Dynamics of Wilson Loops

- Wilson Loop Distributions and Diffusion
- Casimir Scaling and Screening in Confining Phase
- Wilson Loop Effective Action
- Approximate Center Symmetry
- Correlation functions of large Wilson loops
- Summary

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Wilson Loop Distributions and Diffusion

Wilson-loops

$$W = \frac{1}{2}\text{tr}\, P\, \exp\{ig \oint_{\mathcal{C}} dx^{\mu} A_{\mu}(x)\}$$

Vacuum expectation values of Wilson loops

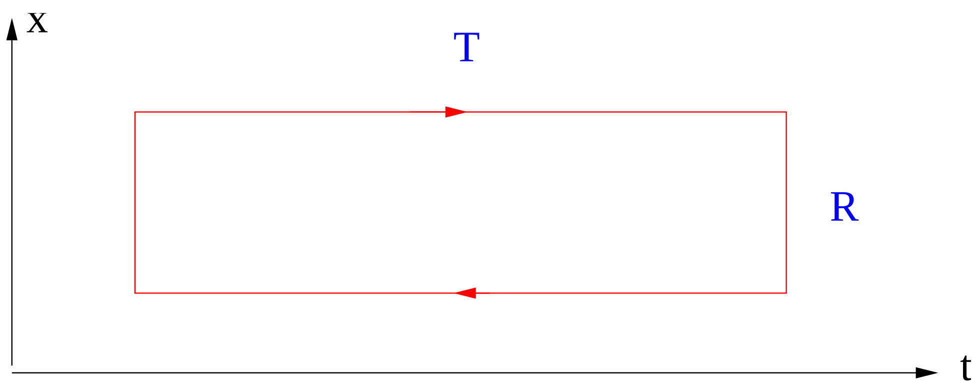
$$\langle 0|W|0\rangle = \int d[A] e^{-S[A]} W[A] \;\; , \quad S = -\frac{1}{4}\int d^4x\, F^{a}_{\mu\nu} F^{a\,\mu\nu}$$

of sufficiently large size satisfy in the confining phase an area law

$$\langle 0|W|0\rangle \sim e^{-\sigma\mathcal{A}}$$

with string tension σ.

For rectangular loops $\mathcal{A} = RT$

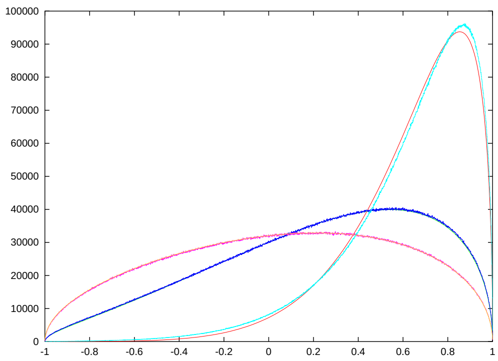


and for sufficiently large times $T \gg R$ Wilson loops determine the interaction energy $V(R)$ of static quarks

$$\langle 0|W|0\rangle \sim e^{-T\,V(R)}\,, \qquad V(R) \sim \begin{cases} g^2\dfrac{1}{R}\,, & R \to 0 \\ \sigma R\,, & R \to \infty \end{cases}$$

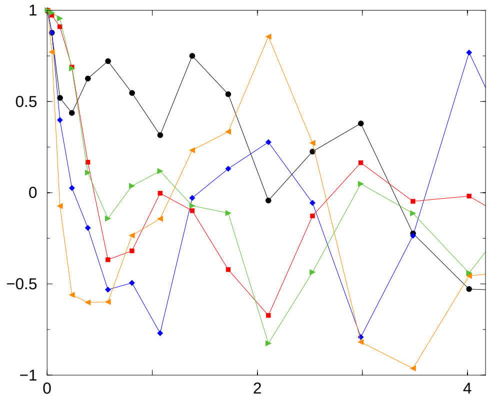
Distribution of Wilson loops

$$p(\omega) = \langle 0 | \delta\left(\omega - W\right) | 0 \rangle$$



SU(2) lattice gauge theory

Wilson loop distribution for rectangular loops of different areas



Values of Wilson loops in different planes centered at the origin as function of the area (fm^2) for a single configuration

