Nucleon Structure from Lattice QCD

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– QCDSF Collaboration –

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Outline

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- **Some Basics**
- **Lattice QCD**
- **Nucleon Structure**
  - Form Factors
  - Spin Asymmetries
  - (Orbital) Angular Momentum
  - Generalized Parton Distributions
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Introduction

Much of our knowledge about QCD and the structure of hadrons has been derived from DIS experiments and measurements of elastic form factors

Perturbative QCD allowed us to extract quark and gluon distribution functions from the experimental data. However, a quantitative understanding of the observed phenomena is still missing

For example, we would like to know how quarks and gluons provide the binding and spin of the nucleon

Lattice QCD is the only fundamental formulation of QCD allowing the calculation of all its consequences in both the high and low energy regimes

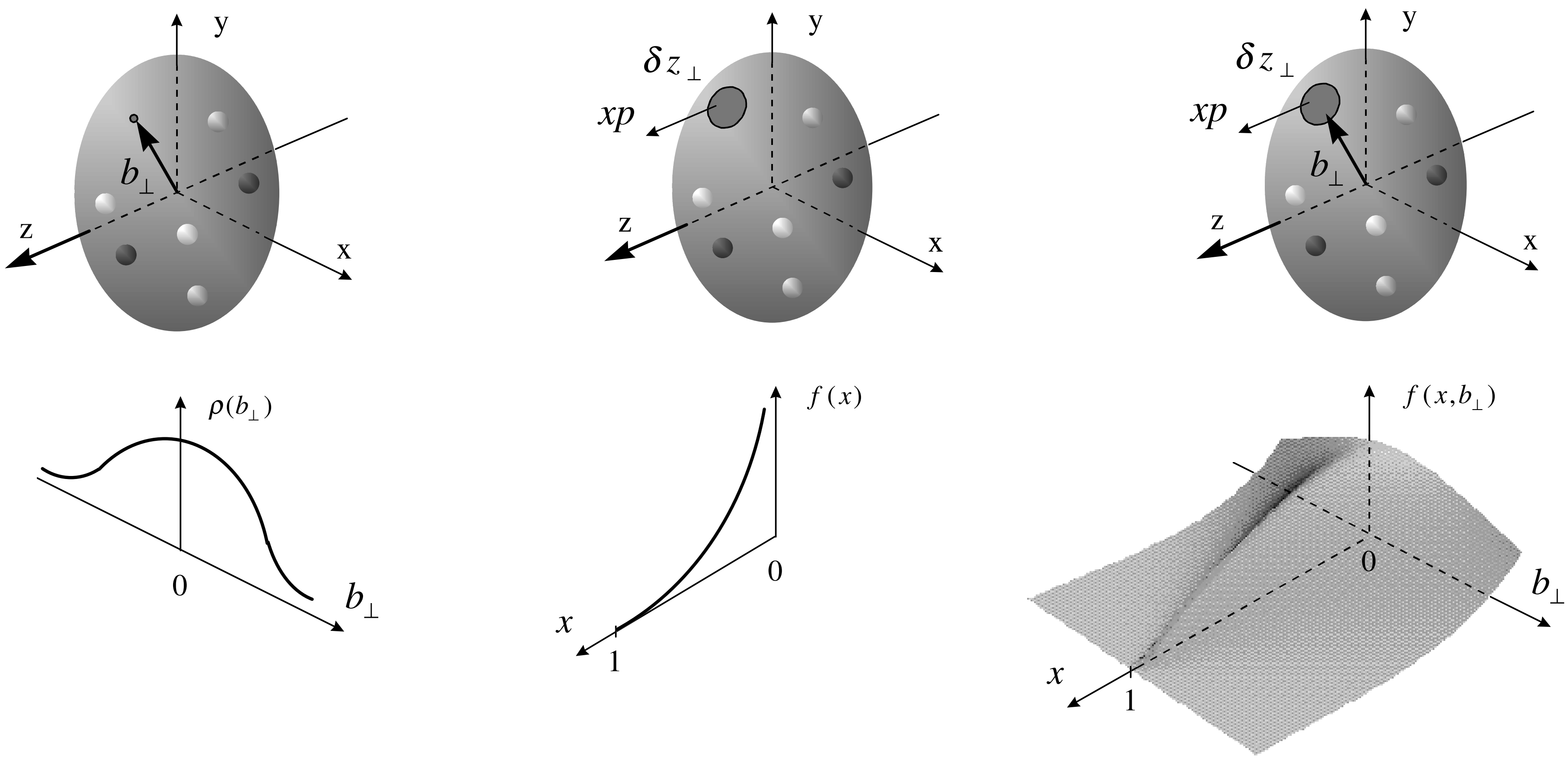
Continuing advances in computing power & algorithms, and theoretical developments, such as

- $O(a)$ Improvement of the action and the operators, to reduce finite cut-off effects and to facilitate the extrapolation to the continuum limit,
- (Non)-perturbative renormalization and matching of the (bare) lattice operators,
- Chiral perturbation theory, to extrapolate reliably from the masses where the lattice calculations are performed to the physical pion mass,

have now brought us to the point that definitive quantitative calculations of a host of hadron observables are becoming possible

Key quantities

- Form factor
- Parton density
- Generalized parton distribution at η=0



Spatial resolution: $\delta z_\perp \sim 1/Q$

Some Basics

OPE

$$\langle p, s|\, \mathcal{O}^{q}_{\{\mu_1\cdots\mu_n\}}\, |p, s\rangle = v^q_n\, \bar{u}(p, s)\, \left(\gamma_{\mu_1} p_{\mu_2} \cdot \cdots \cdot p_{\mu_n}\right) u(p, s)$$

$$\langle p, s|\mathcal{O}^{5q}_{\{\mu\,\mu_1\cdots\mu_n\}}|p, s\rangle = a^q_n\, \bar{u}(p, s)\, \left(\gamma_{\{\mu}\gamma_5 p_{\mu_1} \cdots p_{\mu_n\}}\right) u(p, s)$$

$$\langle p, s|\mathcal{O}^{5q}_{[\mu\{\mu_1]\cdots\mu_n\}}|p, s\rangle = d^q_n\, \bar{u}(p, s)\, \left(\gamma_{[\mu}\gamma_5 p_{\{\mu_1]} \cdots p_{\mu_n\}}\right) u(p, s)$$

$$\langle p, s|\mathcal{O}^{Tq}_{\mu\nu\{\mu_1\cdot\cdot\mu_n\}}|p, s\rangle = t^q_n\, \bar{u}(p, s)\, \left(\sigma_{\mu\nu}\gamma_5 p_{\{\mu_1} \cdots p_{\mu_n\}}\right) u(p, s)$$

$$\mathcal{O}^{q}_{\mu_1\cdots\mu_n} = \left(\frac{i}{2}\right)^{n-1} \bar{q}\gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n} q$$

$$\mathcal{O}^{5q}_{\sigma\mu_1\cdots\mu_n} = \left(\frac{i}{2}\right)^{n} \bar{q}\gamma_{\sigma}\gamma_5 \overleftrightarrow{D}_{\mu_1} \cdots \overleftrightarrow{D}_{\mu_n} q$$

$$\mathcal{O}^{Tq}_{\mu\nu\mu_1\cdots\mu_n} = \left(\frac{i}{2}\right)^{n} \bar{q}\sigma_{\mu\nu}\gamma_5\, \overleftrightarrow{D}_{\mu_1} \cdots \overleftrightarrow{D}_{\mu_n} q$$

In particular

$$v_n^q(\mu) = \int_0^1 dx\, x^{n-1} q(x,\mu^2) = \langle x^{n-1}\rangle^q$$

$$a_n^q(\mu) = \int_0^1 dx\, x^n \Delta q(x,\mu^2) = \Delta^n q \qquad \Rightarrow\rightarrow - \Rightarrow\leftarrow \qquad a_0^q = \Delta q, \quad \boxed{g_A = \Delta u - \Delta d}$$

$$t_n^q(\mu) = \int_0^1 dx\, x^n \delta q(x,\mu^2) = \delta^n q \qquad \overset{\uparrow}{\Uparrow} - \overset{\downarrow}{\Uparrow} \qquad t_0^q = \delta q, \quad \boxed{g_T = \delta u - \delta d}$$

$$2\int_0^1 dx\, x^n g_1(x,Q^2) = e_{1,n}(Q^2/\mu^2, g(\mu^2))\, a_n(\mu)$$

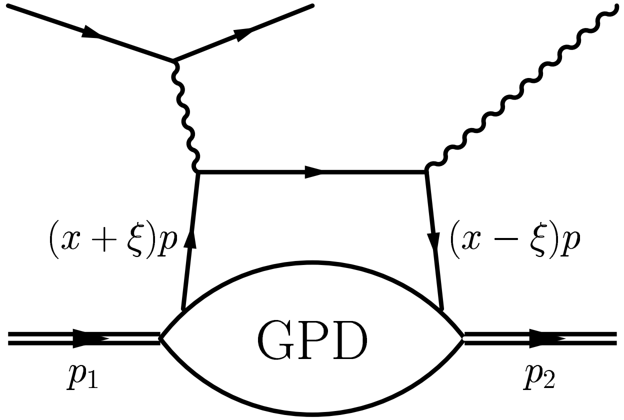
$$2\int_0^1 dx\, x^n g_2(x,Q^2) = \frac{n}{n+1}\left[e_{2,n}(Q^2/\mu^2, g(\mu^2))\, d_n(\mu) - e_{1,n}(Q^2/\mu^2, g(\mu^2))\, a_n(\mu)\right]$$

$\uparrow$

Twist-3

No parton model interpretation

Off forward



$p = \frac{1}{2}(p_1+p_2)$, $\Delta = p_2 - p_1$, $Q = \frac{1}{2}(q_1+q_2)$

$\xi = 0$: Momentum transfer of the struck parton purely transverse, i.e. $\Delta = \mathbf{\Delta}_\perp$

