Discretization errors in the spectrum of the Wilson-Dirac operator

Work in progress!

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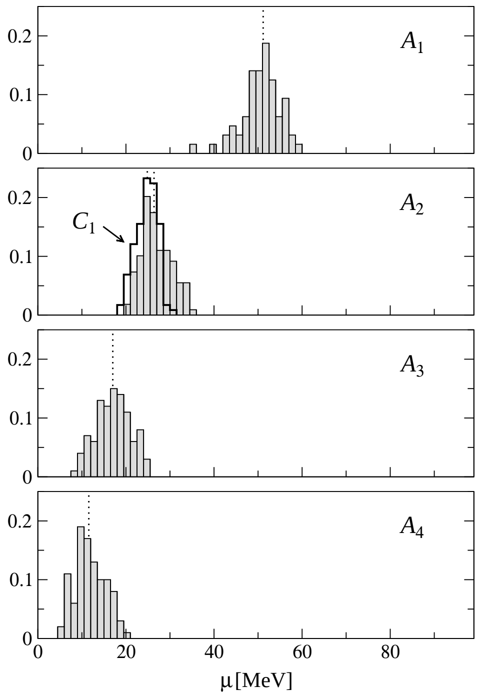
Outline

- Introduction and motivation
- Method
- Results
- Conclusions and outlook

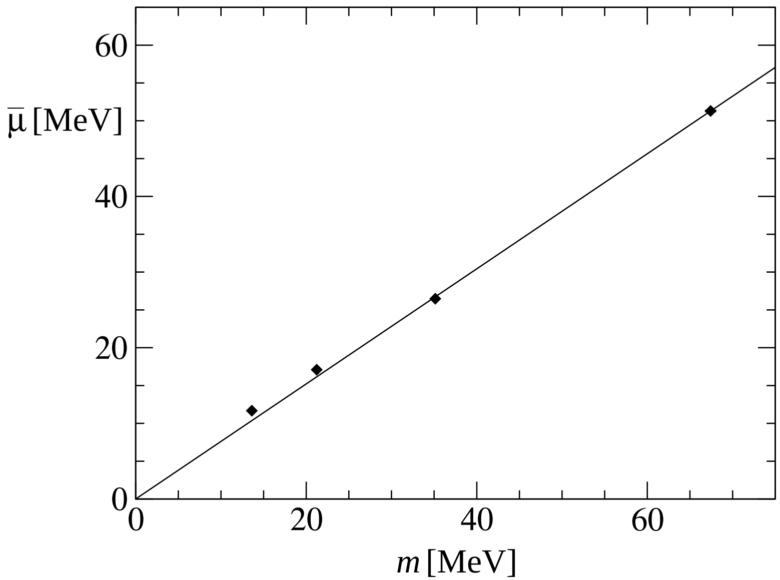
Background

- Simulations with Wilson-like fermions can now enter the chiral regime
- Important practical issue is size of spectral gap μ_Q in Hermitian Wilson-Dirac operator $Q_m = \gamma_5(D_W + m_0)$
  - Small gap can lead to instabilities and enhanced errors

- Gap bounded $\mu_Q > |m|$ if have chiral symmetry, but not for Wilson-like fermions
- [Del Debbio *et al*, hep-lat/0512021] study distribution of gap:
  - Median $\bar{\mu}_Q$ proportional to m
  - Distribution of has width $\sigma \approx a/\sqrt{V}$
- Provides constraints on simulation parameters



Background (continued)



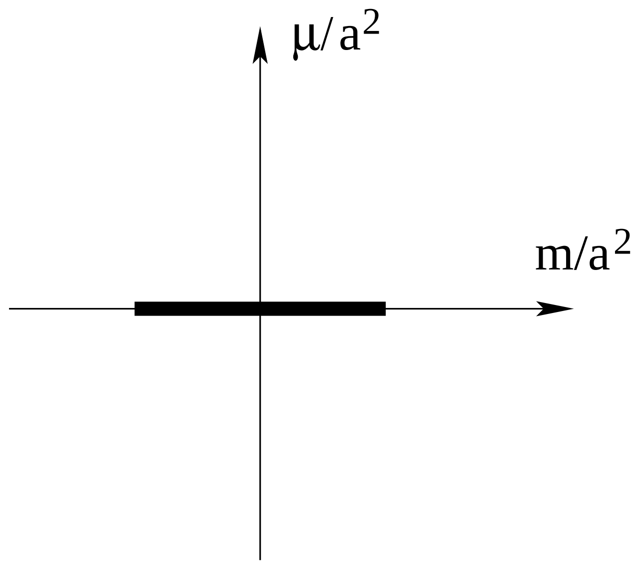
Median gap $\bar{\mu}_Q$ close to linear in quark mass—what are the discretization errors?

Motivation of present work

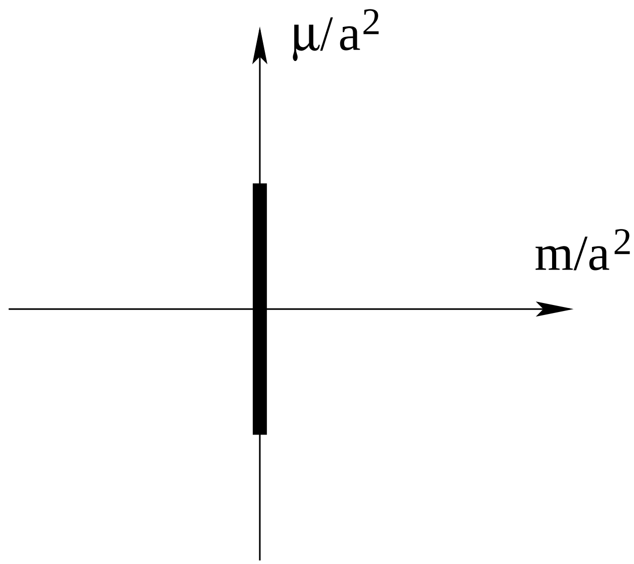
- Leading order discretization effects on gap vanish when $V \to \infty$
- What about $O(a^2)$ effects?
  - Observed to be large with Wilson fermions at $a \approx 0.2$ fm [Farchioni *et al*]
  - First-order transition with $m_\pi^{\rm min} \approx 500$ MeV
  - $m_\pi^{\rm min} \propto a$ implies $m_\pi^{\rm min} \approx 250, 200, 150$ MeV at $a = 0.1, 0.08, 0.06$ fm
  - Not unreasonably large: if $m_\pi^{\rm min} = a\Lambda^2$ then $\Lambda \approx 700$ MeV
- Motivates study of $O(a^2)$ effects on gap and spectral density of Q_m
  - As a first step, work in infinite volume

Impact of $O(a^2)$ errors

$O(a^2)$ discretization errors lead to phase transitions in $m - \mu$ plane:
[SS & Singleton, Münster, Scorzato, SS & Wu]



Aoki-phase scenario



First-order scenario

- Phase boundaries have length $\sim a^2$
- Focus on Aoki-phase scenario, since method developed here works in that case (slight over-simplification, more later)
  - Gap vanishes at end-point of Aoki phase
  - Study gap as approach end-point along Wilson axis

