\pi i \mu_3}, e^{-\pi i \mu_3}, e^{2\pi i \mu_3} \right)
$$

$$
\mathcal{P}(\vec{z}_2) = \text{diag} \left( e^{2\pi i \mu_1}, e^{-\pi i \mu_1}, e^{-\pi i \mu_1} \right)
$$

$$
\mathcal{P}(\vec{z}_3) = \text{diag} \left( -e^{-\pi i \mu_2}, e^{2\pi i \mu_2}, -e^{-\pi i \mu_2} \right)
$$

where the last constituent is carrying Taubes winding

**Signature of the monopoles:** degeneracy of 2 of three eigenvalues

**To practically localize monopoles:**

$\Rightarrow$ minimal distance from the envelope of the Polyakov loops
Figure 6: Topological charge density (left) and modulus of Polyakov loop (right) of a non-trivial holonomy $SU(3)$ caloron with well-sparated constituents.
A cooled $Q = 1$ calororon with nontrivial holonomy on $20^3 \times 4$

**Figure 7:** Holonomy eigenphases vs. action density (upper left), Polyakov loop scatter plot with action cut (upper right), action density (bottom left) and magnetic monopole locations (bottom right).
Figure 8: Two generic examples of charge-one $SU(3)$ calorons with non-trivial holonomy: mapped out by the zero mode (left), by the action density (right).
A $SU(3)$ caloron in the maximal Abelian and Center gauges

Figure 9: The Abelian components of the field strength (left) and the center vortices (right) inside and around a $Q = 1$ caloron.
Figure 10: The center vortex spanned by the two monopoles (top and bottom) of an $SU(2)$ caloron stretched in $z$-direction (shown is the intersection with a time slice).
The cooled sample represents a broad spectrum of holonomy $|P_\infty|$ and $Q$.

Figure 11: Ensemble of $12^3 \times 4$ calorons (excluding $Q = 0$): histogram of asymptotic holonomy (left) and distribution of topological charges (right).
Figure 12: Non-trivial $Q = 2$ caloron: as seen by fermion zero-modes (upper left), plotted in Polyakov loop (upper right), with top. charge density (bottom left) and monopole positions (bottom right).
Monopole counting

A $SU(2)$ multicaloron ring with $Q = 3$ containing 6 dyons

Figure 13: A $Q = 3$ caloron with non-trivial holonomy in $SU2$: Iso-surfaces of the Polyakov loop at positive ($L$ dyons, red) and negative values ($M$ dyons, green). Regions of non-staticity (in dark) are separating the $L$ and $M$ dyons (alternating along a multicaloron ring) from each other.
5. Global Non-staticity (here for $SU(2)$ calorons) measures the separation into lumps, PRD 69, 114505

Figure 14: Getting more and more instanton-like at lower temperature. Histograms of non-staticity without cut (left) and with cut (right) restricting holonomy to $P_\infty \approx 0$. The vertical line marks the bifurcation into two dyon constituents.

lower temperature at fixed volume $\Rightarrow$ more overlap of bigger constituents $\Rightarrow$ more instanton-like
Correlation with static Abelian monopoles

(PRD 69, 114505)

Figure 15: Histograms of the full sample compared with the subsample with static monopoles at high $T$ (left) and lower $T$ (right). The vertical line marks the bifurcation into two dyon constituents.

more instanton-like configurations
⇒ static Abelian monopoles becoming rare
see also R. C. Brower et al., Lattice 1998

This general tendency is also seen for $SU(3)$ calorons

The influence of bigger volume and lower temperature

Figure 16: Distribution of non-staticity for various $Q$, compared to $12^3 \times 4$ (middle), for larger volume $20^3 \times 4$ (left) and for lower temperature $12^3 \times 6$ (right).

bigger volume at fixed temperature $\Rightarrow$ less overlap of constituents
$\Rightarrow$ more static (dyonic) configurations
$\Rightarrow$ almost all contain static Abelian monopoles
III. Semiclassical arguments applied near the transition

1. Where can calorons be semiclassically dealt with?

"Quantum weights of dyons and of instantons with nontrivial holonomy",

$SU(2)$ calorons become unstable with respect to decay into dyons at $T_{\text{dec}}$!

Asymptotic holonomy: $P_\infty = \cos(\phi/2T)$,
is described by $\phi = \sqrt{A_4^a A_4^a}$ and $\bar{\phi} = 2\pi T - \phi$.