$\Uparrow$

Of interest to us here only

$$\langle p_1, s|\, \mathcal{O}^q_{\{\mu_1\cdots\mu_n\}}\, |p_2, s\rangle = \bar{u}(p_1, s)\left[A^q_n(\Delta^2)\gamma_{\{\mu_1} + \frac{\mathrm{i}\Delta^\alpha}{2m_N} B^q_n(\Delta^2)\sigma_{\alpha\{\mu_1}\right] p_{\mu_2}\cdot\cdot p_{\mu_n\}}\, u(p_2, s) + \cdot\cdot$$

$$\langle p_1, s|\mathcal{O}^{5q}_{\{\mu\mu_1\cdot\cdot\mu_n\}}|p_2, s\rangle = \bar{u}(p_1, s)\left[\tilde{A}^q_{n+1}(\Delta^2)\gamma_{\{\mu}\gamma_5 p_{\mu_1}\cdot\cdot p_{\mu_n\}}\right] u(p_2, s) + \cdot\cdot$$

$$\langle p_1, s|\mathcal{O}^{Tq}_{\{\mu\nu\mu_1\cdot\cdot\mu_n\}}|p_2, s\rangle = \bar{u}(p_1, s)\left[A^{Tq}_{n+1}(\Delta^2)\, \sigma_{\mu\nu}\gamma_5\, p_{\{\mu_1}\cdot\cdot p_{\mu_n\}}\right] u(p_2, s) + \cdot\cdot$$

$$A_n^q(\Delta^2) = \int_0^1 dx\, x^{n-1} H^q(x, \Delta^2)$$

$$H^q(x, 0) = q(x)$$

$$B_n^q(\Delta^2) = \int_0^1 dx\, x^{n-1} E^q(x, \Delta^2)$$

$$\tilde{A}_n^q(\Delta^2) = \int_0^1 dx\, x^{n-1} \tilde{H}^q(x, \Delta^2)$$

$$\tilde{H}^q(x, 0) = \Delta q(x)$$

$$A_n^{Tq}(\Delta^2) = \int_0^1 dx\, x^{n-1} H^{Tq}(x, \Delta^2)$$

$$H^{Tq}(x, 0) = \delta q(x)$$

↑ GFFs

↑ GPDs

$$\frac{1}{2}\left(A_2^q(0) + B_2^q(0)\right) = J^q$$

Ji

$$A_1^q(\Delta^2) = F_1^q(\Delta^2)$$

$$B_1^q(\Delta^2) = F_2^q(\Delta^2)$$

$$\tilde{A}_1^q(\Delta^2) = g_A^q(\Delta^2)$$

$$A_1^{Tq}(\Delta^2) = g_T^q(\Delta^2)$$

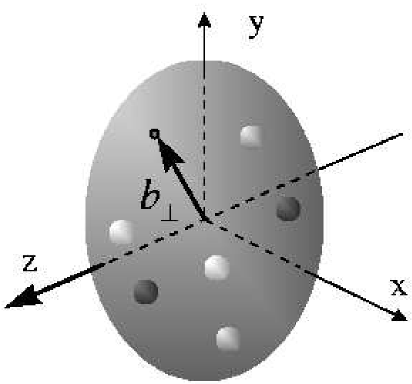
In impact parameter space

Generically

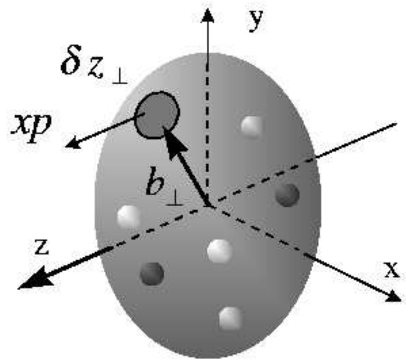
$$A_n^q(\mathbf{b}_\perp^2) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2} \, \mathrm{e}^{\,\mathrm{i}\,\mathbf{b}_\perp \mathbf{\Delta}_\perp} A_n^q(\mathbf{\Delta}_\perp^2) \qquad \Longleftrightarrow \qquad \boxed{\langle p_+, s | \bar{q}(\mathbf{b}_\perp) \cdots q(\mathbf{b}_\perp) | p_+, s \rangle}$$

$$|p_+, s\rangle = \mathcal{N} \int \frac{d^2\mathbf{p}_\perp}{(2\pi)^2} |p_+, \mathbf{p}_\perp, s\rangle$$

$$H^q(x, \mathbf{b}_\perp^2) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2} \, \mathrm{e}^{\,\mathrm{i}\,\mathbf{b}_\perp \mathbf{\Delta}_\perp} H^q(x, \mathbf{\Delta}_\perp^2)$$



$F_1(\mathbf{b}_\perp^2) \equiv A_1(\mathbf{b}_\perp^2)$



$H(x, \mathbf{b}_\perp^2)$

$\Longrightarrow$ Probability interpretation Burkardt

Lattice QCD

Action

$$S = S_G + S_F$$

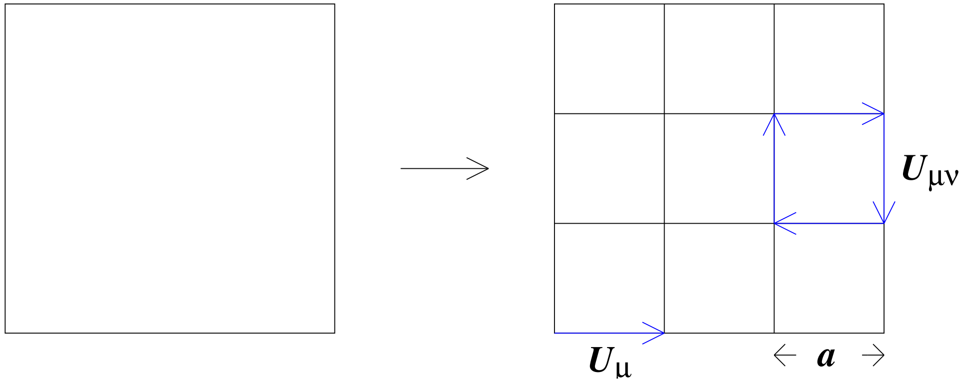
$$S_G = \beta \sum_{x,\mu<\nu} \left(1 - \frac{1}{3} \mathrm{Re}\,\mathrm{Tr}\, U_{\mu\nu}(x)\right)$$

$$S_F = \sum_x \Big\{ \bar\psi(x)\psi(x) - \kappa\, \bar\psi(x) U^\dagger_\mu(x-\hat\mu)[1+\gamma_\mu]\psi(x-\hat\mu) - \kappa\, \bar\psi(x) U_\mu(x)[1-\gamma_\mu]\psi(x+\hat\mu) - \frac{1}{2}\kappa\, c_{SW}\, g\, \bar\psi(x)\sigma_{\mu\nu}F_{\mu\nu}(x)\psi(x) \Big\}$$

Nonperturbatively $O(a)$ improved

Sheikholeslami & Wohlert

The simulation

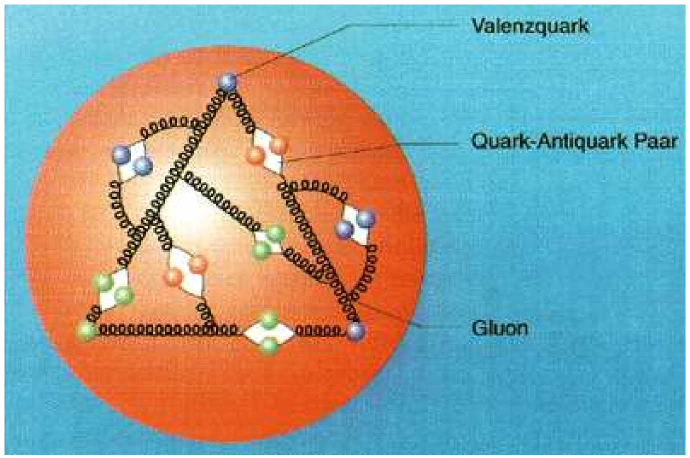


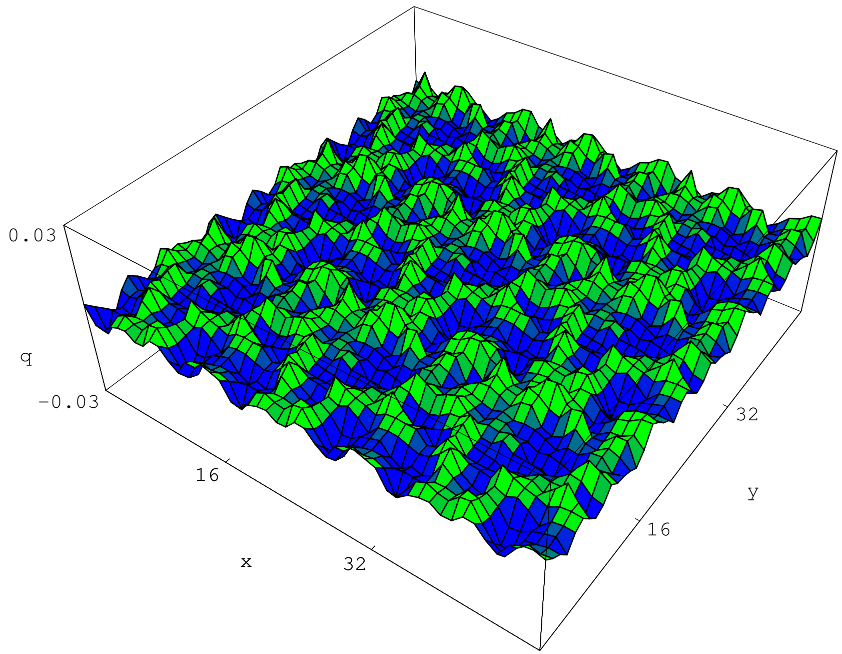
Generate sequence of configurations: HMC

$$\mathcal{P}\{U_\mu^{(i)}\} \propto \int \prod_x \mathcal{D}\bar{\psi}(x)\mathcal{D}\psi(x)\, \mathrm{e}^{-S_F - S_G}$$

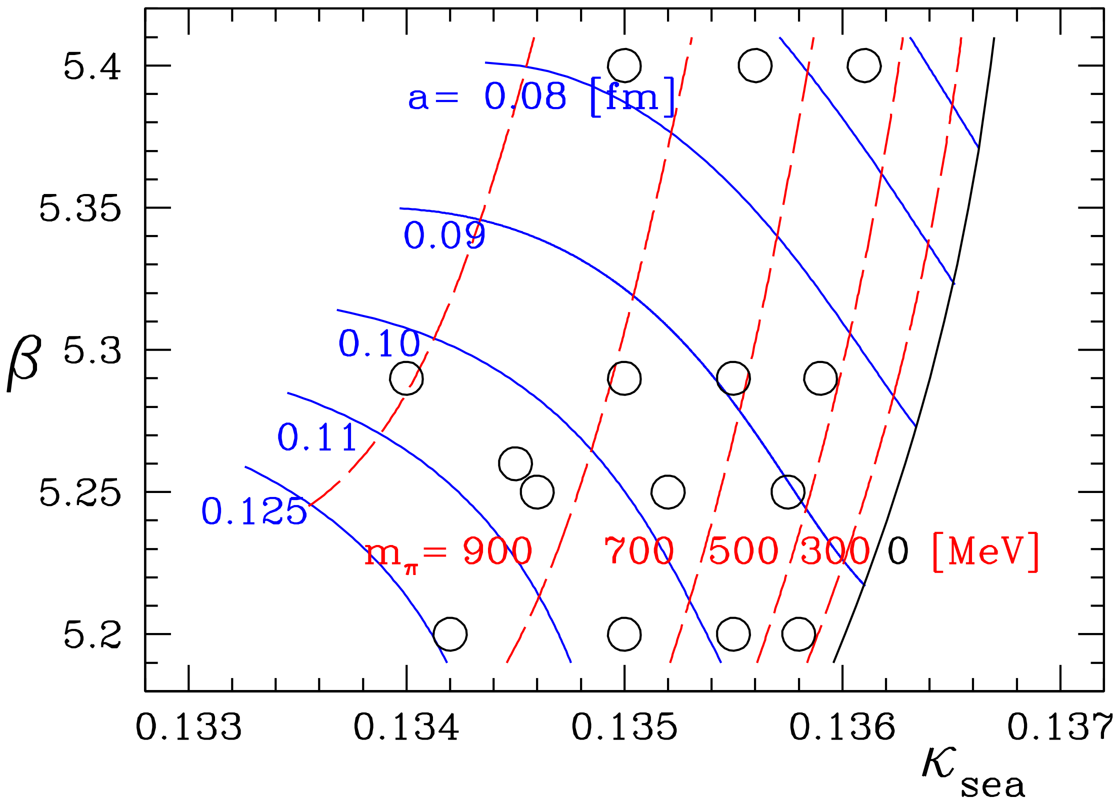
$$= \det\left(D(U_\mu^{(i)}) + am\right) \mathrm{e}^{-S_G}$$

$$\langle \mathcal{O} \rangle = \frac{1}{N}\sum_{i=1}^{N} \mathcal{O}(U_\mu^{(i)})$$