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Overview of method

- Low "energy" part of spectrum related to long distance physics
  - Classic example is Banks-Casher relation: $\rho_{\not{D}}(0) = -\langle \bar{q}q \rangle / \pi$
- Use chiral perturbation theory to represent long-distance physics
  - In Banks-Casher example, $-\langle \bar{q}q \rangle \longrightarrow f^2 B_0$
  - [Damgaard *et al*] determined microscopic spectral density using chiral perturbation theory in $\epsilon-$regime (justifying use of RMT)
- Discretization errors can be included in χPT: $\Rightarrow$ WχPT [SS & Singleton, . . . , Bär *et al*]

$\Rightarrow$ **Use WχPT for long-distance physics and convert to prediction for low-energy spectrum**

Connecting χPT to $\rho_{\not{D}}(\lambda)$ in continuum QCD

- Use partially quenched (PQ) extension of QCD [Osborn *et al*]

$$\langle \bar{q}_V q_V \rangle (m_V) = -\int d\lambda \frac{\rho_{\not{D}}(\lambda)}{i\lambda + m_V}$$

- Invert using

$$\begin{aligned} \text{Disc } \langle \bar{q}_V q_V \rangle \Big|_{m_V = -i\lambda} &= \lim_{\epsilon \to 0} \left[\langle \bar{q}_V q_V \rangle (-i\lambda + \epsilon) - \langle \bar{q}_V q_V \rangle (-i\lambda - \epsilon) \right] \\ &= -2\pi \rho_{\not{D}}(\lambda) \end{aligned}$$

- Need to calculate valence condensate in PQχPT for imaginary mass
- To obtain microscopic spectral density requires integration over appropriate Goldstone super-manifold

Generalization of method to lattice QCD at $a > 0$

- Consider spectrum of $Q_m = \gamma_5(D_w + m_0)$ (rather than $\not{D}$)
- Need to consider parity violating valence condensate (to obtain γ_5) and work at complex twisted mass ($i\mu\bar{q}_V\gamma_5\tau_3 q_V$) [Golterman *et al*]

$$\mathrm{Disc}\, \langle \bar{q}_V\gamma_5\tau_3 q_V\rangle\Big|_{\mu=i\lambda} = 2i\pi\left[\rho_Q(\lambda) + \rho_Q(-\lambda)\right]$$

⇒ **Use PQWχPT to calculate valence condensate for complex μ?**

- However, there are subtleties in studying condensates in PQWχPT related to presence of ghosts
  - Discussed for quenched case by [Golterman, SS & Singleton]
  - Made more complicated by need to use a complex mass?
- Avoid these subtleties by a trick which works at leading order in χPT
- Start from a reformulation due to [Del Debbio *et al*] rather than calculate the valence condensate directly

Formulation of [Del Debbio *et al*]

- Consider spectrum $\rho(\alpha)$ of Q_m^2: real, and positive with gap $\bar\alpha = \mu_Q^2$
- In continuum $Q_m^2 = -\not{D}^2 + m^2$ so gap is $\bar\alpha = m^2$ and

$$\rho(\alpha)_{a\to 0} \longrightarrow -\frac{\langle \bar q q\rangle}{\pi\sqrt{\alpha - m^2}} + O(1) \qquad \alpha \geq m^2$$

- Introduce resolvent

$$R(z) = \int_{\bar\alpha}^{\infty} d\alpha\, \frac{\rho(\alpha)}{\alpha^2(z-\alpha)} \qquad \text{Disc } R\Big|_{z=\alpha\geq\bar\alpha} = \frac{-2i\pi\rho(\alpha)}{\alpha^2}$$

- Convergent expansion about $z=0$ with radius of convergence $\bar\alpha$

$$R(z) = \sum_{k=0}^{\infty} z^k M_k\,, \qquad M_k = -\int_{\bar\alpha}^{\infty} d\alpha\, \frac{\rho(\alpha)}{\alpha^{3+k}}$$

- Coefficients are PQ correlators

$$M_k = \sum_{x_1,\ldots,x_{n-1}} \langle P_{12}(x_1)P_{23}(x_2)\ldots P_{n-1\,n}(x_{n-1})P_{n1}(0)\rangle_{PQ} \equiv \langle P^n\rangle_{PQ}\,, \qquad n = 2k+6$$

Strategy of calculation

- [Del Debbio *et al*] construct renormalized spectral density

$$\rho_R(\alpha) = Z_P^2\ \rho(Z_P^2 \alpha)\,; \qquad \bar\alpha_R = \bar\alpha / Z_P^2$$

with expansion in terms of renormalized pseudoscalar densities

$$R_R(z) = \sum_{k=0}^{\infty} z^k M_{k,R}\,, \qquad M_{k,R} = Z_P^{2k+6} M_k \equiv \langle P_R^n \rangle_{PQ}\,,$$

- **STRATEGY: Calculate $\langle \mathbf{P_R^n} \rangle_{\mathbf{PQ}}$ using PQWχPT at leading order, sum series, analytically continue beyond cut along real axis, determine discontinuity which gives $\rho(\alpha)$**
- In practice, easier to calculate auxiliary function

$$F(z) = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell} \langle P_R^{2\ell} \rangle_{PQ}\,, \qquad R_R(z) = \frac{F'(z) - F'(0) - zF''(0)}{z^2}$$

whose discontinuity is proportional to the integrated density

$$\mathrm{Disc}\ F\Big|_{z=\alpha>\bar\alpha} = -2i\pi N_R(\alpha) \equiv -2i\pi \int_{\bar\alpha}^{\alpha} d\alpha' \rho_R(\alpha')$$

PQ Wilson χPT for $m \sim a^2$

- Assume "Aoki regime" power counting: $m \sim p^2 \sim a^2$
- Leading order Lagrangian given by [Bär *et al*]

$$\begin{aligned} \mathcal{L}_\chi &= \frac{f^2}{4}\mathrm{Str}\left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right) - \frac{f^2}{4}\mathrm{Str}\left(\chi'_+ \Sigma^\dagger + \Sigma \chi'_-\right) \\ &\quad - \widehat{a}^2 W_6'' \left[\mathrm{Str}\left(\Sigma + \Sigma^\dagger\right)\right]^2 - \widehat{a}^2 W_7'' \left[\mathrm{Str}\left(\Sigma - \Sigma^\dagger\right)\right]^2 \\ &\quad - \widehat{a}^2 W_8''\ \mathrm{Str}\left(\Sigma^2 + [\Sigma^\dagger]^2\right) \end{aligned}$$

  - $\Sigma = \exp(2i\Phi/f) \in SU(N_S + N_V|N_V)$ contains pseudo-Goldstones
  - LECs: $f \approx f_\pi = 93$ MeV; $\widehat{a} = 2W_0 a$; W''_{6-8}
  - Source terms: $\chi'_\pm = 2B_0(M' \pm p)$ with $\delta/\delta p_{jk} \to P_{jk,R}$
  - $O(a)$ term has been absorbed into shift of M' $\Rightarrow$ **Leading correction here is $\mathbf{O(a^2)}$ irrespective of improvement of Wilson action**
  - In unquenched $SU(2)$ sector, W_7'' term vanishes, while W_6'' and W_8'' terms combine $\Rightarrow$ **PQ theory introduces extra LECs**
  - However, $W_6''/W_8'' \propto 1/N_c$ so expect W_8'' dominant

Extent of Partial Quenching?