The perturbative free energy density (N. Weiss)

$$P(\phi) = \frac{1}{3T(2\pi)^2} \phi^2 \bar{\phi}^2$$

would favour trivial holonomy $\phi = 0$ or $\phi = 2\pi T$. 
2. The single-caloron path integral on the background $\phi$, with dyon coordinates $\vec{z}_1$ and $\vec{z}_2$ and dyon separation $r_{12} = |\vec{z}_1 - \vec{z}_2| >> 1/T$

$$\mathcal{Z}_{KvB} = C T^6 \int d^3 z_1 \, d^3 z_2$$

$$\times \left( \frac{8\pi^2}{g^2} \right)^4 \left( \frac{\Lambda e^{\gamma E}}{4\pi T} \right)^{\frac{22}{3}} \left( \frac{\phi}{2\pi T} \right)^{\frac{4\phi}{3\pi T}} \left( \frac{\bar{\phi}}{2\pi T} \right)^{\frac{4\bar{\phi}}{3\pi T}}$$

$$\times \exp \left[ -2\pi r_{12} \, P''(\phi) \right] \exp \left[ -V^{(3)} P(\phi) \right]$$

$$\left( \phi/(2\pi T) \right)^{4\phi/3\pi T} \text{ and } \left( \bar{\phi}/(2\pi T) \right)^{4\bar{\phi}/3\pi T} = \text{fugacities}$$

fugacities of the two sorts ($L$ and $M$) of dyons
Main results

- for $|P_\infty| > 0.78$ attractive interaction,
  
  the $r_{12}$ integral converges, giving a finite fugacity of an entire caloron

- but for $T < T_{\text{dec}} = 1.125$ trivial holonomy instable minimum of free energy moves to vanishing Polyakov loop

- the $r_{12}$ integral diverges, dissociation into dyons

- How then to model the confined phase?
3. Free energy density vs. holonomy near $T_{\text{dec}}$

Figure 17: Free energy density in units of $T^4$ as a function of the asymptotic value of $A_4$ in units of $2\pi T$, with the temperature as parameter. Dotted line: $T = 1.3\Lambda$, solid line: $T_{\text{dec}} = 1.125\Lambda$ (transition temperature), dashed line: $T = 1.05\Lambda$, where the caloron gas would "roll down" to non-trivial holonomy, thereby dissociating in dyons.
IV. Have KvB calorons already been observed in Monte Carlo configurations?

1. $SU(3)$: the fermion zero-mode method


- analyzed equilibrium lattice configurations
- restricted the analysis to $Q = \pm 1$ configurations
- tested the hypothesis that fermionic zero-modes behave as known for KvB calorons

Two finite-$T$ samples generated using Lüninger-Weisz action on a $20^3 \times 6$ lattice $\Rightarrow$ confinement and deconfinement:

$\beta = 8.20$ (in confinement) \hspace{1cm} $a = 0.115$ fm \hspace{1cm} $T = 287$ MeV

$\beta = 8.45$ (deconfinement) \hspace{1cm} $a = 0.094$ fm \hspace{1cm} $T = 350$ MeV
Figure 18: Slices of the scalar density for $6 \times 20^3, \beta = 8.20$, configuration 125: $x, y$-slices are shown at $t = 5, z = 9$ (left column), at $t = 2, z = 19$ (center column) and $t = 5, z = 18$ (right column). The values for $\zeta$ are $\zeta = 0.05, 0.3, 0.65$ (from top to bottom).
Two technical advances have made this feasible:

- **Chirally improved lattice Dirac operator** with good locality properties (Ginsparg-Wilson condition) due to Gattringer, Hip, Lang
- **Zero-mode counting** allows to determine the topological charge $Q$ without cooling

Localization of the single zero mode was studied depending on the periodicity angle $z$:

- **Pattern of localization and delocalization** of zero-mode depending on background Polyakov loop
- **Flow** with $z$ of the gap in the spectrum between zero and lowest non-zero eigenvalue
- **Jumping of the zero-mode** at appropriate phases observed for a sizeable fraction of confined configurations
2. **Comparison with the topological content revealed by smearing**

Regensburg-Berlin collaboration (reported at Lattice 2003)

4-dimensional APE smearing has good properties

- Dirac spectrum is not changed by moderate APE smearing,
- topological density shows lumps at the (one, two or three) positions of the zero-mode and also where the near-zero modes are localized;
- however, fractional charge of the lump at the zero-mode position was impossible to establish,
- it was argued that for finite $T$ jumping between non-fractional calorons less likely (van Baal, M. Garcia-Perez et al.)
Figure 19: Topological density profile in cuts through the three peaks (marked by crosses) of the zeromode. Typical confinement configuration.
Figure 20: Topological density profile in cuts through the two peaks (marked by crosses) of the zeromode. Typical deconfinement configuration.
3. Abelian projection method