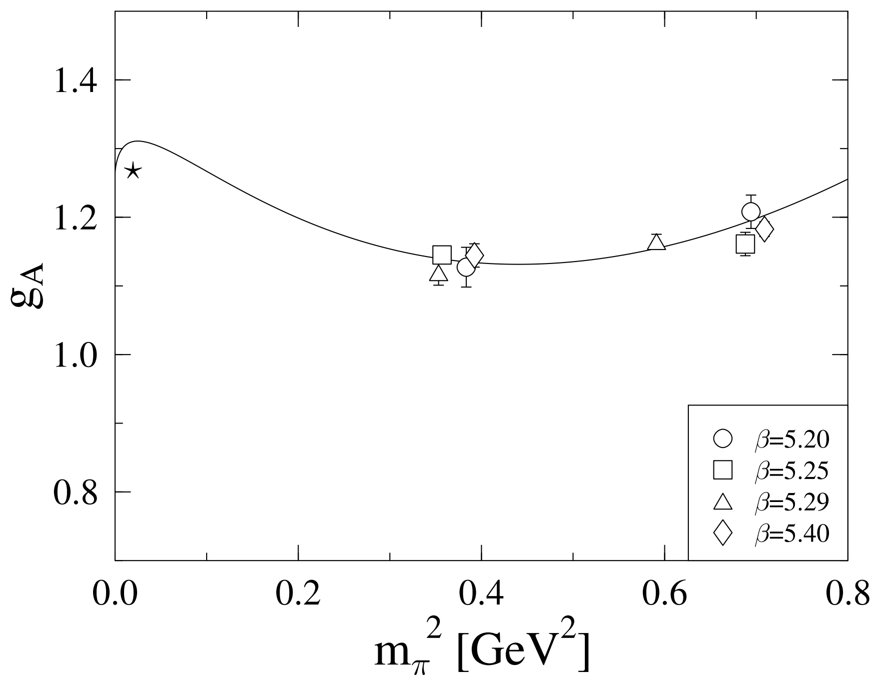
State of the calculation

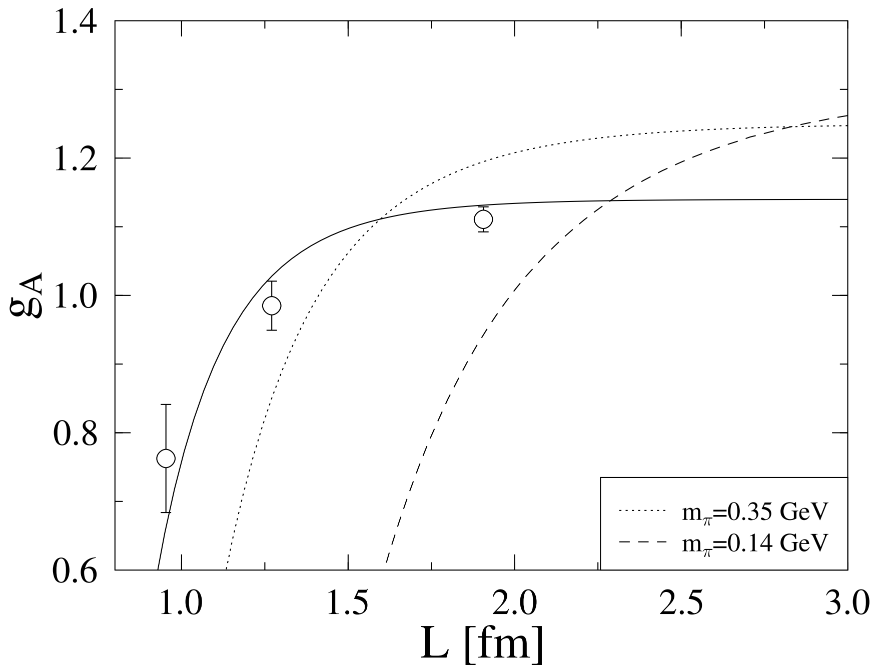


$0.07\ \text{fm} \lesssim a \lesssim 0.12\ \text{fm}\ , \quad 1\ \text{fm} \lesssim L \lesssim 2.2\ \text{fm}$

Chiral limit

Extrapolation to the chiral limit needs theoretical guidance, which may be provided by Chiral Perturbation Theory (χPT)



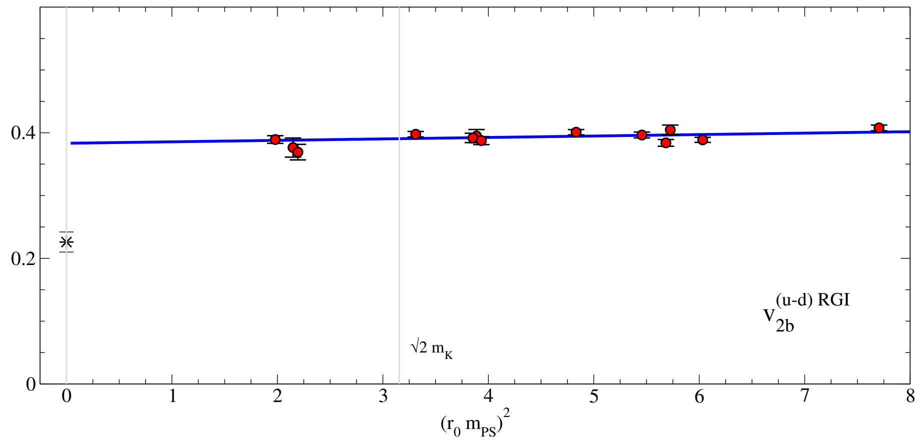


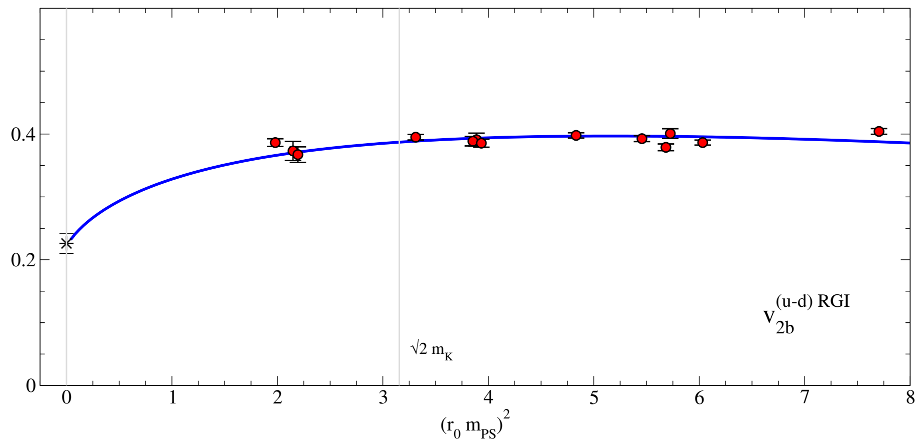
χPT $O(p^3)$

including finite volume corrections

An intractable case

$$\langle x \rangle = \int_0^1 dx\, x\, q(x, \mu^2)$$

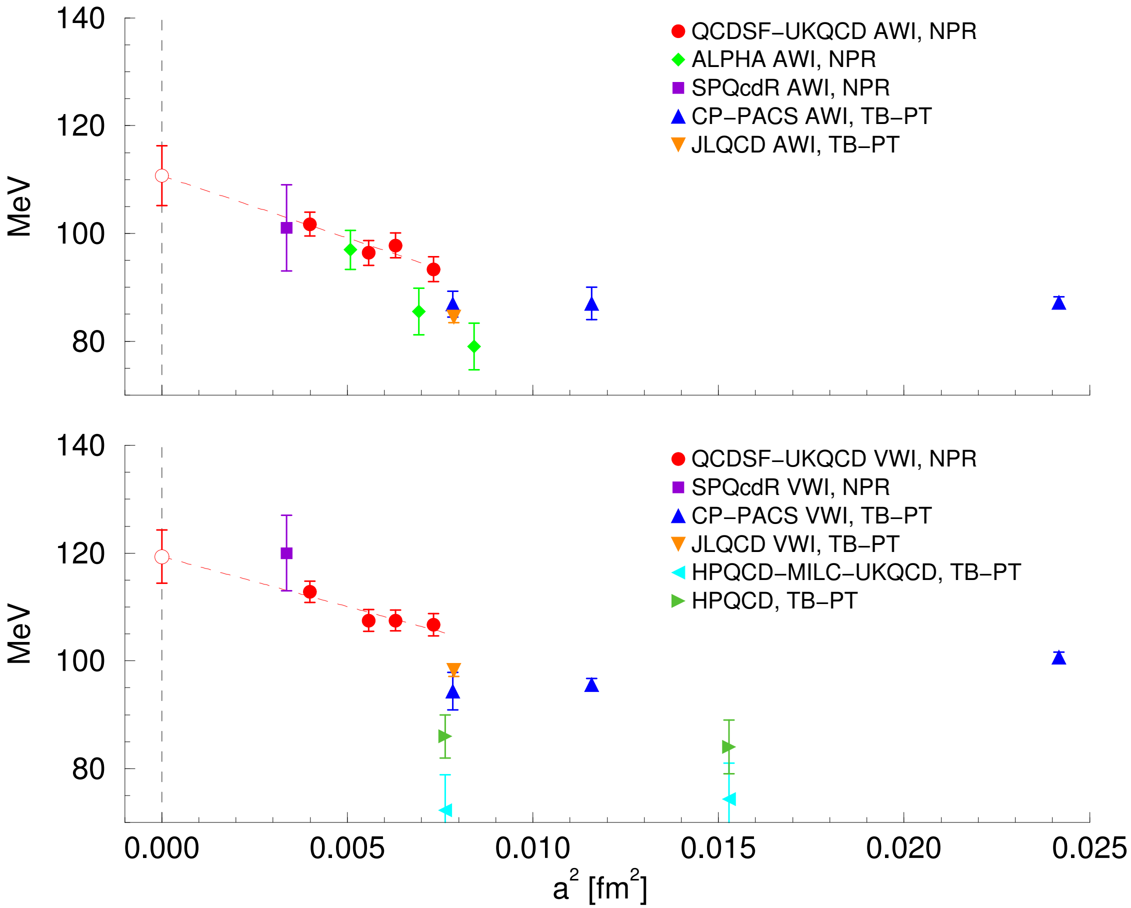




· · · but there is hope

Continuum limit

$m_s^{\overline{MS}}(2\text{ GeV})$



Nucleon Structure

Focus on

Form Factors	$A_1(\Delta^2)$, $B_1(\Delta^2)$
Spin Asymmetries	$A_1^T(\Delta^2)$, $\tilde{A}_1^T(\Delta^2)$, $B_1^T(\Delta^2)$
(Orbital) Angular Momentum	$A_2(\Delta^2)$, $B_2(\Delta^2)$
Generalized Parton Distributions	$A_1(\Delta^2)$, $A_2(\Delta^2)$, $A_3(\Delta^2)$, $A_4(\Delta^2)$