- Start with $2+1$ flavor PQ QCD:

$$M = \mathrm{diag}(m_u, m_d, m_s, \underbrace{m_V, m_V, \ldots}_{N_V\, terms}, \underbrace{m_V, m_V, \ldots}_{N_V\, terms})$$

- Need $N_V \to \infty$ to calculate $\langle P_R^{2\ell} \rangle_{PQ}$ for all ℓ
- Choose $0 < m_u = m_d = m_\ell \ll m_s$, with m_s fixed to physical value, and m_ℓ varying
- Interested in spectral density for up and down quarks
  - Integrate out strange quark (so don't need to worry about status in χPT)
  - Form of $\mathcal{L}_\chi$ same, with different unknown coefficients
  - Results apply also for $N_f = 2$ theory (though coefficients will differ)
- Choose $m_V = m_\ell$ since want spectral density of unquenched Hermitian Wilson-Dirac operator
- Minimal partial quenching with no double-pole contributions

Phase structure

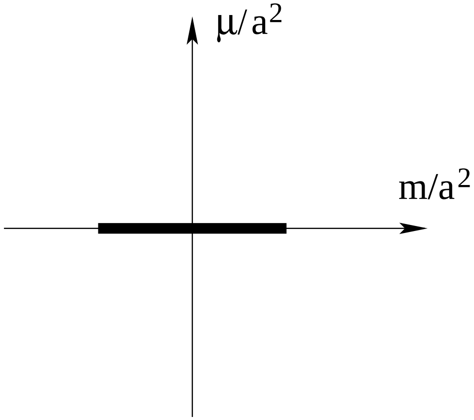
- Determined by unquenched $SU(2)$ sector [SS & Singleton] :

$$-\widehat{a}^2 W_6'' \left[\mathrm{Str}\left(\Sigma+\Sigma^\dagger\right)\right]^2 - \widehat{a}^2 W_7'' \left[\mathrm{Str}\left(\Sigma-\Sigma^\dagger\right)\right]^2 - \widehat{a}^2 W_8'' \,\mathrm{Str}\left(\Sigma^2+[\Sigma^\dagger]^2\right)$$

$$\xrightarrow[SU(2)]{} -\widehat{a}^2 (W_6''+W_8''/2)\left[\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)\right]^2$$

- Aoki-phase if $W' = W_6'' + W_8''/2 < 0$, first-order if $W' > 0$
- Useful notation

$$\widehat{m} = 2B_0 m\,, \quad w_6 = \frac{16\widehat{a}^2 W_6''}{f^2}\,, \quad w_8 = \frac{16\widehat{a}^2 W_8''}{f^2}\,, \quad w' = \frac{16\widehat{a}^2 W'}{f^2} = w_6 + \frac{w_8}{2}\,.$$



- Aoki-phase end-points at $\widehat{m} = \mp 2w'$
- Pion mass: $m_\pi^2 = |\widehat{m}| + 2w'$
- Gap vanishes inside Aoki-phase

Calculating $\langle P_R^n \rangle_{PQ}$ at leading order

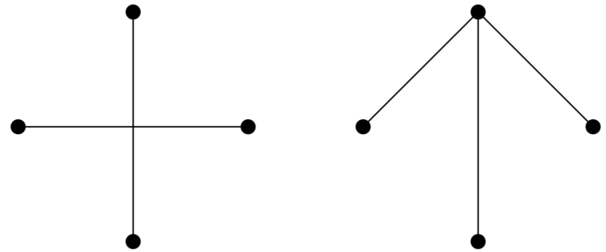
$$\langle P^n \rangle_{PQ,\mathrm{lat}} = \int d^4x_1 \dots d^4x_{n-1} \langle P_{R,12}(x_1) \dots P_{R,n-1n}(x_{n-1}) P_{R,n1}(0) \rangle_{PQ,\mathrm{cont}} + O(a^2)$$

$$\begin{aligned} \mathcal{L}_\chi^{a^2} &= \frac{f^2}{4} \mathrm{Str}\left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right) - \frac{f^2}{4} \mathrm{Str}\left(\chi'_+ \Sigma^\dagger + \Sigma \chi'_-\right) \\ & -\hat{a}^2 W_6'' \left[\mathrm{Str}\left(\Sigma + \Sigma^\dagger\right)\right]^2 - \hat{a}^2 W_8'' \,\mathrm{Str}\left(\Sigma^2 + [\Sigma^\dagger]^2\right) + \dots \end{aligned}$$

- Consider $m > 0$ so expand about $\Sigma = 1$
- $P_{jk,R}(x) = -\frac{f^2 B_0}{2} (\Sigma^\dagger - \Sigma)_{kj}$ creates any odd number of pions
- $\langle P_R^n \rangle_{PQ} = 0$ if n is odd, by parity
- Cyclic arrangement of flavor indices restricts contractions
  - No disconnected contractions
  - Each vertex can contain only one supertrace (true only at LO)
- External momenta vanish so kinetic term does not contribute

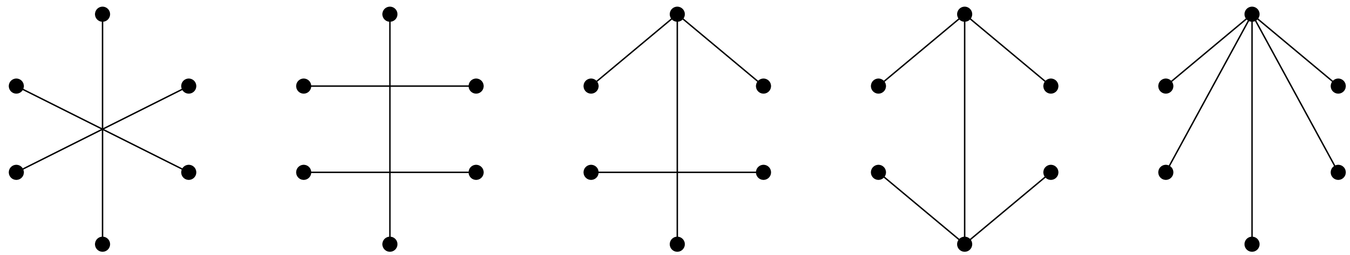
Examples of diagrams contributing to $\langle P^n \rangle_{PQ}$

$n = 4$



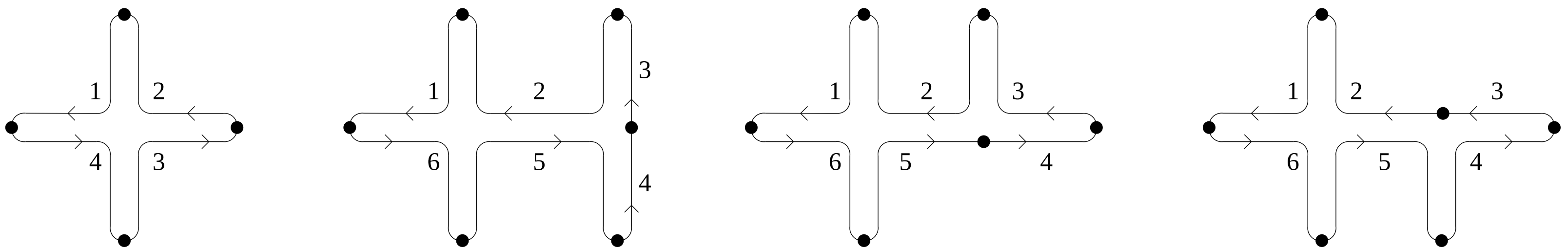
(a) (b)