Diffusion of Wilson Loops

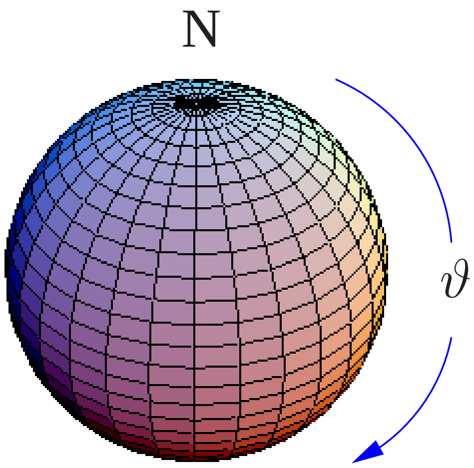
When increasing size of loop in 4-dimensional Euclidean space as function of parameter t - ”time ”, value of the (untraced) Wilson loop carries out Brownian motion on the group manifold, S^3 ($SU(2)$). The motion of an ensemble of these degrees of freedom is in turn described by a diffusion equation. Due to gauge invariance of Wilson-loops diffusion is independent of 2-nd polar and of azimuthal angle on S^3. With

$$\omega = \cos\vartheta$$

Laplacian given by

$$\Delta = \frac{1}{\sin^2\vartheta}\frac{d}{d\vartheta}\sin^2\vartheta\frac{d}{d\vartheta}$$

Diffusion on the sphere

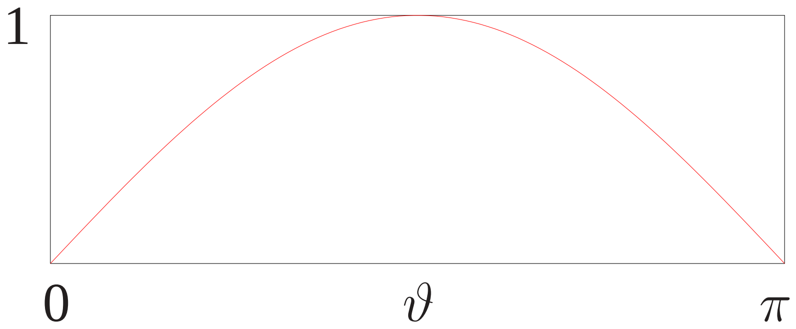


Laplacian

$$\Delta = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\,\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$$
$$= -L^2$$

Uniform Distribution

on the sphere



Diffusion equation on S^3

$$\left(\frac{d}{dt} - \Delta\right) G(\vartheta, t) = \frac{1}{\sin^2 \vartheta}\, \delta(\vartheta)\delta(t)$$

Eigenfunctions of Laplace operator

$$\psi_n = \sqrt{\frac{2}{\pi}}\, \frac{\sin n\vartheta}{\sin \theta}, \qquad -\Delta\, \psi_n = (n^2 - 1)\psi_n$$

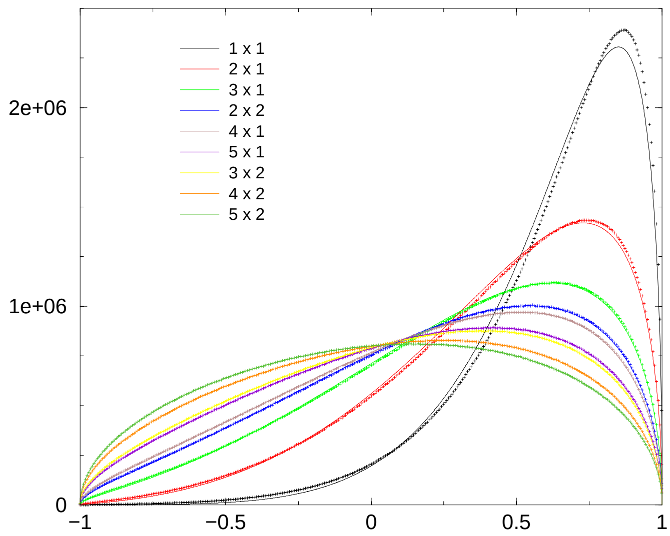
Solution of diffusion equation

$$G(\vartheta, t) = \frac{2}{\pi \sin \vartheta}\, \theta(t) \sum_{n=1}^{\infty} n\, \sin n\vartheta\, e^{-(n^2-1)t}$$

$$G(\vartheta, t) \sim \begin{cases} t^{-3/2} \exp -\dfrac{\vartheta^2}{4t}\,, & t \to 0 \\[2ex] \dfrac{2}{\pi}\,, & t \to \infty \end{cases}$$

QED-behavior for small t, uniform distribution on S^3 asymptotically.