Renormalization of operators is done nonperturbatively

Form Factors

$$F_1(\Delta^2) = A_1(\Delta^2)$$

$$F_2(\Delta^2) = B_1(\Delta^2)$$

Sachs form factors

$$G_e(\Delta^2) \ = F_1(\Delta^2) + \frac{\Delta^2}{4m_N^2} F_2(\Delta^2)$$

$$G_m(\Delta^2) = F_1(\Delta^2) + F_2(\Delta^2)$$

$$F_1(0) = e_N$$

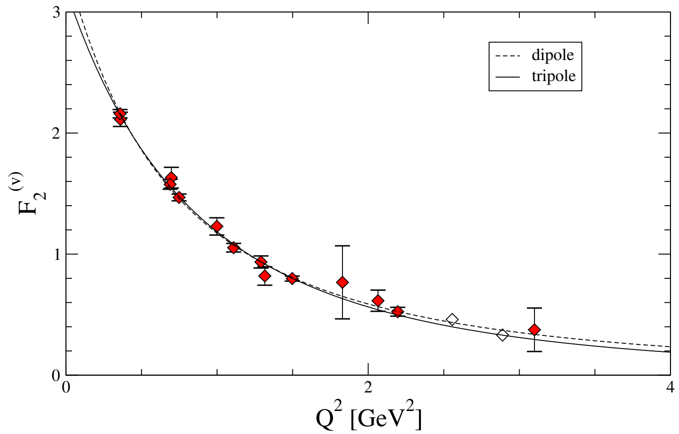
$$F_2(0) = \mu_N - e_N = \kappa_N$$

$$G_e(0) \ = e_N$$

$$G_m(0) = \mu_N = 1 + \kappa_N$$

Benchmark calculation

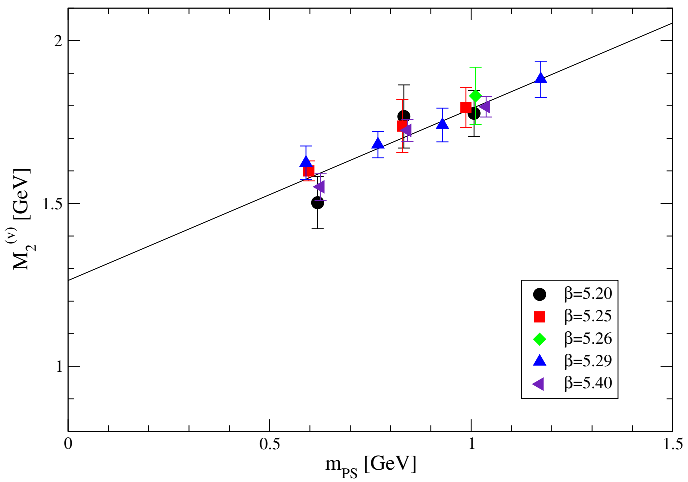
Proton – Neutron

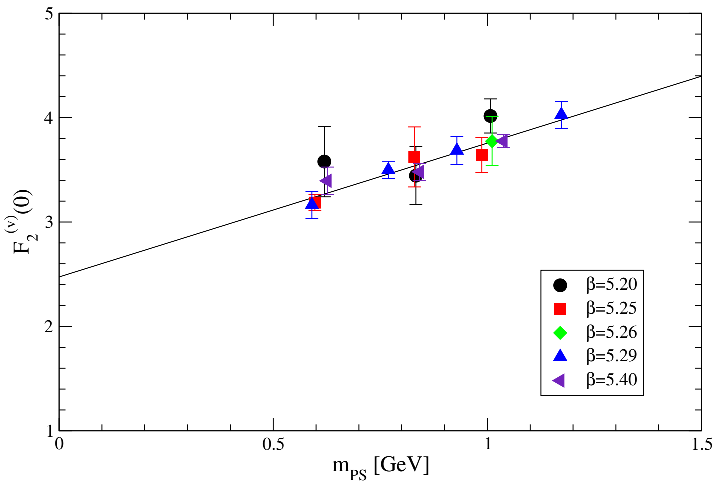


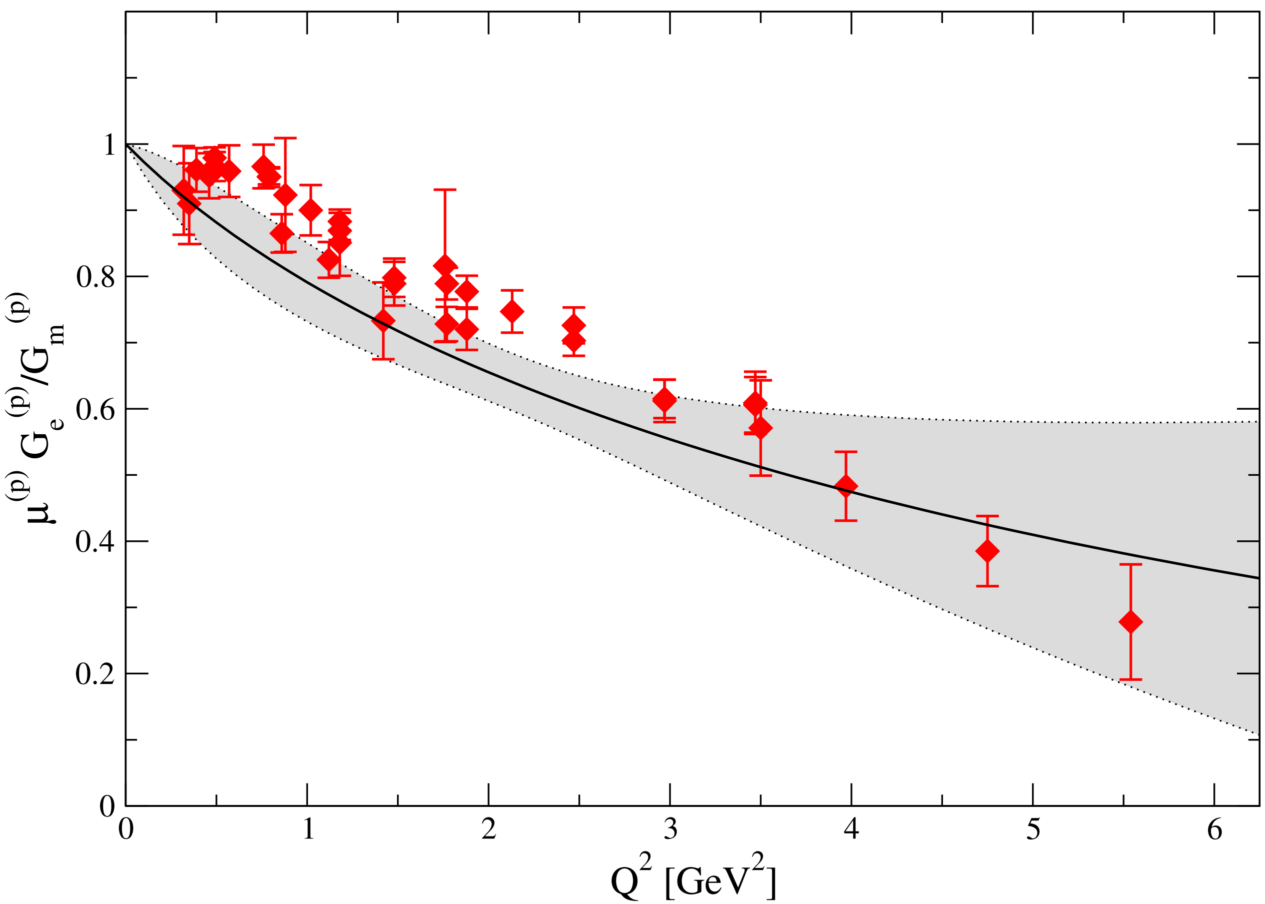
$\beta = 5.25, \kappa_{\rm sea} = 0.13575$

$$F(Q^2) = \frac{F(0)}{(1 + Q^2/M^2)^2}$$

$$Q^2 = -\Delta^2$$





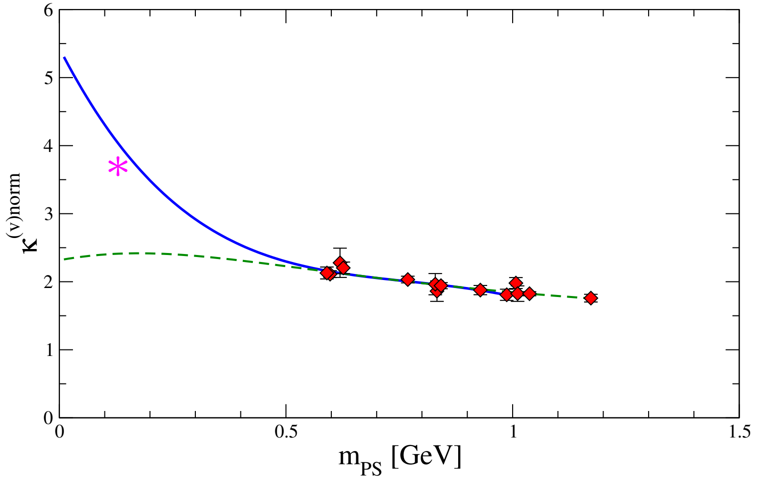


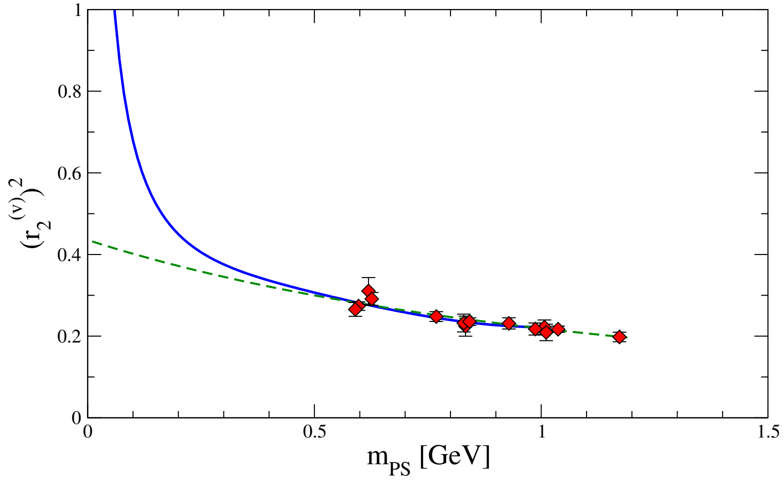
Data from JLab

Comparison with χPT

Combined fit to radii and magnetic moment

$$F_i(Q^2) = F_i(0)\left[1 - \frac{1}{6} r_i^2 Q^2 + O(Q^4)\right]$$





For example

$$\kappa_v(m_\pi) = \kappa_v^0 - \frac{g_A^2\, m_\pi M_N}{4\pi F_\pi^2} + \frac{2c_A^2 \Delta M_N}{9\pi^2 F_\pi^2}\left\{\sqrt{1-\frac{m_\pi^2}{\Delta^2}}\log R(m_\pi) + \log\left[\frac{m_\pi}{2\Delta}\right]\right\}$$
$$- 8E_1^{(r)}(\lambda) M_N m_\pi^2 + \frac{4c_A c_V g_A M_N m_\pi^2}{9\pi^2 F_\pi^2}\log\left[\frac{2\Delta}{\lambda}\right] + \frac{4c_A c_V g_A M_N m_\pi^3}{27\pi F_\pi^2 \Delta}$$
$$- \frac{8c_A c_V g_A \Delta^2 M_N}{27\pi^2 F_\pi^2}\left\{\left(1-\frac{m_\pi^2}{\Delta^2}\right)^{3/2}\log R(m_\pi) + \left(1-\frac{3m_\pi^2}{2\Delta^2}\right)\log\left[\frac{m_\pi}{2\Delta}\right]\right\}$$

$$R(m) = \frac{\Delta}{m} + \sqrt{\frac{\Delta^2}{m^2} - 1}$$

Spin Asymmetries

Tensor operator (transversity) $\Longrightarrow$ 0^{th} moments

$$\langle p_1, s|\bar{q}\sigma_{\mu\nu}\gamma_5 q|p_2, s\rangle = \bar{u}(p_1, s)\left\{\sigma_{\mu\nu}\gamma_5 A_1^{Tq}(\Delta^2) + \frac{p_{[\mu}\Delta_{\nu]}}{m_N^2}\tilde{A}_1^{Tq}(\Delta^2)\right.$$