Idea: topological cluster search with 4-dimensional APE smearing (varying the number of smearing steps)

"The monopole content of topological clusters: have KvB calorons been found?", E.-M. I., B. V. Martemyanov, M. Müller-Preussker, A. I. Veselov, Phys. Rev. D71, 034505 (2005)

"Calorons and monopoles from smeared SU(2) lattice fields at non-zero temperature", E.-M. I., B. V. Martemyanov, M. Müller-Preussker, A. I. Veselov, hel-lat/0602002

First study:

500 $SU(2)$ configurations $20^3 \times 6$ at $\beta = 2.3$

confinement phase ($T = \frac{2}{3}T_{\text{dec}}$)
Our procedure:

We applied 50 or 100 APE smearing steps, then we

- identified connected clusters of pos. and neg. charge density,
- identified time-like MAG Abelian monopoles within the clusters,
- selected clusters which are likely to contain single static monopoles or monopole-antimonopole loops,
- studied the correlation between time-like Abelian monopoles and local Polyakov loop inside the cluster,
- identified the respective cluster charges $Q_{\text{cluster}}$. 
Second study: refined procedure

200 $SU(2)$ configurations $24^3 \times 6$

at $\beta = 2.2$, $\beta = 2.3$ and $\beta = 2.4$ (confinement)

at $\beta = 2.5$ and $\beta = 2.6$ (deconfinement, $\beta_{\text{dec}} = 2.42$)

smearing:

- in confinement 50 smearing steps,
- in deconfinement 25 and 20 steps
- We monitor the timelike/spacelike string tension

MAG more carefully performed by simulated annealing

More precise selection of topological clusters

Results:
• strong correlations between local Polyakov loop and Abelian monopoles:

Figure 21: The distribution of the Polyakov loop over points where time-like Abelian monopole currents are detected (thick line). For comparison, the distribution of Polyakov loops over all points is shown (thin line). Upper row: $\beta = 2.3, 2.4$ (from left to right, confinement). Lower row: $\beta = 2.5, 2.6$ (from left to right, deconfinement).
also after smearing, the asymptotic holonomy characterizes the confinement or deconfinement phase

Figure 22: The distribution of the asymptotic holonomy $H$. Upper row: $\beta = 2.2$, 2.3, 2.4 (from left to right, confinement). Lower row: $\beta = 2.5$, 2.6 (from left to right, deconfinement).
• cluster-oriented approach: find all clusters of $q(x)$ in all configurations of the sample (threshold to allow for max. number of clusters)

• accept only those two sorts of topological clusters that have a clear monopole content: (a very selective cut!)
  — static monopole covered by the cluster: dyon
  — closed monopole loop covered by the cluster: caloron

• characterize the selected clusters by two cluster variables:
  — cluster charges estimated according to the type of cluster
  — average Polyakov loop averaged over the timelike monopoles

⇒ scatterplot showing the selected clusters w.r.t. the two cluster variables
Figure 23: Scatter plots in the \((Q_{\text{cluster}}, \langle PL(\text{Abelian monopoles})\rangle_{\text{cluster}})\) plane. Upper row: \(\beta = 2.2, 2.3, 2.4\) (from left to right, confinement). Lower row: \(\beta = 2.5, 2.6\) (from left to right, deconfinement). Circles = dyon clusters, triangles = undissociated calorons.
The composition at highest temperature, now specified according to $Q = \pm 1$ or $Q = 0$.

**Figure 24**: Scatter plots in the $(Q_{\text{cluster}}, \langle PL(\text{Abelian monopoles})\rangle_{\text{cluster}})$ plane for $\beta = 2.6$, the highest temperature. Left: all 200 configurations, Middle: 45 charge $Q = \pm 1$ configurations, Right: 155 charge $Q = 0$ configurations.
Conclusion:
Topological structure at $T < T_{\text{dec}}$ appears as a cocktail:

- complete calorons:
  
  $$\overline{PL} \approx 0 \quad Q_{\text{cluster}} \approx \pm 1$$

- half-calorons:

  $$\overline{PL} \approx \pm 1 \quad Q_{\text{cluster}} \approx \pm \frac{1}{2}$$

- the dyon/caloron ratio increases as $T \to T_{\text{dec}}$ from below
- at low $T$ only undissociated calorons, no isolated dyons!