Relation between "time" t and Wilson-loop size R not determined. "Time" t obtained from fit of expectation value of Wilson loop, $\langle 0|W(R)|0\rangle = \exp(-3t)$



Wilson loop distribution in SU(2) lattice gauge theory

Casimir Scaling and Screening in Confining Phase

Higher representation Wilson loops

Expectation value of an observable $\mathcal{O}(\vartheta)$ in a diffusion process on the S^3

$$\langle \mathcal{O}(\vartheta) \rangle = \frac{2}{\pi} \int_0^\pi \sin\vartheta \, d\,\vartheta \, \mathcal{O}\,(\vartheta) \sum_{n=1}^{\infty} n \sin n\,\vartheta \, e^{-(n^2-1)t}$$

Wilson loops in the 2 j + 1 representation of SU(2)

$$W_j(\vartheta) = \frac{1}{2j+1} \mathrm{tr}\; \exp 2i\vartheta \begin{pmatrix} -j & & \\ & \ddots & \\ & & j \end{pmatrix}$$

$$= \frac{1}{2j+1} \frac{\sin(2j+1)\vartheta}{\sin\vartheta} \sim \psi_{2j+1}(\vartheta)$$

$$\langle W_j \rangle = e^{-4j(j+1)\cdot t}$$

Laplace-Beltrami = Quadratic Casimir

Approximate Casimir Scaling SU(3) Lattice Gauge Theories

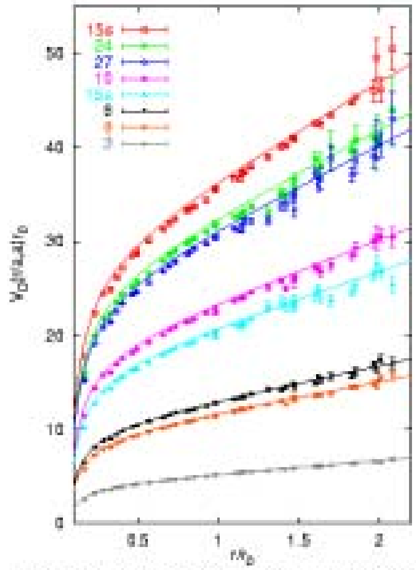


FIG. 4. The potentials for all measured representations, obtained at $\beta = 6.2$. Note that we did not subtract any self energy pieces but just rescaled the raw lattice data in units of r_0.

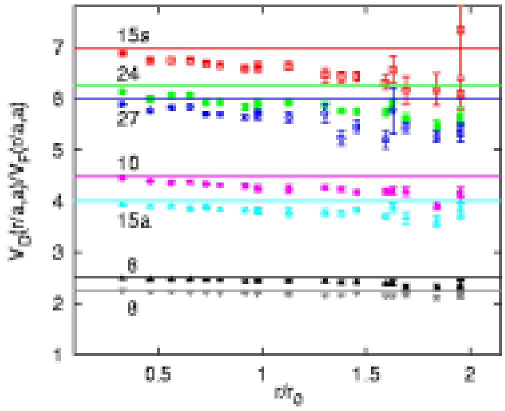


FIG. 5. The potentials normalized to the fundamental potential at $\beta = 5.8$, in comparison to the expectations from Casimir scaling (horizontal lines).

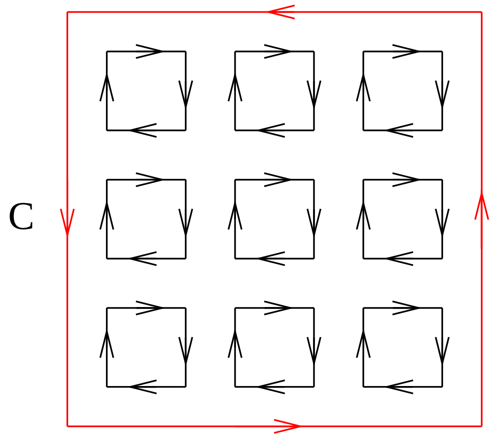
Potentials and their ratios in different representations of SU(3), G.S. Bali

Casimir Scaling $\Leftrightarrow$ Wilson Loop Diffusion

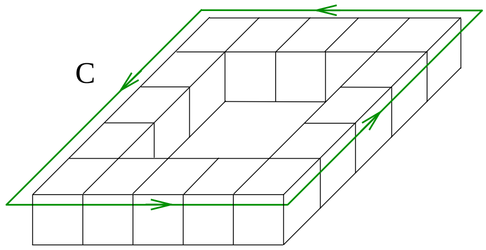
Two regimes in confining phase

- Regime of Casimir Scaling
  Confinement of fundamental and adjoint charges
  Wilson loop distribution result of diffusion
- Screening Regime
  Confinement of half-integer (fundamental) charges
  and screening of integer (adjoint) charges