$n = 6$



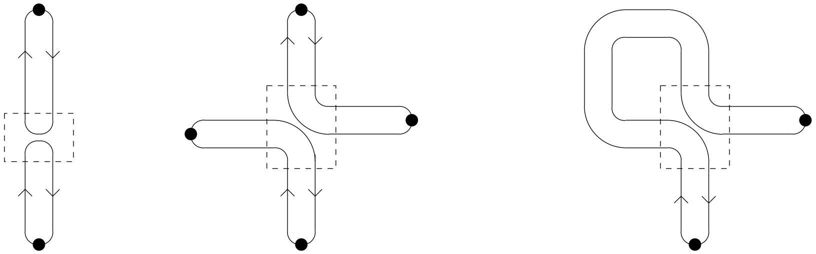
(a) (b) (c) (d) (e)

Examples of corresponding quark flow diagrams:



(a) (b) (c) (d)

Restrictions on quark-flows



(a) (b) (c)

(a,b) Disallowed contributions from two supertrace vertices

(c) *Allowed* contribution with two supertrace vertex, but at one-loop

$\Rightarrow$ **Contributions from two supertrace operators**

$$-\widehat{a}^2 W_6'' \left[\mathrm{Str}\left(\Sigma + \Sigma^\dagger\right)\right]^2 - \widehat{a}^2 W_7'' \left[\mathrm{Str}\left(\Sigma - \Sigma^\dagger\right)\right]^2$$

involve contractions in which only one supertrace connected to pions

$\Rightarrow$ Can replace $\mathcal{L}_\chi$ *at tree-level* with

$$\mathcal{L}_\chi^{\mathrm{eff}} = -f^2 \left\{ \frac{\mathrm{Str}\left([\widehat{m}^{\mathrm{eff}} + \widehat{p}]\,\Sigma^\dagger + \Sigma\,[\widehat{m}^{\mathrm{eff}} - \widehat{p}]\right)}{4} + w_8 \frac{\mathrm{Str}\left(\Sigma^2 + [\Sigma^\dagger]^2\right)}{16} \right\}$$

with $\widehat{m}^{\mathrm{eff}} = \widehat{m} + 2w_6$ and $\widehat{p} = 2B_0 p$

Examples of results

$$\langle P_R^2\rangle_{PQ,LO} = -f^2(B_0)^2\frac{2}{m_\pi^2}$$

$$\langle P_R^4\rangle_{PQ,LO} = -f^2(B_0)^4\frac{4\,\widehat{m}^{\rm eff}}{(m_\pi^2)^4}$$

$$\langle P_R^6\rangle_{PQ,LO} = -f^2(B_0)^6\frac{12\,\widehat{m}^{\rm eff}(\widehat{m}^{\rm eff}-w_8)}{(m_\pi^2)^7}\,,$$

with $\widehat{m}^{\rm eff} = \widehat{m} + 2w_6$ and $m_\pi^2 = \widehat{m}^{\rm eff} + w_8$

- Allows check of subsequent manipulations
- **$\mathbf{w_6}$ and $\mathbf{w_8}$ do not appear in combination $\mathbf{w' = w_6 + w_8/2}$ as in unquenched theory**

⇒ **Results for gap and spectral density involve an additional parameter (unless $\mathbf{w_6 \ll w_8}$)**

Key trick to allow summation

$$\mathbf{2(n-1)!\langle P_R^n\rangle_{PQ,LO} = \langle (P_{R,12}+P_{R,21})^n\rangle_{CONN,LO}\,, \qquad (n\geq 1)}$$

where the **unquenched** correlator on RHS is

$$\mathbf{\langle (P_{R,12}+P_{R,21})^n\rangle_{CONN,LO} \equiv \frac{1}{V}\langle \left\{\int d^4x[P_{R,12}(x)+P_{R,21}(x)]\right\}^n\rangle_{CONN,LO}}$$

- Unquenched correlator must be **connected**
- Result holds only at tree-level and only if there are only single supertrace vertices (as in $\mathcal{L}_\chi^{\rm eff}$)
- Requires $m_\ell = m_V$
- Follows by showing that all quark-flow diagrams are the same, aside from overall common contraction factor

Making use of the trick

- Introduce another auxilliary function

$$\mathbf{G}(\mu) = \sum_{\mathbf{n}=1}^{\infty} \frac{(-\mathbf{i}\mu)^{\mathbf{n}}}{\mathbf{n}!} \langle (\mathbf{P}_{\mathbf{R},12} + \mathbf{P}_{\mathbf{R},21})^{\mathbf{n}} \rangle_{\mathbf{CONN}}$$

  - All powers of n, though only even contribute

- Then, using trick, G and F are related at LO

$$\begin{aligned} \mathbf{G}(\mu)_{\mathbf{LO}} &= \sum_{n=1}^{\infty} \frac{(-i\mu)^n}{n!} 2(n-1)!\, \langle P_R^n \rangle_{PQ,LO} \\ &= \sum_{n=1}^{\infty} \frac{(-i\mu)^n}{n/2}\, \langle P_R^n \rangle_{PQ,LO} \\ &= \sum_{\ell=1}^{\infty} \frac{(-\mu^2)^\ell}{\ell}\, \langle P_R^n \rangle_{PQ,LO} \quad = \quad \mathbf{F}(-\mu^2)_{\mathbf{LO}} \end{aligned}$$

- **Thus if can calculate $\mathrm{G}(\mu)_{\mathrm{LO}}$ for imaginary μ (twisted mass) obtain desired function $\mathrm{F}(\mathrm{z})$, at leading order, for positive z**

Summing the series

$$
\begin{aligned}
G(\mu) &= \sum_{n=1}^{\infty} \frac{(-i\mu)^n}{n!} \langle (P_{R,12} + P_{R,21})^n \rangle_{CONN} \\
&= \sum_{n=1}^{\infty} \frac{(-i\mu)^n}{n!} \frac{1}{V} \left[-\int \left(\frac{\delta}{\delta p_{12}} + \frac{\delta}{\delta p_{21}} \right) \right]^n \ln Z_{\chi,\mathrm{eff}}(p) \Big|_{p=0} \\
&= \frac{1}{V} \ln Z_{\chi,\mathrm{eff}}(\mu)
\end{aligned}
$$

where $Z_{\chi,\mathrm{eff}}(\mu)$ is the unquenched partition function for $\mathcal{L}_\chi^{\mathrm{eff}}$ with a twisted mass (normalized such that $Z_{\chi,\mathrm{eff}}(0) = 1$)

$$
\begin{aligned}
\mathcal{L}_\chi^{\mathrm{eff}}(\mu) &= \frac{f^2}{4} \mathrm{Tr}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) + \mathcal{V}(\mu) + \mathrm{c.t.} \\
\mathcal{V}(\mu) &= -f^2 \left\{ \frac{\mathrm{Tr}\left([\widehat{m}^{\mathrm{eff}} + i\widehat{\mu}\tau_1]\, \Sigma^\dagger + \Sigma\, [\widehat{m}^{\mathrm{eff}} - i\widehat{\mu}\tau_1] \right)}{4} + w_8 \, \frac{\mathrm{Tr}\left(\Sigma^2 + [\Sigma^\dagger]^2 \right)}{16} \right\} \\
\widehat{\mu} &= 2B_0 \mu
\end{aligned}
$$

