GDH sum rule and magnetic moments in chiral EFT

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Topics

- GDH sum rule and its derivatives
- Magnetic moment of the nucleon (chiral behavior)
- Magnetic moment of the Delta-resonance (experimental determination and chiral behavior)
Gerasimov-Drell-Hearn (GDH) sum rule

\[ \frac{e^2}{2M^2} \kappa^2 = \frac{1}{\pi} \int_0^\infty \frac{d\omega}{\omega} \Delta \sigma(\omega) \]

\[ \kappa = (g - 2) s \quad \text{anomalous magnetic moment} \]

\[ \Delta \sigma = \sigma_{1+s} - \sigma_{1-s} \quad \text{doubly-polarized total photoabsorption cross section} \]

(photons circular polarized parallel or anti-parallel to the target’s spin)

Principles/Assumptions:

- **Low-energy theorem for Compton scattering** (gauge-invariance, crossing symmetry, …)
- **Analyticity** (forward Compton amplitude obeys disp. relations along the production cut)
- **Unitarity** (optical theorem: Im forward Compton amplitude = total photoabsorption)
- **Stability??** (neglect the target’s decay)
Derivation steps

Forward Compton scattering amplitude:

\[ \text{Amp}_{t=0} = f(\omega) \vec{e}' \cdot \vec{e} + g(\omega) i \vec{S} \cdot (\vec{e}' \times \vec{e}) + \ldots \]

\[ 2s + 1 \text{ terms } \leftrightarrow \# \text{ of e.m. moments} \]

**Analyticity:**

\[ \text{Re} \ g(\omega) = \frac{2\omega}{\pi} \int_{\omega_{th}}^{\infty} d\omega' \frac{\text{Im} \ g(\omega')}{\omega'^2 - \omega^2} \]

**Unitarity:**

\[ \text{Im} \ g(\omega) = \frac{\omega}{2} \Delta \sigma(\omega) \]

**Low-energy theorem:**

\[ g(\omega) = \frac{e^2 \kappa^2}{4sM^2} \omega + O(\omega^3) \]

**branch cuts:**

\( \pi N, \pi \pi N, \ldots \)

**Stability condition:**

\( \omega_{th} \geq 0 \)
Empirical verification of the GDH sum rule for the proton

\[ \frac{e^2}{2M^2} \kappa^2 = \frac{1}{\pi} \int_0^\infty \frac{d\omega}{\omega} \Delta \sigma(\omega) \]

<table>
<thead>
<tr>
<th>PROTON</th>
<th>$E_\gamma$ [GeV]</th>
<th>GDH [\mu b]</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAID2000 estimate</td>
<td>$&lt; 0.2$</td>
<td>$-28.5 \pm 2$</td>
</tr>
<tr>
<td>MAMI experiment</td>
<td>0.2 - 0.8</td>
<td>226 $\pm 5 \pm 12$</td>
</tr>
<tr>
<td>ELSA experiment</td>
<td>0.8 - 2.9</td>
<td>27.5 $\pm 2.0 \pm 1.2$</td>
</tr>
<tr>
<td>Bianchi-Thomas Simula et al. estimate</td>
<td>$&gt; 2.9$</td>
<td>$-14 \pm 2$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>211 \pm 15</strong></td>
</tr>
<tr>
<td><strong>GDH sum rule</strong></td>
<td></td>
<td><strong>205</strong></td>
</tr>
</tbody>
</table>
Compute both sides of the SR in perturbation theory…

\[
\frac{e^2}{2M^2} \kappa^2 = \frac{1}{\pi} \int_0^\infty \frac{d\omega}{\omega} \Delta \sigma(\omega)
\]

\[O(e^4) \quad 0\]

\[
\Delta \sigma^{(4)} = -\frac{2\pi \alpha^2}{M^2 x} \left[ (1 + \frac{1}{x}) \log(1 + 2x) \right.
- 2 \left( 1 + \frac{x^2}{(1 + 2x)^2} \right)
\]

\[x = \omega/M.\]

[Altarelli, Cabibbo, Maiani, PLB (72)]

\[O(e^6) \quad + \text{other 1-loop graphs}\]

The l.h.s is the Schwinger's correction, the r.h.s. (much harder) done numerically by Dicus and Vega, PLB (2001).
Derivatives of the GDH sum rule

[Introduce a trial a.m.m.: \( \mathcal{L} = \mathcal{L}_{QED} + \frac{\kappa_0}{4M} \overline{\psi} i \sigma_{\mu\nu} \psi F^{\mu\nu} \)]