Strong coupling implies screening



Planar tiling of
fundamental loop

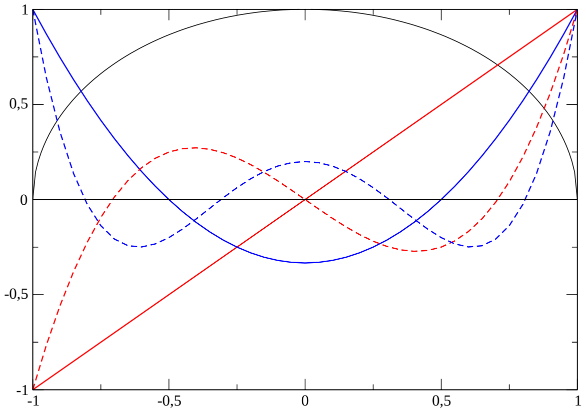


3-d tiling of
adjoint loop

3-d tiling suppressed in regime of Casimir scaling

Screening

Diffusion of Wilson loops in the presence of forces described by potential $V(\vartheta)$



Haar Measure and eigenfunctions of Laplacian $\psi_{2j+1}(\vartheta)$ as a function of $\cos\vartheta$ for $j = 1/2,\ 1,\ 3/2,\ 2$

Vanishing of fundamental Wilson loop for size $\longrightarrow \infty$ guaranteed if $V(\vartheta)$ symmetric around equator of S^3 $\vartheta = \frac{\pi}{2}$

$$V(\pi/2 - \vartheta) = V(\vartheta - \pi/2)$$

Symmetric potentials admix $j = 1/2$ to half-integer states, $j = 0$ to integer Laplace - eigenfunctions

Confinement $\rightarrow$ Symmetric V

Casimir scaling $\rightarrow$ Vanishing V

Wilson Loop Effective Action

Connecting to QCD

Effective action of Wilson loops from diffusion (=Euclidean Quantum-Mechanics of point particle)

$$S \sim \int dt \dot{\vartheta}^2$$

In the confinement regime $t \sim \rho^2$

$$S_{\text{eff}}[\vartheta] = \frac{3}{8\pi\sigma} \int \frac{\mathrm{d}\rho}{\rho} \left(\frac{\mathrm{d}\vartheta}{\mathrm{d}\rho} \right)^2$$

Yang-Mills action in gauge

$$A^a_\varphi = \delta_{a,3} \frac{1}{g\pi\rho} \left(\frac{\pi}{2} - a(t, \rho, z) \right)$$

where (circular) Wilson loops become elementary degrees of freedom

$$W(t,\rho,z) = \sin(a(t,\rho,z))$$

$$S_{\rm YM} = \int \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z\,\mathrm{d}t \times\{\frac{1}{2\pi^2 g^2\,\rho^2}[(\partial_\rho a)^2 + (\partial_z a)^2 + (\partial_t a)^2] + \mathcal{L}'\}\,,$$

Assume

- coupling of Wilson loops at different t, z to each other to be negligible beyond a range $t^2 + z^2 \leq \lambda^2$
- within this range no dependence of the Wilson loops on t, z
- negligible coupling to other degrees of freedom

$$S_{\rm YM} \approx \frac{\lambda^2}{g^2} \int \frac{{\rm d}\rho}{\rho} (\partial_\rho a)^2$$

From comparison with effective action

$$\lambda^2 = \frac{3g^2}{8\pi\sigma}$$

Approximate Center Symmetry

As suggested by the diffusion model Lagrangian is symmetric under reflections of the Wilson loops at the equator of S^3

$$\tilde{Z} : a(t, \rho, z) \to -a(t, \rho, z)$$

accompanied by changes in the other fields. Residual gauge transformation with twist - in analogy with center symmetry if a space time dimension is compact.

Unless for vanishing ρ, $\rho A_\varphi \rightarrow 0$, a singular magnetic field $\sim \delta(\rho^2)$ of infinite action is generated and therefore

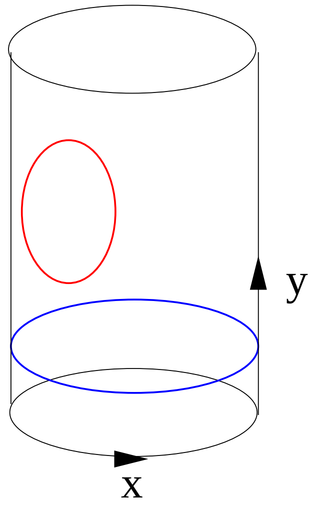
$$\rho \rightarrow 0 : a(t, \rho, z) \rightarrow \frac{\pi}{2}$$

Boundary condition at $\rho = 0$ prevents symmetry $\tilde{Z}$ to be realized and gives rise to distributions of Wilson-loops for small ρ concentrated around $a = \pi/2$. With increasing loop size, the effect of the "boundary condition" at $\rho = 0$ decreases and the symmetry is approximatively restored as in diffusion of Wilson loops. The vacuum expectation value of the Wilson loop is a measure of the restoration of the symmetry

$$\langle 0|W(t, \rho, z)|0\rangle \sim \exp(-\sigma\pi\rho^2)$$

If exact the symmetry $(\tilde{Z})$ would require $\langle 0|W|0\rangle = 0$.