Summing the series (continued)

- At leading order get classical result:

$$Z_{\chi,\text{eff}}(\mu)_{LO} = \exp[-S_{\min}(\mu) + S_{\min}(0)]$$

(assuming μ real so have real action)

- Thus obtain

$$\begin{aligned} G(\mu)_{LO} &= \frac{1}{V} \ln Z_{\chi,\text{eff}}(\mu) \\ &= -\mathcal{V}_{\min}(\mu) + \mathcal{V}_{\min}(0) \\ &= F(-\mu^2)_{LO} \end{aligned}$$

- **Determine minimum analytically for real μ (sums series) then analytically continue to complex μ**
- From [Osborn *et al*] expected form like this. Method has shown how to treat double supertrace operators which were problematic in study of quenched Aoki-phase [Golterman, SS & Singleton]

Summary of method

- Minimize potential for auxiliary unquenched $SU(2)$ theory for real twisted mass μ

$$\mathcal{V}(\mu) = -f^2 \left\{ \frac{\mathrm{Tr}\left([\widehat{m}^{\mathrm{eff}} + i\widehat{\mu}\tau_1]\, \Sigma^\dagger + \Sigma\, [\widehat{m}^{\mathrm{eff}} - i\widehat{\mu}\tau_1] \right)}{4} + w_8 \, \frac{\mathrm{Tr}\left(\Sigma^2 + [\Sigma^\dagger]^2 \right)}{16} \right\}$$

- Due to parity, result depends only on μ^2
- Minimum gives desired function along negative real axis

$$F(z = -\mu^2)_{LO} = -\mathcal{V}_{\mathrm{min}}(\mu) + \mathcal{V}_{\mathrm{min}}(0)$$

- Analytically continue to complex μ^2 and determine start of cut on real axis (giving $\bar{\alpha}$) and discontinuity (giving integrated spectral density)

$$\mathrm{Disc}\ F\Big|_{z=\alpha>\bar{\alpha}} = -2i\pi N_R(\alpha)$$

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Check method for continuum theory

- Set $w_6 = w_8 = 0$

$$\mathcal{V}(\mu) = -f^2 \frac{\mathrm{Tr}\left(\left[\widehat{m} + i\widehat{\mu}\tau_1\right] \Sigma^\dagger + \Sigma \left[\widehat{m} - i\widehat{\mu}\tau_1\right]\right)}{4}$$

- Minimized when

$$\Sigma = \frac{\widehat{m} + i\widehat{\mu}\tau_1}{\sqrt{\widehat{m}^2 + \widehat{\mu}^2}}$$

- So find

$$F(z = -\mu^2) = -\mathcal{V}_{\rm min}(\mu) + \mathcal{V}_{\rm min}(0) = 2f^2 B_0(\sqrt{m^2 - z} - m)$$

- Discontinuity gives correct result for gap and spectrum

$$\bar{\alpha} = m^2\,, \qquad N_R(\alpha) = \frac{2B_0 f^2}{\pi}\sqrt{\alpha - m^2}$$

Applying method for $a \neq 0$

- Analysis identical to that for phase structure of tmLQCD, except with different parameters [Münster, SS & Wu, Bär & Aoki]
- Condensate twisted in τ_1 direction: $\Sigma = \exp(i\omega\tau_1)$

$$\frac{\mathcal{V}(\mu)}{\mathbf{f^2}} = -\widehat{\mathbf{m}}^{\rm eff}\cos\omega - \widehat{\mu}\sin\omega - \frac{\mathbf{w_8}}{\mathbf{4}}\cos(\mathbf{2}\omega)$$

- Extrema are solutions of quartic ($c = \cos\omega$) [Oliver's "gap equation"]

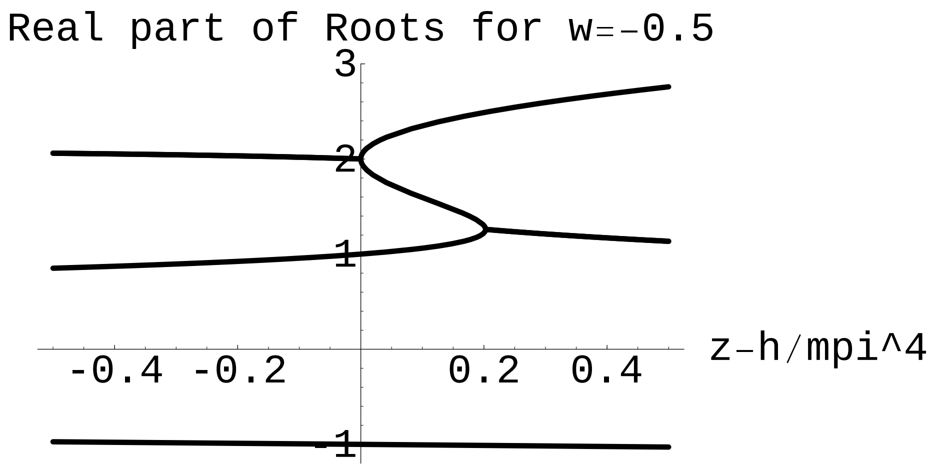
$$\mathbf{w_8^2 c^4} + \mathbf{2w_8}\widehat{\mathbf{m}}^{\rm eff}\mathbf{c^3} + [(\widehat{\mathbf{m}}^{\rm eff})^{\mathbf{2}} + \widehat{\mu}^{\mathbf{2}} - \mathbf{w_8^2}]\mathbf{c^2} - \mathbf{2w_8}\widehat{\mathbf{m}}^{\rm eff}\mathbf{c} = (\widehat{\mathbf{m}}^{\rm eff})^{\mathbf{2}}$$

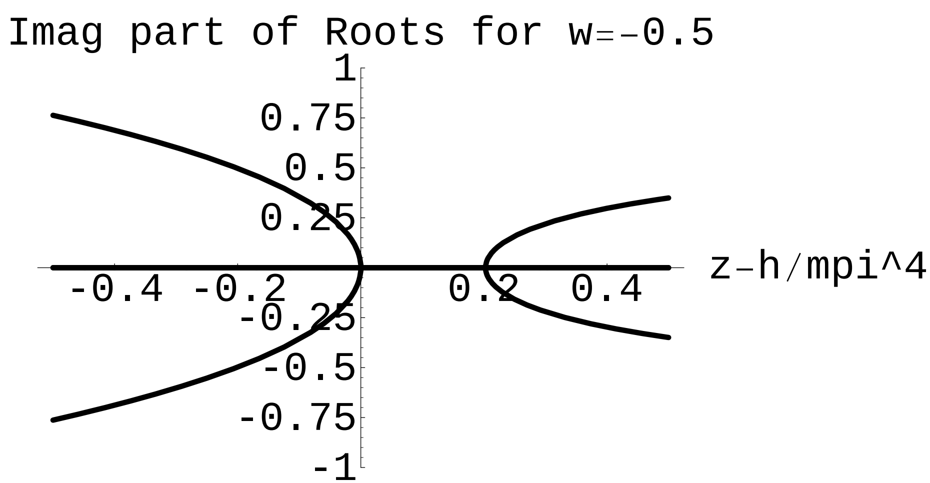
- Roots $\bar{c}$ [$= t(\text{Oliver})$] depend on $\widehat{m}^{\rm eff}/w_8$ and $(\widehat{\mu}/w_8)^2$
- Minimum of potential (explicitly depends on μ^2):

$$\frac{\mathcal{V}(\mu)_{\rm min}}{\mathbf{f^2}} = -\widehat{\mathbf{m}}^{\rm eff}\bar{\mathbf{c}} - \frac{\widehat{\mu}^{\mathbf{2}}\bar{\mathbf{c}}}{\widehat{\mathbf{m}}^{\rm eff} + \mathbf{w_8}\bar{\mathbf{c}}} - \frac{\mathbf{w_8}}{\mathbf{4}}(\mathbf{2}\bar{\mathbf{c}}^{\mathbf{2}} - \mathbf{1})$$