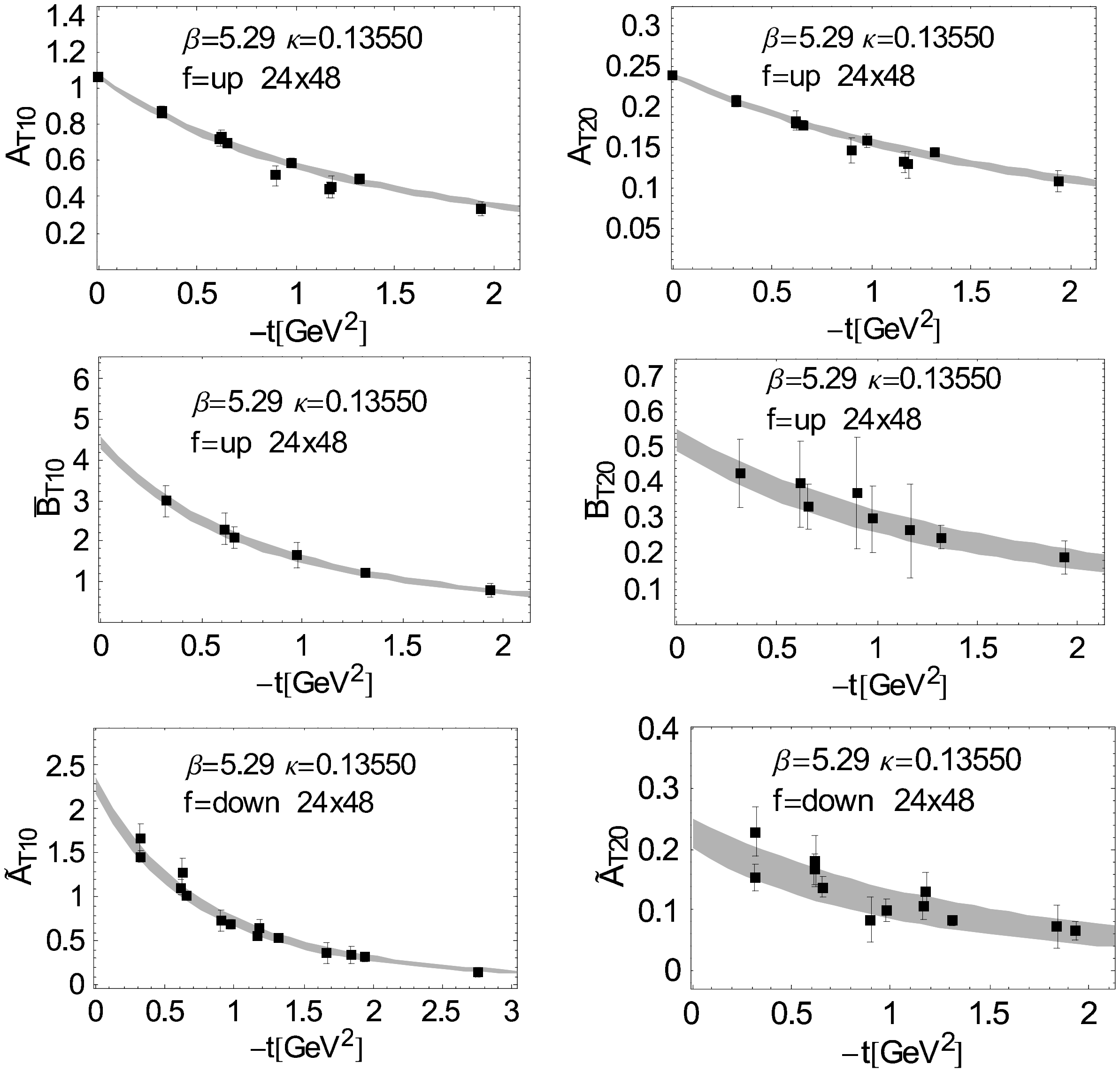
$$\left. + \frac{\gamma_{[\mu}\Delta_{\nu]}}{2m_N}B_1^{Tq}(\Delta^2)\right\} u(p_2, s)$$

Compute

$$A_1^{Tq}(\Delta^2) = \int_0^1 dx\, H^{Tq}(x, \Delta^2)$$

$$\tilde{A}_1^{Tq}(\Delta^2) = \int_0^1 dx\, \tilde{H}^{Tq}(x, \Delta^2)$$

$$B_1^{Tq}(\Delta^2) = \int_0^1 dx\, E^{Tq}(x, \Delta^2)$$



Dipole fit

1^{st} moment

To be extrapolated to chiral limit

In impact parameter space, for both quark and nucleon being polarized in the transverse plane

$\lambda_\perp$ quark spin

$s_\perp$ nucleon spin

Projection operator

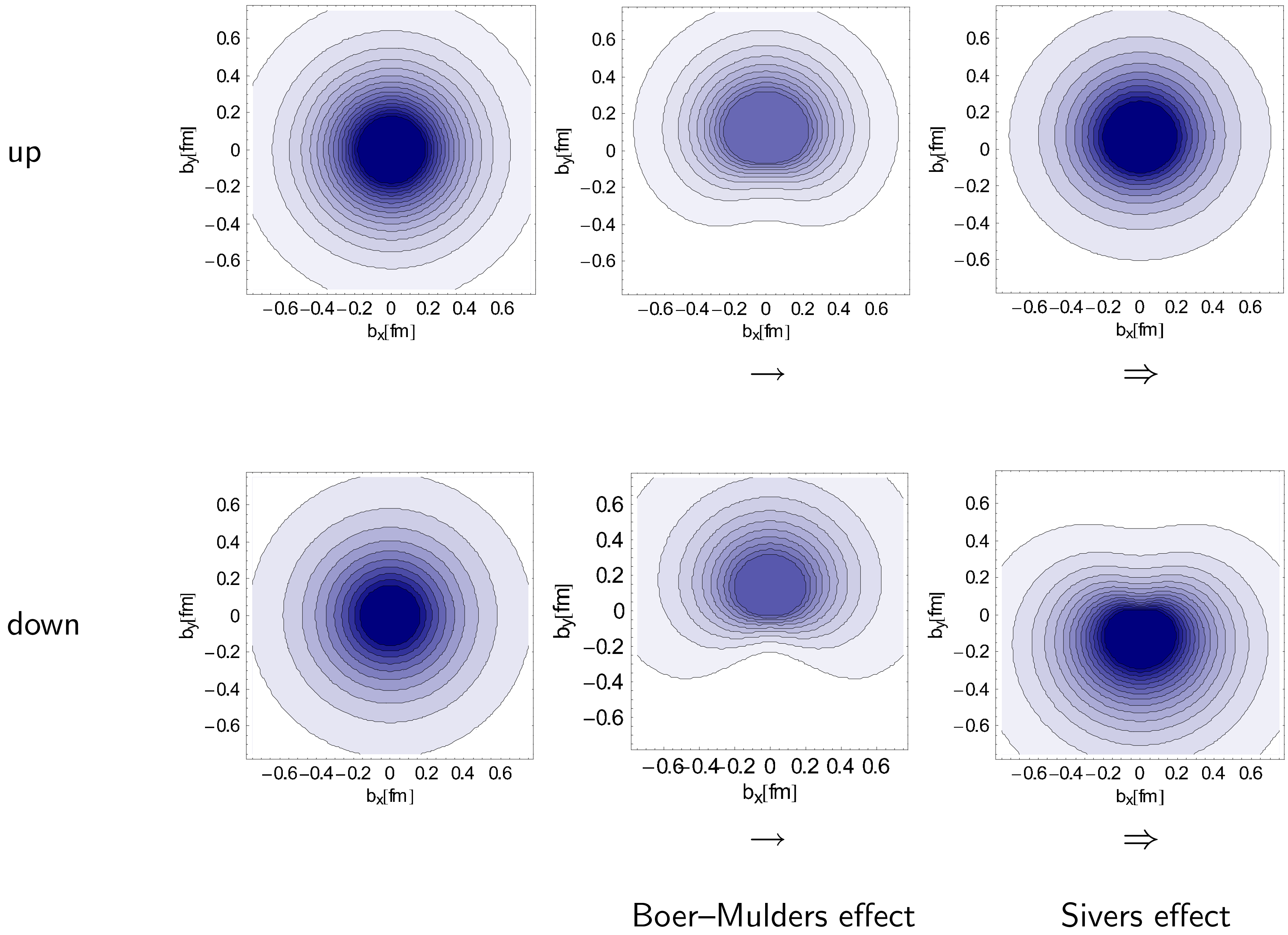
$\downarrow$

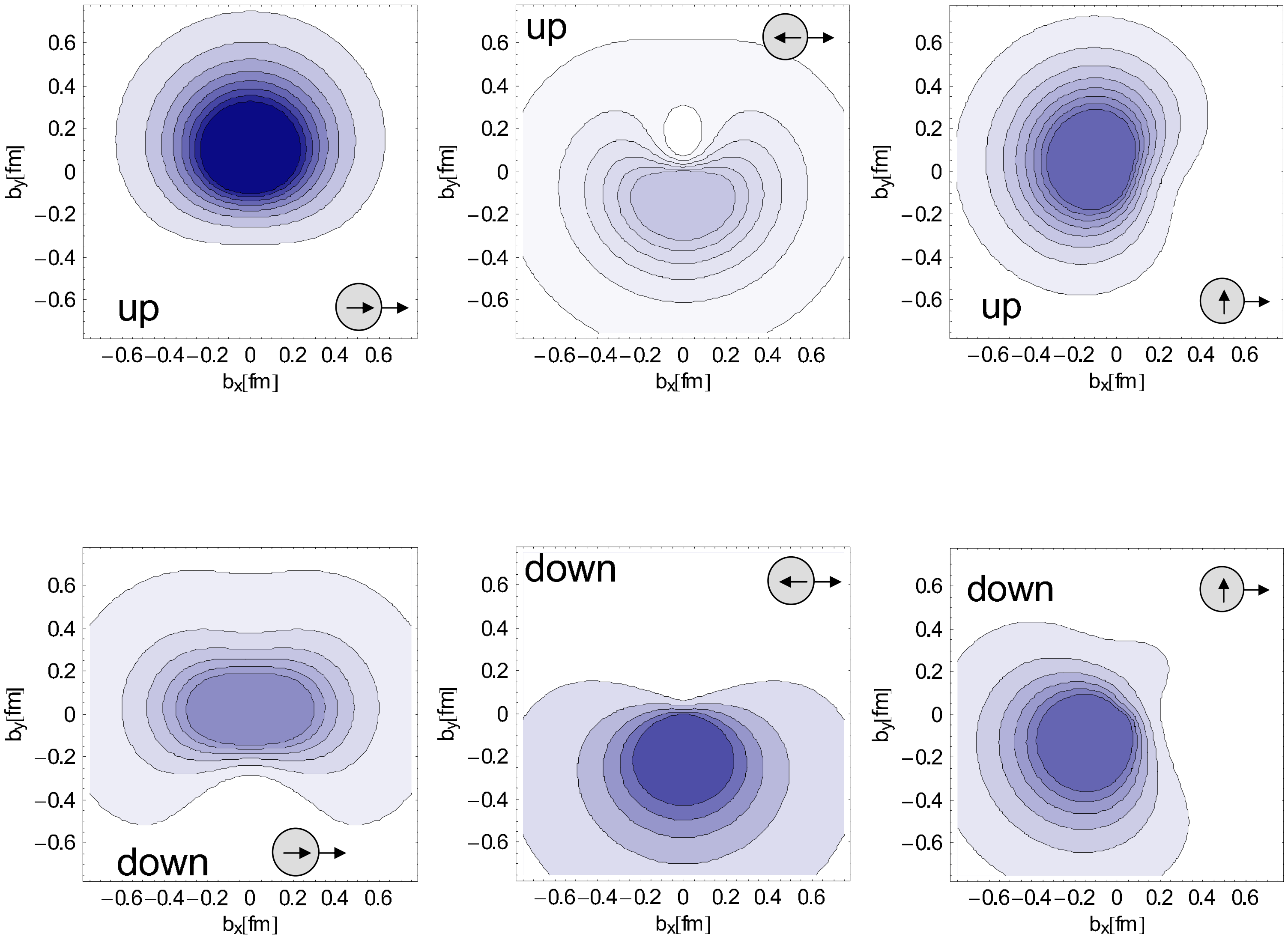
$$\langle p_+, s_\perp | \bar{q}(\mathbf{b}_\perp) \big[\gamma_+ - \lambda_{\perp i}\, \sigma_{+j}\, \gamma_5\big] q(\mathbf{b}_\perp) | p_+, s_\perp \rangle = \Big\{ A_1^q(\mathbf{b}_\perp^2) + \lambda_{\perp i}\, s_{\perp i} \Big[A_1^{Tq}(\mathbf{b}_\perp^2) - \frac{1}{4m_N^2} \Delta_{b_\perp} A_1^{Tq}(\mathbf{b}_\perp^2) \Big] + \frac{1}{m_N} b_{\perp i} \Big[s_{\perp i} + \lambda_{\perp i}\, B_1^{Tq}(\mathbf{b}_\perp^2)' \Big] + \frac{1}{m_N^2} \lambda_{\perp i}\, (2 b_{\perp i}\, b_{\perp j} - \mathbf{b}_\perp^2 \delta_{ij}) s_{\perp j}\, \tilde{A}_1^{Tq}(\mathbf{b}_\perp^2)'' \Big\}$$