Best described as dipoles with $d \neq 0$, instantonlike with new degrees of freedom, 
⇒ need a suitable radial distribution $D(\rho, T)$
Topological structure at $T > T_{\text{dec}}$:

- **complete calorons**: HS calorons in $Q = 1$ lattices only, ca. 50% of configs only, other split in half-calorons
- **half-calorons**: ”heavy dyons” in $Q = 1$ lattices only, but holonomy $\neq \pm 1$, not clear caloron constituents
- **small-charge monopoles**: ”light (nonselfdual) monopoles” mostly in $Q = 0$ lattices, with $Q_{\text{cluster}} \approx 0$ and holonomy $\approx H$

No clear identification of all $Q = \pm 1$ as caloron configs, contrary to expectation!
Nonselfdual monopoles become more abundant

$\Rightarrow$

necessary to confirm this picture using a better fermionic/gluonic topological density
V. An SU(2) KvB caloron gas model and confinement

We have built/are testing a model that extends the RILM (with uncorrelated sampling) to the new class of solutions

\( d \) = dipole separation, the new d.o.f. together with ext. holonomy

- if \( d << 1/T \) single lumps of action, non-static
- if \( d >> 1/T \) two separate, static lumps of action

caloron parameters to be sampled:

- center of mass position (4 random parameters)
- distance \( d \) according to the quantum amplitude, replaces \( \rho^2 = d/(\pi T) \) (1 scale parameter) sampled from a distribution \( D(\rho,T) \) that describes the cocktail
- spatial orientation (2 random angles)
- \( U(1) \) rotation (unbroken \( U(1) \) symmetry, 1 parameter)
model parameters to be chosen:

- **holonomy**: $\omega = 1/4$ in confinement, 
  $\omega(T)$ to be taken from lattice for $T > T_{\text{dec}}$?
- **density**: $n$ constant in confinement, 
  $n(T)$ above $T_{\text{dec}}$ from topolog. susceptibility $\chi_t(T)$?

Consider a fiducial volume (compact, finite $T$) embedded in a larger volume (same $T$, filled randomly with a 4D density $n$)
Problems to be solved in superposing calorons:

- Superposition done in the algebraic gauge (where $A_{4\infty}^{\text{alg}} \to 0$) but with the same holonomy $\vec{\omega} \cdot \vec{\tau}$

- Gauge transformation into periodic gauge corresponding to the holonomy (where $A_{4\infty}^{\text{per}} \to 2\pi T \vec{\omega} \cdot \vec{\tau}$)

- Gauge transformation to avoid eventual overlap between Dirac string and further calorons (flip outside and back)

- ADHM-type local improvement (similar to ratio ansatz) is applied (if action lumps happen to overlap) in order to correct for local selfduality
A caloron as a (electric and magnetic) dipole

Figure 25: Caloron $\omega = \bar{\omega}$ in the process of dissociation, $d$ increasing from 0 to maximal.

Dirac string fine tuned to be free of action

Figure 26: Left: action density for dissociated caloron $\omega = \bar{\omega}$, right: corresponding vector potential $\sum_\mu |A_\mu|^2$. 
Original setting

- continuum field discretized on links
- fiducial volume: $32^3 \times 8$ lattice
- inverse temperature $1/T = 1$ fm
- 3D volume $V_3 = 64$ fm$^3$
- density $n = 1$ fm$^{-4}$
- size $\bar{\rho} = 0.33$ fm $\rightarrow \bar{d} = 0.34$ fm fixed
- distribution $D(\rho) \propto \rho^{7/3} \exp(-c\rho^2)$

What is the influence of the holonomy $\omega$ alone?
Figure 27: Left: the potential for fundamental and adjoint static charges from the corresponding Polyakov loop correlators; right: linearly rising $-\log W(R, R_1)/R_1$ for fundamental spatial Wilson loops $W$. Note the influence of various asymptotic holonomies $\omega$. 
More about the $\rho$ distribution $D(\rho, T)$:

- For $T \ll T_{\text{dec}}$

  \[ D(\rho, T) \propto \rho^{b-5} \exp(-c \rho^2) \]

  with constant $\bar{\rho} = 0.33 \text{ fm} \ll 1/T$, due to caloron interactions; $\rho^{b-5}$ corresponds also to the caloron amplitude (Diakonov).