Roots for $w_8 < 0$ (I)

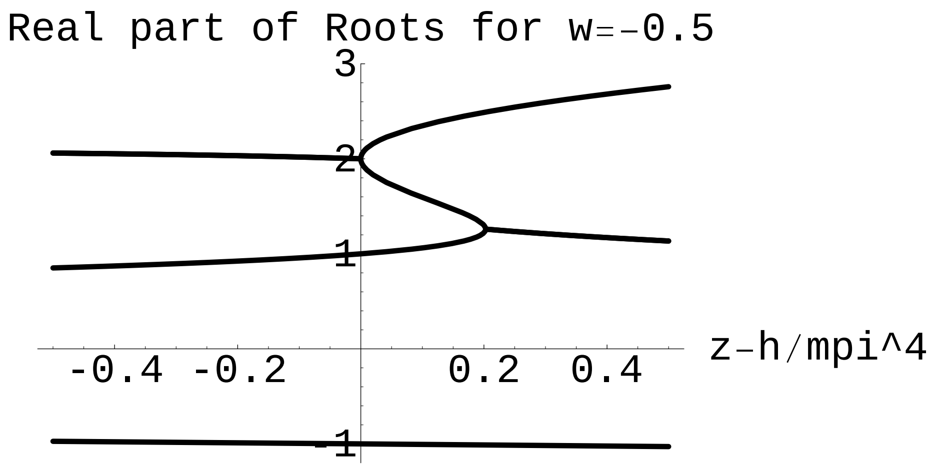
- $w_8 < 0$ corresponds to Aoki-phase if $|w_6| \ll |w_8|$
- Here $\widehat{m}^{\rm eff}/w_8 (\equiv 1/w) = -2$
- $-\widehat{m}^{\rm eff}/w_8 \approx \widehat{m}/|2w'|$ is quark mass in units of Aoki-phase half-width
- Roots plotted versus $(2B_0)^2 z/m_\pi^4 \equiv \widehat{z}/m_\pi^4$
- In continuum, gap at $\widehat{z}/m_\pi^4 = 1$
- Here reduced to $\widehat{z}/m_\pi^4 \approx 0.2$

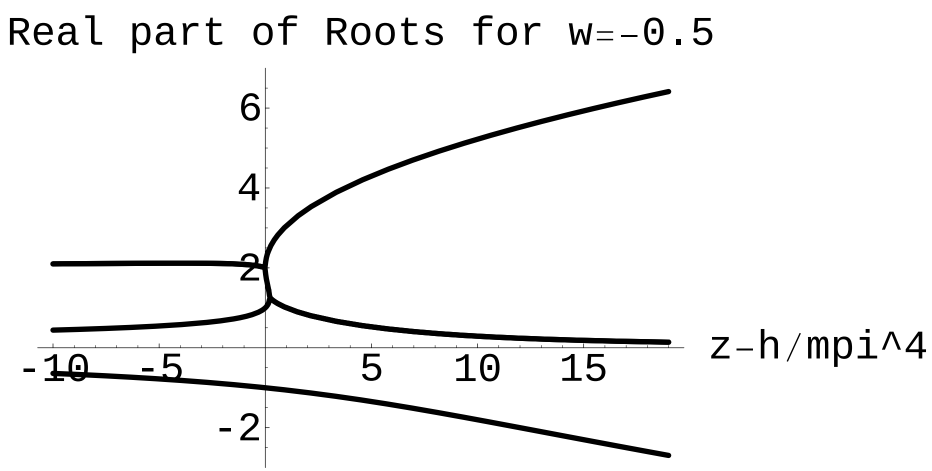




Roots for $w_8 < 0$ (II)

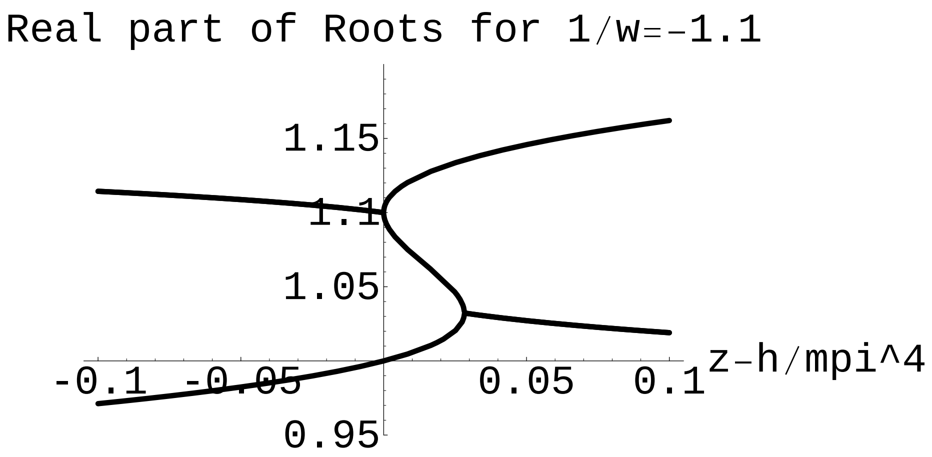
- Again $\widehat{m}^{\text{eff}}/w_8 = -2$
- Shows dependence for larger $\widehat{z}$





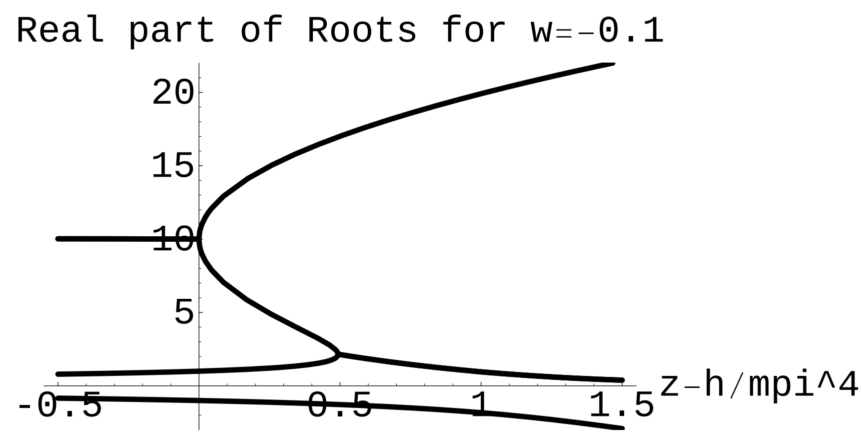
Roots for $w_8 < 0$ (III)