Correlation function of large Wilson loops



Approximate center symmetry makes the dynamics of Polyakov and Wilson loops of large size similar. Symmetries protect gauge strings from decaying.
In particular from

$$\langle 0|P(d)P(0)|0\rangle \sim \exp(-\sigma\beta d)$$

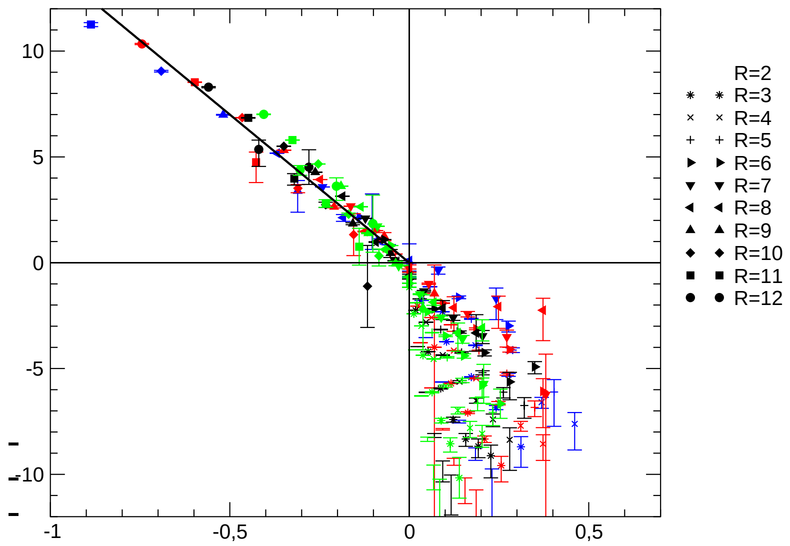
one expects

$$\langle 0|W(d,\rho)W(0,\rho)|0\rangle \sim \exp(-\sigma 2\pi\rho d)$$

Accounting for $\exp(-\pi\rho^2)$ suppressed contributions from the violation of the $\tilde{Z}$ symmetry

$$\langle 0|\, W(d,\rho)\, W(0,\rho)|0\rangle \;\approx\; e^{-2\sigma\pi\rho^2}(c_0 + c_1 e^{-m_{gb}d} + ..) \\ +\; d_1 e^{-\sigma 2\pi\rho d}$$

- The first class of contribution arises from the intermediate ground-state and glueballs. Not present for Polyakov loops - forbidden by exact center symmetry Z
- Second class of contributions - common to Wilson and Polyakov loops - arises from intermediate states which are odd under Z and $\tilde{Z}$ respectively



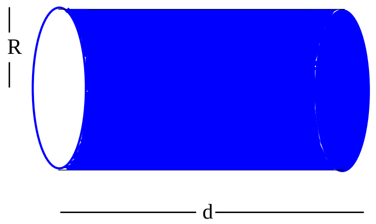
Logarithm of correlation function of quadratic Wilson-loops $(R \times R)$ as a function of $\xi = 2Rd - R^2$

For sufficiently large Wilson loops we expect

$$\left[\langle 0|W(d,R)\,W(0,R)|0\rangle - \langle 0|\,W(0,R)|0\rangle^2\right]/\langle 0|W(0,R)|0\rangle^2$$

$$\approx\; c_1 e^{-m_{gb}\frac{\xi+R^2}{2R}} + d_1 e^{-2\sigma\xi} + \ldots$$

Geometrical interpretation



$\sim \exp\{-2\pi\sigma Rd\}$

$\sim \exp\{-2\pi\sigma R^2 - m_{gb} d\}$

For large loops, up to the glueball correction, the surface of minimal area dominates. Change in configuration around $d \approx R$ - Gross-Ooguri transition.

Wilson loop operators applied to the vacuum generate approximative stationary states of energy

$$E[W(R)] \approx \sigma 2\pi R$$

Single (gauge) string states are approximative eigenstates of the Hamiltonian.

Summary

- Wilson loop distribution - result of diffusion process
- Casimir scaling: a new regime in the confinement phase, different from strong coupling limit
- Approximate center symmetry associated with confinement
- Sufficiently large Wilson loops generate approximative stationary single string states