Diehl & Hägler

$\Uparrow$

Probability


up
down
b_y[fm]
b_x[fm]
−0.6 −0.4 −0.2 0 0.2 0.4 0.6
→
⇒
Boer–Mulders effect
Sivers effect



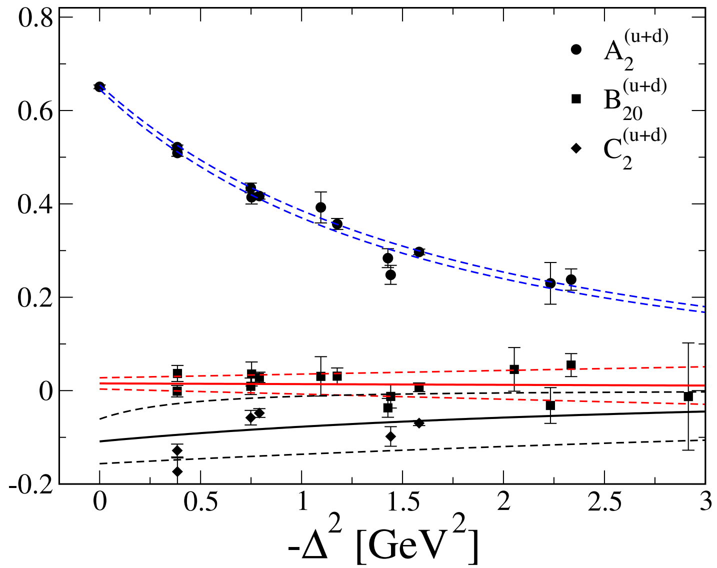
up
down
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
b_y[fm]
b_x[fm]

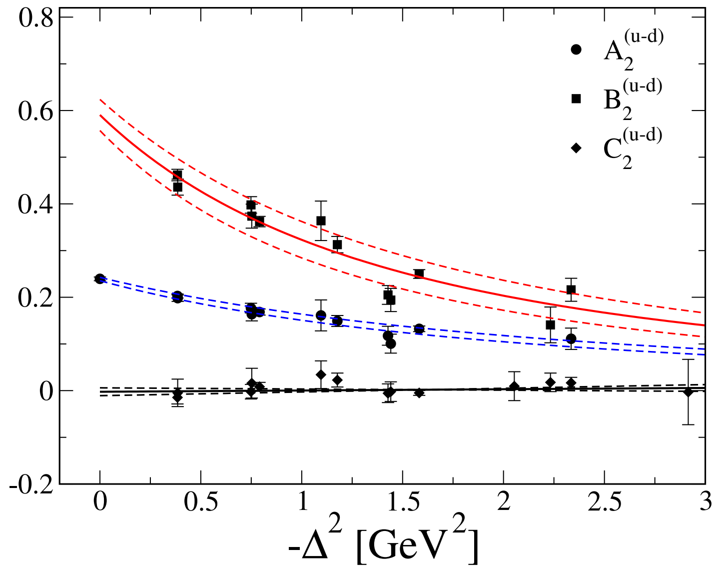
(Orbital) Angular Momentum

$$\frac{1}{2} = \underbrace{\frac{1}{2}\Delta\Sigma + L^q}_{J^q} + J^g\,, \quad \Delta\Sigma = \sum_q \Delta q$$

$$\text{EMT}: J^q = \frac{1}{2}\left(A_2^q(0) + B_2^q(0)\right)$$

$\beta = 5.40, \kappa_{sea} = 0.1350$

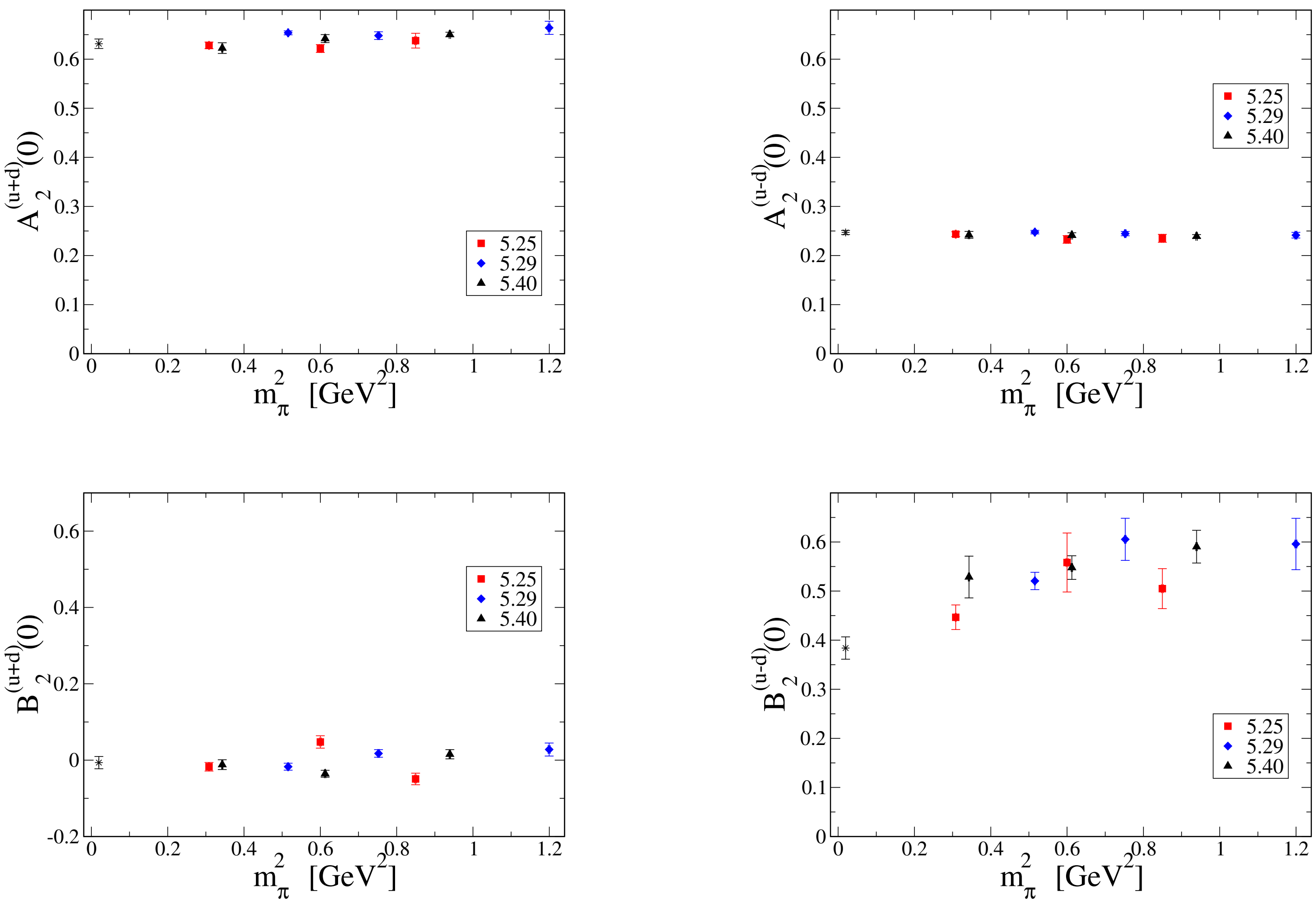




$$A_2(\Delta^2) = \frac{A_2(0)}{(1 - \Delta^2/M_2^2)^2}$$

$$B_2(\Delta^2) = \frac{B_2(0)}{(1 - \Delta^2/\hat{M}_2^2)^2}$$

Chiral extrapolation



$$J^{u+d} \approx \frac{1}{2} \langle x \rangle^{u+d} \qquad J^{u-d} \approx \frac{5}{4} \langle x \rangle^{u-d}$$

$$J^q = L^q + S^q, \quad S^q = \frac{1}{2}\Delta q$$

$$J^u = \quad 0.32(4), \quad J^d \approx 0$$

$\overline{MS}$, $Q^2 = 4$ GeV^2

$$L^u = -0.21(4), \quad L^d = 0.24(4)$$

$$\boxed{L^{u+d} = 0.03(7)}$$

$\uparrow$

Valence quarks only

$$\boxed{L^{u-d} = -0.45(6)}$$

$\cdots$ but strong cancellations

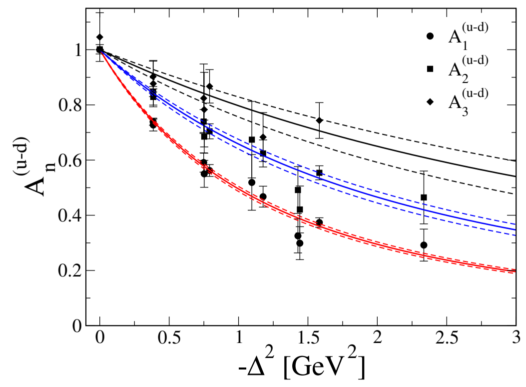
$$B_1^{u+d} \approx B_2^{u+d} \approx 0 \quad \Longrightarrow E^{u+d} \approx 0$$