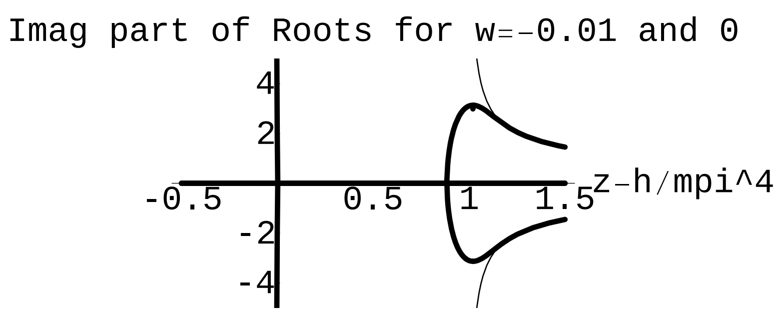
- Closer to Aoki end-point: $\widehat{m}^{\rm eff}/w_8 = -1.1$
- Gap nearly vanishes **even in units of** m_π^4

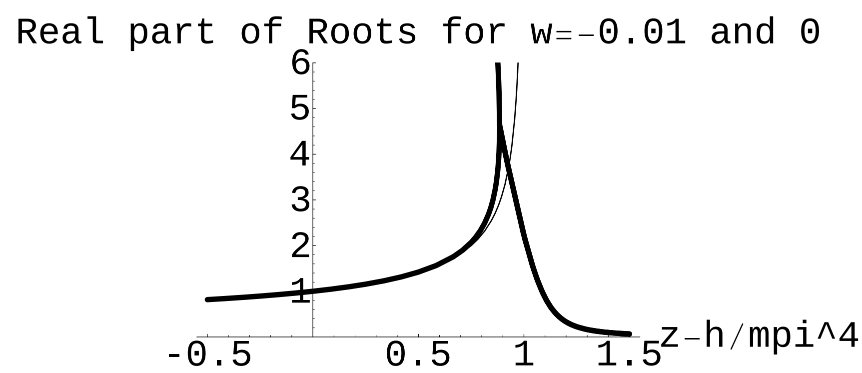


Roots for $w_8 < 0$ (IV)

- Towards continuum limit: $\widehat{m}^{\text{eff}}/w_8 = -10$ & -100
- Gap and spectrum approach continuum result

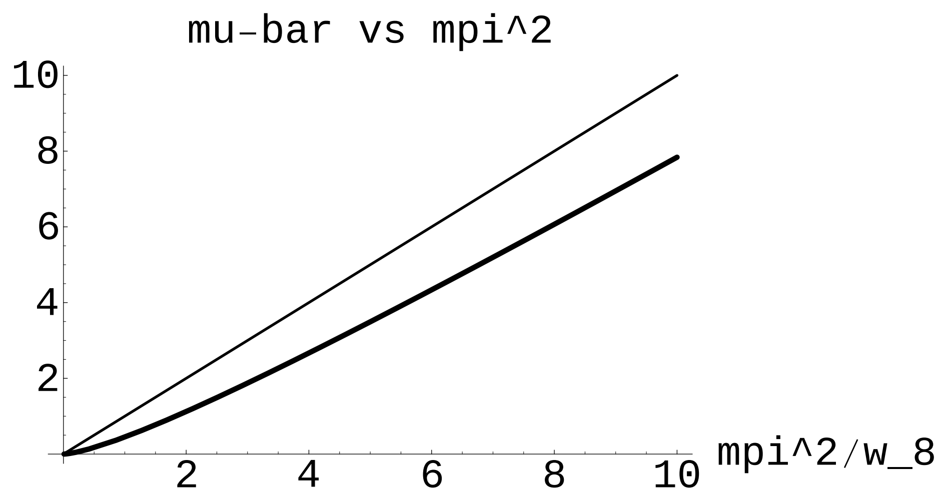




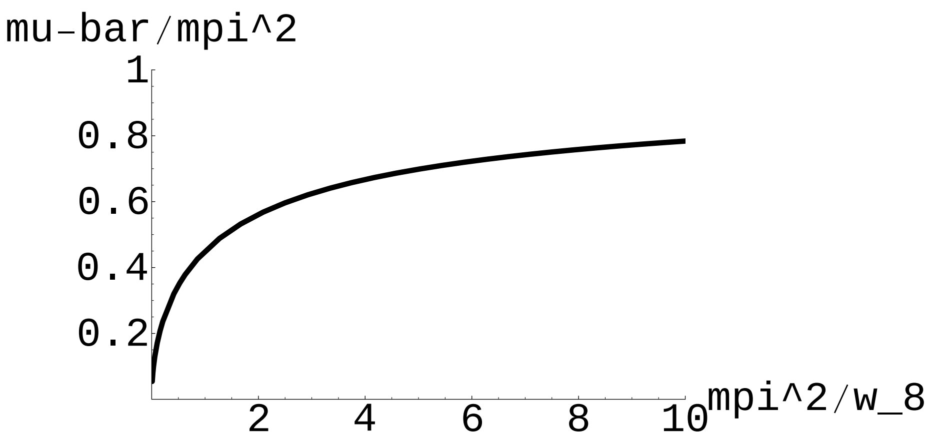


Gap for $w_8 < 0$

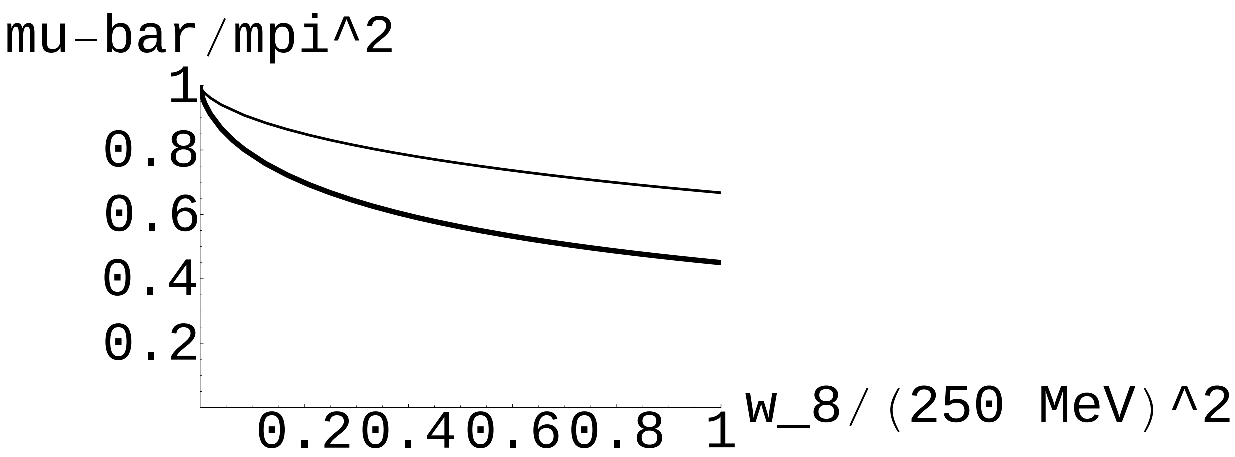
- Gap in Q_m $(2B_0\bar{\mu}_Q = 2B_0\sqrt{\bar{\alpha}})$ vs m_π^2
- Both quantities in units of $|w_8|$



- Gap in Q_m/m_π^2 vs m_π^2
- Square-root behavior can be understood analytically
- Simulations less safe near Aoki-phase end point