- The total value of amm and the cross-sections are dependent on \( \kappa_0 \).
- By taking derivatives of the GDH SR w.r.t. \( \kappa_0 \) at \( \kappa_0 = 0 \) we obtain a new set of SRs, e.g., 1st derivative:

\[
\frac{e^2}{M^2} \kappa = \frac{1}{\pi} \int_0^\infty \frac{d\omega}{\omega} \Delta \sigma'(\omega) \bigg|_{\kappa_0 = 0}
\]

\[
\Delta \sigma'(\omega) \bigg|_{\kappa_0 = 0} = \frac{2\pi \alpha_{em}^2}{M \omega} \left[ 6 - \frac{2M \omega}{(M+2\omega)^2} - \left( 2 + \frac{3M}{\omega} \right) \ln \left( 1 + \frac{2\omega}{M} \right) \right]
\]

\[
\kappa = \frac{\alpha_{em}}{2\pi}
\]
GDH and GDH’ in Chiral Effective Field Theory

To lowest order in $g_{\pi NN} (= g_A M / f_\pi)$

$$O(g^2): \quad \frac{e^2}{M^2} \kappa_0 \delta \kappa = \frac{1}{\pi} \int \frac{d\omega}{\omega} \left( \Delta \sigma_E + \kappa_0 \Delta \sigma_\kappa \right)$$

$$0 = \int \frac{d\omega}{\omega} \Delta \sigma_E$$

$$\delta \kappa = \frac{M^2}{\pi e^2} \int \frac{d\omega}{\omega} \Delta \sigma_\kappa$$
\[ 0 = \int \frac{d\omega}{\omega} \Delta \sigma_E \]

Verified

only in the fully relativistic calculation. Any “heavy-baryon” type of expansion does not do it.
\[ \delta \kappa_p = \frac{M^2}{\pi e^2} \int_{\omega_\text{th}}^\infty \frac{d\omega}{\omega} \left( \Delta \sigma^{(\pi^0 p)}_\kappa + \Delta \sigma^{(\pi^+ n)}_\kappa \right) \]

\[ = \frac{g^2}{(4\pi)^2} \left\{ 1 - \frac{\mu \left( 4 - 11\mu^2 + 3\mu^4 \right)}{\sqrt{1 - \frac{1}{4}\mu^2}} \arccos \frac{\mu}{2} - 6\mu^2 + 2\mu^2 \left( -5 + 3\mu^2 \right) \ln \mu \right\} \]

\[ \delta \kappa_n = \frac{M^2}{\pi e^2} \int_{\omega_\text{th}}^\infty \frac{d\omega}{\omega} \left( \Delta \sigma^{(\pi^0 n)}_\kappa + \Delta \sigma^{(\pi^- p)}_\kappa \right) \]

\[ = \frac{-2g^2}{(4\pi)^2} \left\{ 1 - \frac{\mu \left( 2 - \mu^2 \right)}{\sqrt{1 - \frac{1}{4}\mu^2}} \arccos \frac{\mu}{2} - 2\mu^2 \ln \mu \right\} \]

where \( \mu = m_\pi/M_N \).
Chiral behavior of the nucleon magnetic moment

For small $\mu$:

$$\delta \kappa_p = \frac{g^2}{(4\pi)^2} \left\{ 1 - 2\pi \mu - 2(2 + 5 \ln\mu) \mu^2 + \frac{21\pi}{4} \mu^3 + O(\mu^4) \right\}$$

$$\delta \kappa_n = \frac{g^2}{(4\pi)^2} \left\{ -2 + 2\pi \mu - 2(1 - 2 \ln\mu) \mu^2 - \frac{3\pi}{4} \mu^3 + O(\mu^4) \right\}$$

For large $\mu$:

$$\delta \kappa_p = \frac{g^2}{(4\pi)^2} \left( 5 - 4 \ln\mu \right) \frac{1}{\mu^2} + O(\mu^{-4})$$

$$\delta \kappa_n = \frac{g^2}{(4\pi)^2} \left( 2(3 - 4 \ln\mu) \frac{1}{\mu^2} + O(\mu^{-4}) \right)$$

Goes as $1/m_q$ \hspace{1cm} ($m_{\pi}^2 \sim m_q$)

Exactly as expected from the quark model.
Relativistic effects in chiral behavior of magnetic moments

Relativistic chiral loops (SR) compared with heavy-baryon expansion (HB) or Infrared-Regularized ChPT (IR)