- For higher $T \lesssim T_{\text{dec}} \rightarrow \rho \lesssim 1/T$, transversal size limited by the dyon size alone, rôle of $\rho$ goes to $d$.

- Then the $d$ distribution following Diakonov is derivable from the interaction with the environment holonomy

  \[ D(\rho, T) \propto \exp\left(-V P(\phi) - 2 \pi d \ P''(\phi)\right) \]

  with the free energy density $P(\phi) = \frac{\phi^2 \bar{\phi}^2}{12\pi^2 T}$ and $\phi = 4\pi T \omega$. 
• For large packing fraction, the volume term wins over the repulsive force \(P'' < 0\), resulting in

\[
D(\rho, T) \propto \exp \left( -\frac{2N}{3} (\pi \rho T)^2 G(\omega) \right)
\]

with

\[
G(\omega) = 16C_0 \pi^2 \omega^2 \bar{\omega}^2 + 4(\omega^2 + \bar{\omega}^2 - 4\omega \bar{\omega}) .
\]

• For \(T < T_{\text{dec}}\) and \(\omega = \frac{1}{4}\), this does not change \(\bar{\rho} \approx 0.33\)

• For \(\omega \to 0\) or \(\frac{1}{2}\) (trivial holonomy), this turns for \(T > T_{\text{dec}}\) into the well-known suppression factor for HS calorons

\[
D(\rho, T) \propto \exp \left( -\frac{2N}{3} (\pi \rho T)^2 \right) .
\]
Parameters in the deconfined phase

Figure 28: susceptibility $\chi_t(T)$ (left) and size parameter $\bar{\rho}$ (right) as functions of $T$.

Figure 29: The renormalized Polyakov loop $L_{\text{ren}} = \cos(2\pi \omega(T))$ for $T > T_{\text{dec}}$. 
Various potentials

**Figure 30:** Color average potential from Polyakov loop correlators, left: fundamental charges, right: adjoint charges.

**Figure 31:** Spatial string "potential" from Wilson loops, left: fundamental charges, right: adjoint charges.
Cluster properties of Abelian monopole networks generated in the caloron gas:

Figure 32: Monopole clusters: maximal 3D size (left) and length per cluster (right), changing with the density of the caloron gas.
Quantitatively, with the standard choice:

- density $n = 1 \text{ fm}^{-4}$
- size $\bar{\rho} = 0.33 \text{ fm}$ corresponding to $\bar{d} = 0.34 \text{ fm}$

we get in the confining phase

$$\sigma_{Pol} \approx \sigma_{Wil} \approx 200 \text{ MeV/fm}$$

Uncertainties:

- higher density, $n = \frac{b}{4} \chi_t$ in the confined phase?
- more dumbbell-like calorons, $d > 0.3 \text{ fm}$?

Then the above procedures for dense caloron systems will become important.

Eventually the direct sampling will have to be abandoned, importance sampling be needed.
Discussion :

- Remarkable that a linear potential is generated that extends to distances $R \gg \rho$ or $R \gg d$.
- The adjoint potential does not rise linearly!
- Spatial string tension for fundamental charges is proportional to the caloron density, independent of $T$.
- The description of the deconfined phase is unrealistic as far as calorons (density is assumed to drop $\sim \chi_t(T)$) are the only source of monopoles!
- Other magnetic monopoles are unaccounted!
- We are now more systematically studying the dependence on density $n$ and $\rho$.

Analytical calculations being done by F. Bruckmann.
Summary and outlook

- We have systematically studied the characteristic features of classical $SU(2)$ and $SU(3)$ calorons with general holonomy.
- Although they are classical solutions in a finite volume (depending on the aspect ratio) they are well-described as KvB calorons.
- Although the lump structure changes (depending on $T$ and available space), they always contain $N$ monopoles per caloron.
- Jumping and breathing of zero-modes suggest an underlying presence of KvB calorons, also in Monte Carlo configurations.
• The monopole content (in Abelian projections) of topological charge clusters indicates that calorons present in the confined phase appear as a mixture of dissociated and non-dissociated calorons.

• In the deconfined phase (having \( Q = 0 \) and \( Q = \pm 1 \)) this technique has only partly found non-dissociated calorons. An increasing number of nonselfdual monopoles is observed.

• A verification using independent techniques is necessary (in preparation for \( SU(2) \) and \( SU(3) \)).

• A dipole model of the caloron gas (for \( SU(2) \)) has first results probing the confining properties of caloron gases.