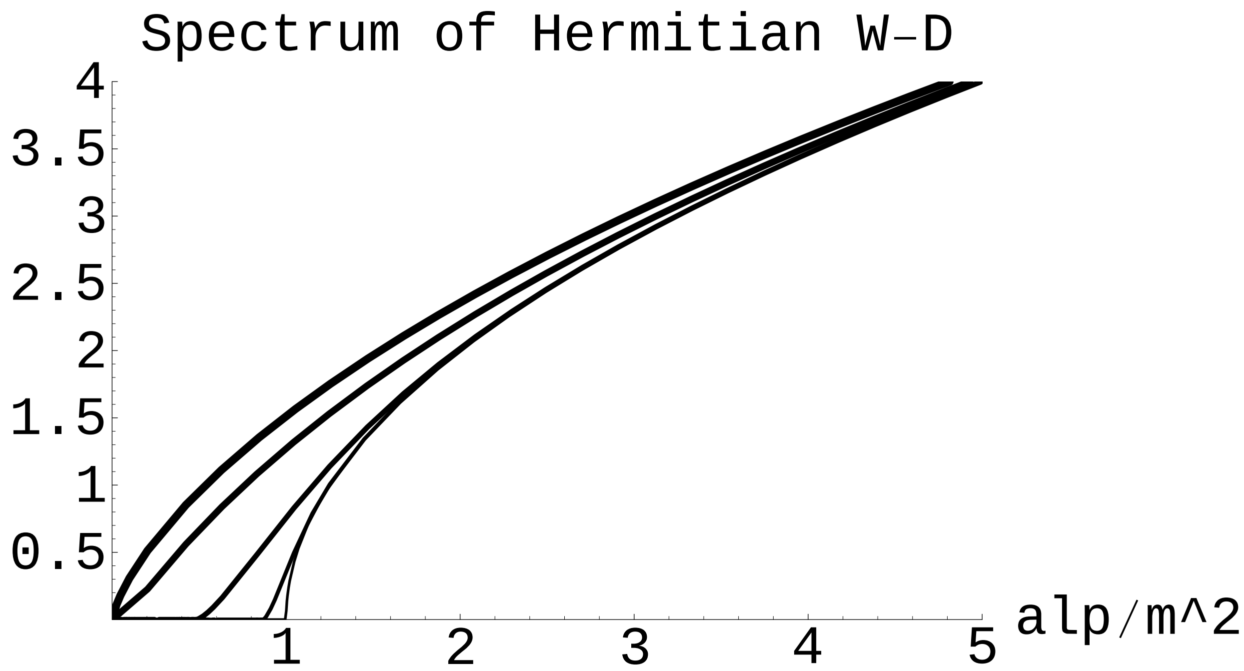


Scaling behavior of gap for $w_8 < 0$



- Hold physical pion mass fixed at 250 MeV (lower curve) and 500 MeV, and send $w_8 \propto a^2 \to 0$
- $w_8 = (250\ \mathrm{MeV})^2$ corresponds to roughly $a = 0.1$ fm with Wilson gauge action

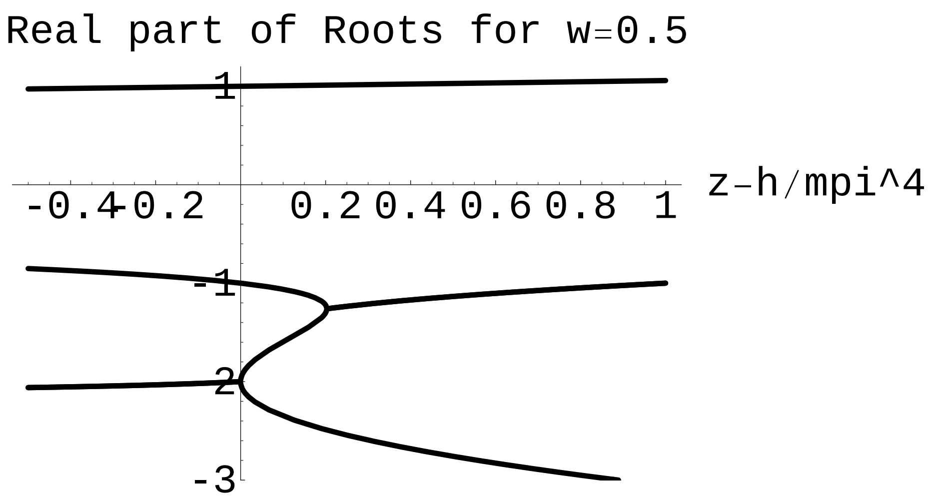
Scaling of spectral density for $w_8 < 0$

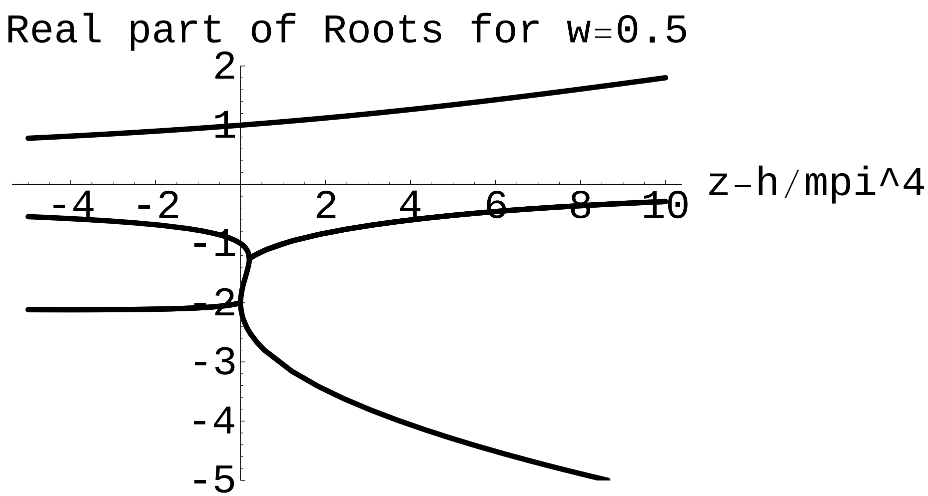


- Integrated spectral density $N_R(\alpha)$ (in arb. units) for $-\widehat{m}^{\rm eff}/w_8 = 1.1,\ 2,\ 10,\ 100,\ \infty$
- Horizontal scale is $\alpha/(m + w_6/B_0)^2$
- For fixed quark mass, spectrum does not translate as $a^2 \to 0$; instead, small eigenvalues move towards zero

Method breaks down for $w_8 > 0$

- $\widehat{m}^{\mathrm{eff}}/w_8 = +2$
- First-order scenario if $w_6 \ll w_8$
- Cut(s) move to complex plane
- $R_R(z)$ cannot satisfy initial dispersion relation
- Failure of LO χPT?
- Failure of method in general? But dispersion relation is satisfied for $w_8 < 0$





Conclusions and Outlook

- Developed method for studying discretizion errors in spectrum of lattice Dirac operator
- Avoids many subtleties of partially quenched χPT
- Works for $w_8 < 0$, and implies that gap is reduced by discretization effects
  - No sign of this in reduction in data of [Del Debbio *et al*]
  - $\Rightarrow$ w_8 small?
- Failure of method for $w_8 > 0$ is nagging worry
  - $\Rightarrow$ Check by using approach of [Osborn *et al*] or use replica trick Now done!
- Interesting to attempt extension to finite volume