Red curve is the single-parameter fit to lattice data
The parametrization is based on SR result

\[ \mu = \frac{\mu_0}{1 + a m_{\pi}^2} + \delta K \]
Motivation for $\mu_\Delta$

- Observation of the magnetic moment of a strongly unstable particle.
- $\Delta$-resonance is the best-studied example of such a particle.
- Excitation energy of the $\Delta$-resonance, $M_\Delta - M_N \lesssim 300$ MeV is relatively low, which allows to treat it as a low-energy scale in an EFT expansion, the $\Delta$ can be incorporated in ChPT.
\[ \gamma p \rightarrow \gamma \pi^0 p : \text{ChEFT calculation} \]

[VP & Vanderhaeghen, PRL 94 (2005)]

Power counting [VP&Phillips, PRC(2003)]:
\[ \delta = (M_\Delta - M_N)/\Lambda_{\chi \PT}, \quad m_\pi/\Lambda_{\chi \PT} \gg \delta^2 \]

To next-to-leading order in the resonance region
(counting \( \omega' \gg m_\pi, \omega \gg M_\Delta - M_N \))

\[ S_{\mu\nu}(p) = \frac{-P^{(3/2)}_{\mu\nu}(p)}{(p \cdot \gamma - M_\Delta)[1 - i\text{Im } \Sigma'(M_\Delta)] - i\text{Im } \Sigma(M_\Delta)^2}, \]

\[ \bar{u}_\alpha(p') \Gamma^{\mu\alpha\beta}_{\gamma\Delta\Delta}(p', p) u_\beta(p) \epsilon_\mu \]

\[ = e \bar{\eta}_\alpha(p') \left[ \epsilon \cdot \gamma F(q^2) + \frac{(p' + p) \cdot c}{2M_\Delta} G(q^2) \right] u^\alpha(p), \quad c = F(0), \quad Z_\Delta = 1 - \Sigma'(M_\Delta) \]

The Ward-Takahashi identity,
\[ q_\mu \Gamma^{\mu\alpha\beta}(p', p) = e \left[ (S^{-1})^{\alpha\beta}(p') - (S^{-1})^{\alpha\beta}(p) \right], \text{ demands } F(0) + G(0) = 1 - \Sigma'(M_\Delta). \]
$\gamma p \rightarrow \gamma \pi^0 p$ observables

Exp. data points from TAPS@MAMI 2002

1. Ratio of the angle-integrated cross-section to the soft-photon limit:

$$R \equiv \frac{1}{\sigma_{\pi}} \cdot \frac{d\sigma}{dE'_{\gamma}}$$

2. $\Sigma$ – linear-pol. photon beam asymmetry.

3. $\Sigma_{\text{circ}}$ – circular-pol. photon beam asymmetry.
Chiral behavior of the $\Delta^{++}$ and $\Delta^+$ magnetic moments

Lattice data points from
I.C. Cloet, D.B. Leinweber and
[2] F.X. Lee et al., hep-lat/0410037
Summary

- GDH: \( \frac{e^2}{2M^2} \kappa^2 = \frac{1}{\pi} \int_0^\infty \frac{d\omega}{\omega} \Delta \sigma(\omega) \), and GDH': \( \frac{e^2}{M^2} \kappa = \frac{1}{\pi} \int_0^\infty \frac{d\omega}{\omega} \Delta \sigma'(\omega) \mid_{\kappa_0=0} \)

  analyticity constraint

- Magnetic moment of the nucleon (chiral behavior)
  Exact analyticity extends the limit of applicability of ChEFT

- Magnetic moment of the Delta-resonance:
  experimental determination and chiral behavior

  ChEFT provides framework for both
Power counting for the $\Delta(1232)$ ($\delta$-expansion)

- The excitation energy of the $\Delta$ resonance, $\Delta = M_\Delta - M_N \sqrt[4]{290}$ MeV resonance can also be treated as small: $\delta = \Delta / \Lambda_\chi \approx 1$.

- Feature of the “$\delta$-expansion”: scale hierarchy $m_\pi \ll \Delta \ll \Lambda_\chi$. $\Rightarrow \delta = \Delta / \Lambda$, $m_\pi / \Lambda = \delta^2$.

- This distinguishes the low-energy ($p \approx m_\pi$) and the resonance ($p \approx \Delta$) regions.

- Crucial for correct counting of the One-$\Delta$-reducible (O$\Delta$R) graphs:

\[
\text{O}\Delta\text{R propagator}
\]

\[
\begin{array}{l}
p \gg m_\pi, \quad 1/\Delta = O(1/\delta) \\
p \gg \Delta, \quad 1/(p-\Delta-\Sigma) = O(1/\delta^3)
\end{array}
\]

\[
\Sigma = + \ldots = O(p^3) = O(\delta^3)
\]