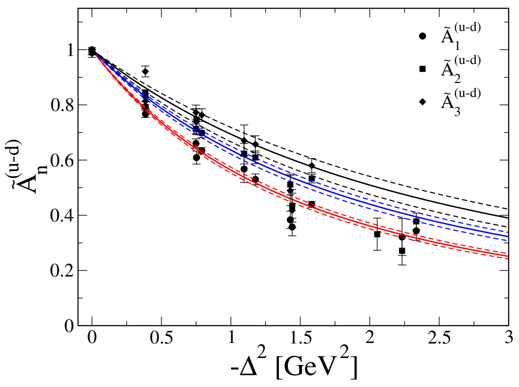
Generalized Parton Distributions

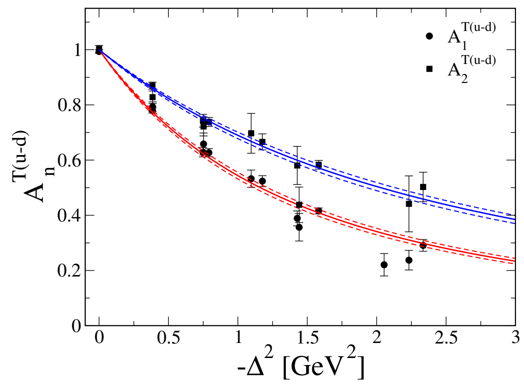
3D images of the nucleon

Axial

Tensor

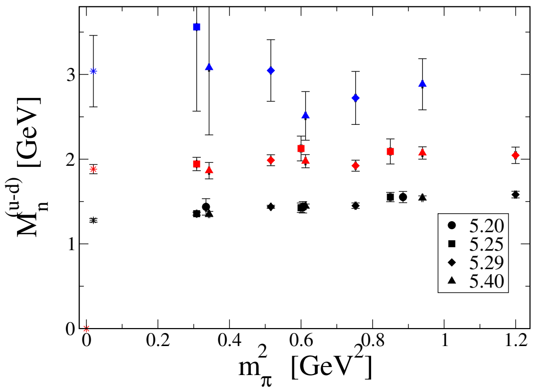


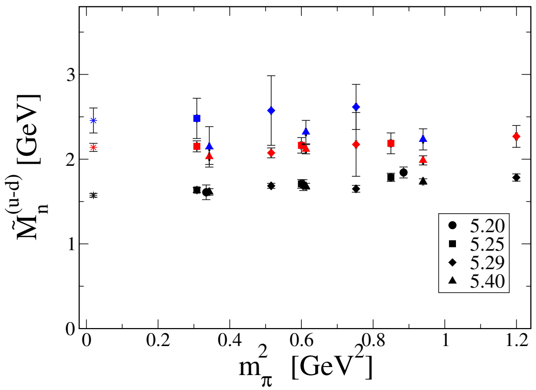


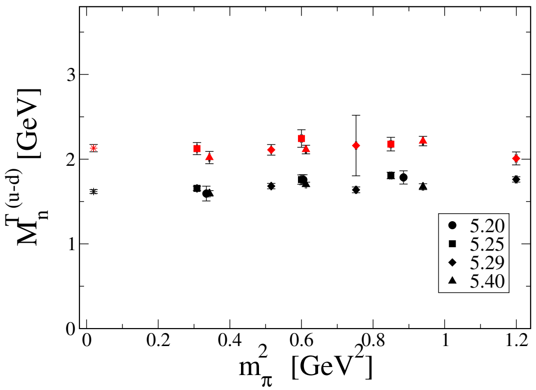


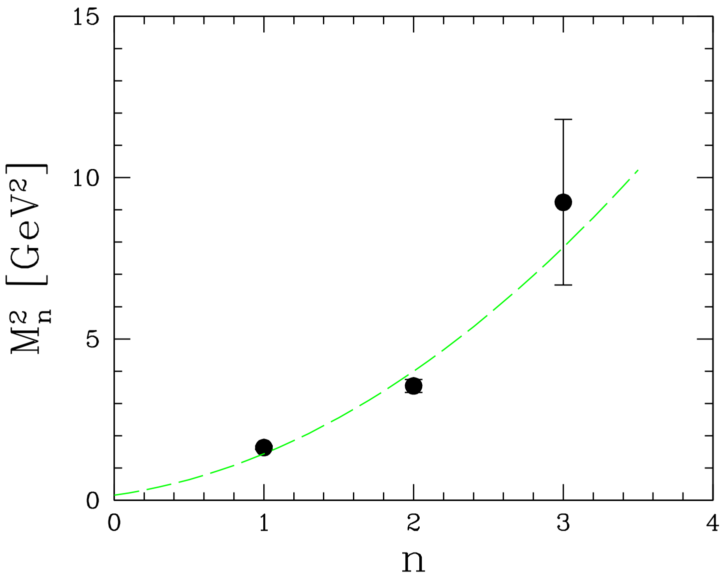
$$A_n(\Delta^2) = \frac{A_n(0)}{(1-\Delta^2/M_n^2)^2}$$

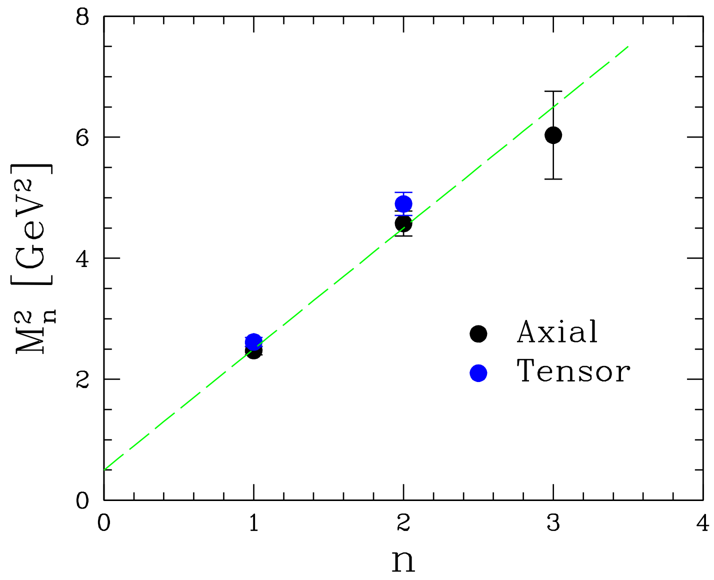
Chiral extrapolation









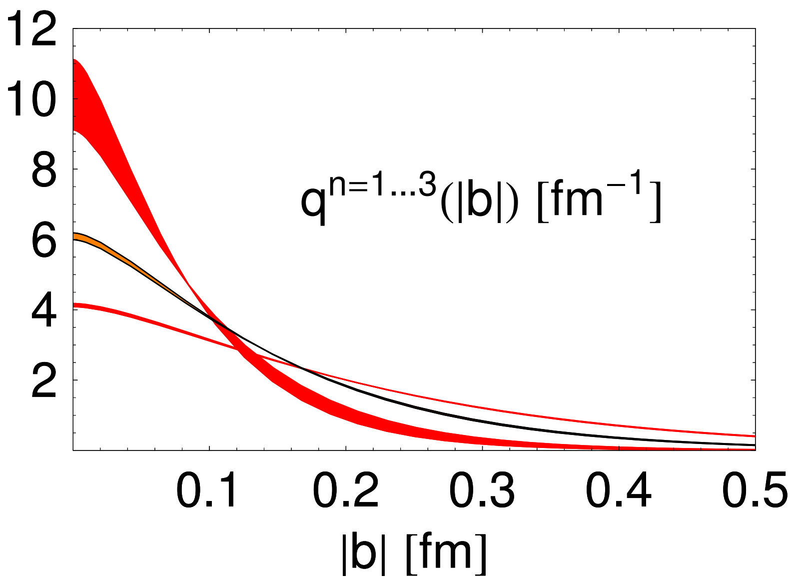


$$n = \alpha_0 + \alpha^{\vee} \sqrt{M_n^2}$$

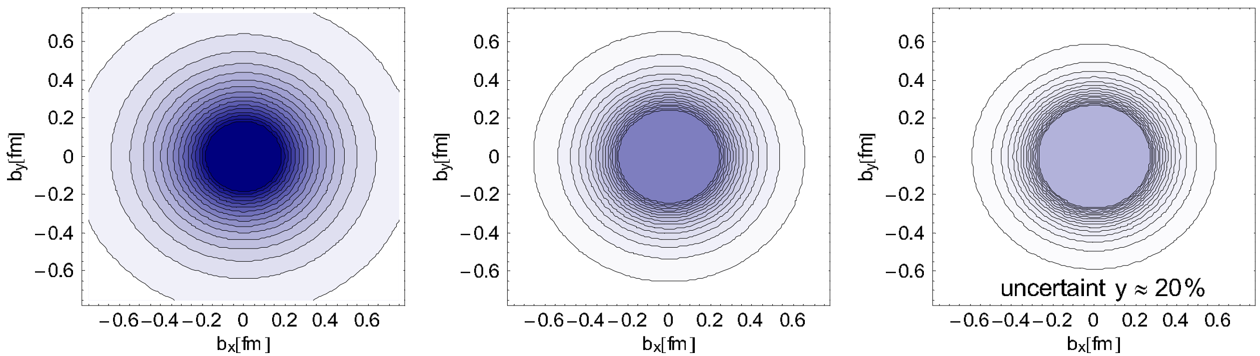
square root

$$n = \alpha_0 + \alpha' \, M_n^2$$

linear trajectory



$$\int_0^1 dx\, x^{n-1}\, H^u(x, \mathbf{b}_\perp^2)$$



$\langle x \rangle = 0.26(1)$ $\langle x \rangle = 0.37(2)$ $\langle x \rangle = 0.40(5)$

Generically

$$H^q(x,\Delta^2) = \int_x^1 \frac{dy}{y} C\left(\frac{x}{y},\Delta^2\right) q(x)$$

$$\alpha(t) = \alpha_0 + \alpha^\vee \sqrt{t}$$

$$\int_0^1 dx\, x^n C(x,\Delta^2) = \frac{A_{n+1}(\Delta^2)}{A_{n+1}(0)} = \frac{1}{(1-\Delta^2/M_n^2)^2}$$

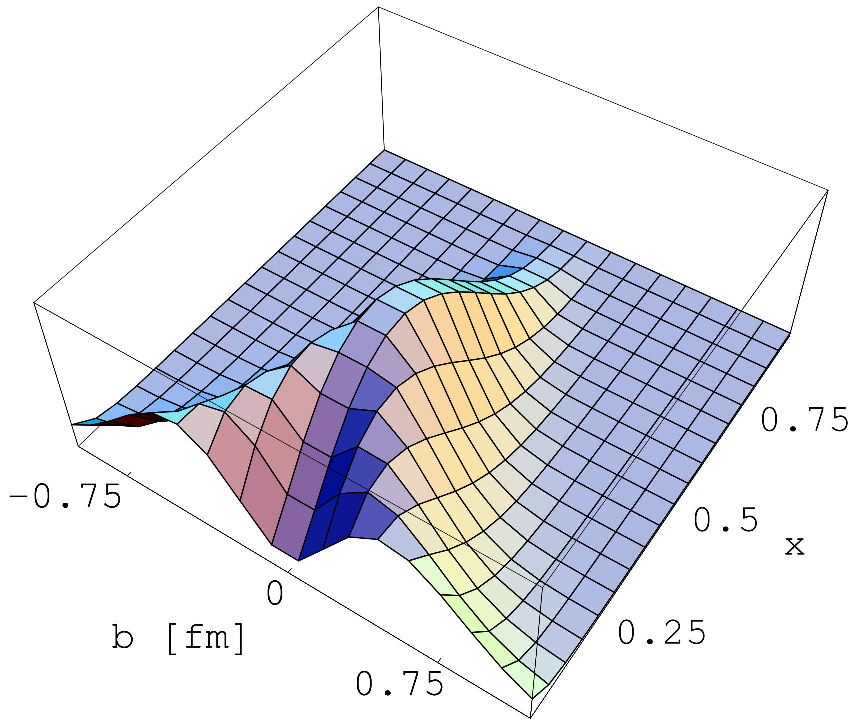
$$\left[\frac{(n+1-\alpha_0)^2}{(n+1-\alpha_0)^2 - \alpha^{\vee\,2}\Delta^2}\right]^2$$

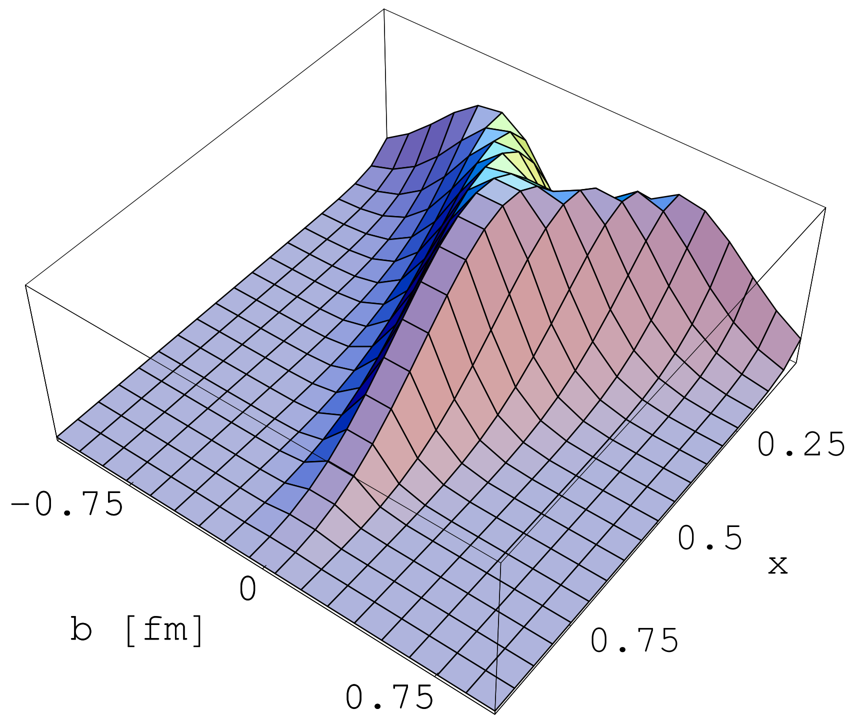
By inverse Mellin transform

$$H^q(x,\mathbf{b}_\perp^2) = \int_x^1 \frac{dy}{y} C\left(\frac{x}{y},\mathbf{b}_\perp^2\right) q(x)$$

$$C(x,\mathbf{b}_\perp^2) = \int \frac{d^2\boldsymbol{\Delta}_\perp}{(2\pi)^2} e^{i\mathbf{b}_\perp \boldsymbol{\Delta}_\perp} C(x,\boldsymbol{\Delta}_\perp^2)$$

$H^u(x, \mathbf{b}_\perp^2)$ Valence $Q^2 = 4\ \text{GeV}^2$





$$\langle b^2 \rangle = \frac{7}{2} \alpha^{\vee\, 2} (1-x)^2 + \mathcal{O}\left((1-x)^3\right)$$

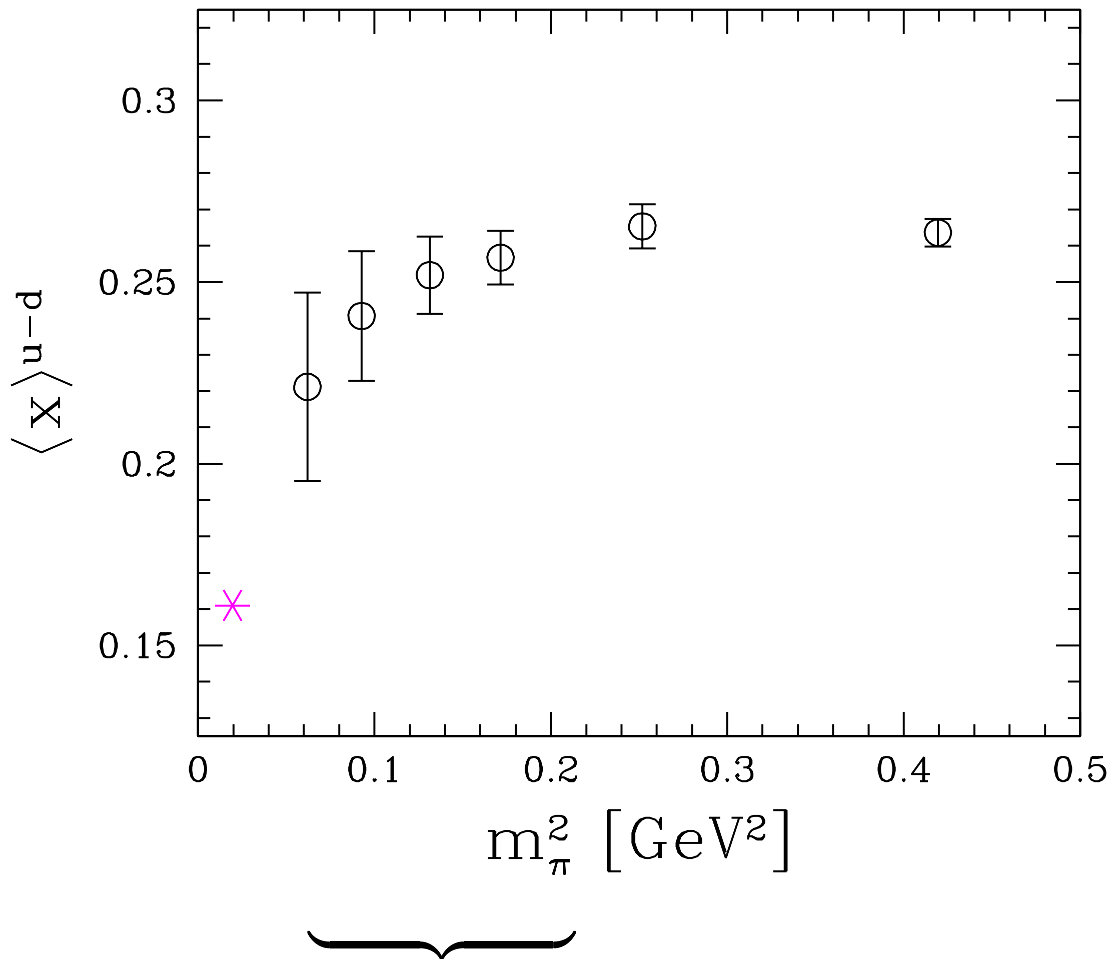
$$\langle r^2 \rangle = \frac{7}{2} \alpha^{\vee\, 2} + \mathcal{O}\,(1-x)$$

$$\approx (0.4\,\text{fm})^2$$

Summary and Outlook

- Lattice QCD is becoming a quantitative tool for exploring hadron structure
  - Computer power
  - Improved action
  - Renormalization
  - Chiral perturbation theory
  - Improved algorithms
- Largest systematic uncertainty is currently coming from extrapolation to the chiral limit
- Most immediate challenges are to extend the calculations to the chiral regime of physical quark masses, and to develop techniques for evaluating disconnected diagrams
- Simulations at $m_\pi \lesssim 300$ MeV on large lattices are possible now and will begin shortly

Quenched



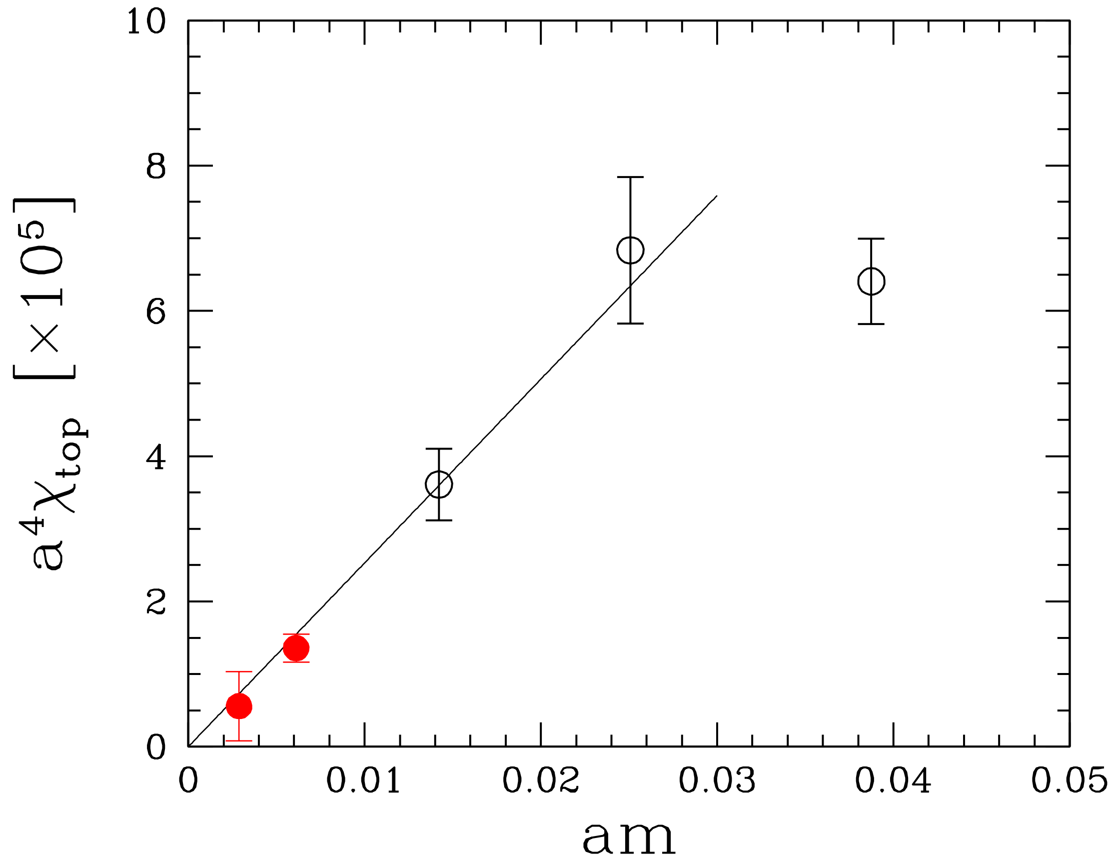
Softening due
to pion cloud

New runs $N_f = 2$

Run	β	$\kappa_{\rm sea}$	V	a [fm]	m_π [MeV]	Cost [TFlops $\times$ y]
1	5.29	0.1362	$16^3 \times 48$	0.082	400	0.01
2	5.29	0.1362	$24^3 \times 48$	0.082	400	0.04
3	5.29	0.13632	$24^3 \times 64$	0.080	250	0.21
4	5.29	0.13632	$32^3 \times 64$	0.080	250	0.60
5	5.40	0.1364	$24^3 \times 48$	0.071	400	0.05
6	5.40	0.13658	$32^3 \times 64$	0.069	250	0.69
7	5.70	0.1360	$32^3 \times 64$	0.051	500	0.11
8	5.70	0.1363	$32^3 \times 64$	0.048	400	0.25
9	5.70	0.1363	$48^3 \times 64$	0.048	400	1.08
10	5.70	0.13648	$48^3 \times 64$	0.045	250	4.33

1000 independ.
configurations

Effect of sea quarks (?) Expect $\chi_{\rm top} = m\langle\bar{\psi}\psi\rangle/N_f$ Leutwyler & Smilga



$$\langle\bar{\psi}\psi\rangle^{\overline{MS}}(2\ \mathrm{GeV}) = \left(255(9)\ \mathrm{MeV}\right)^3$$

Nucleon mass