Dynamics of Wilson Loops

• Wilson Loop Distributions and Diffusion
• Casimir Scaling and Screening in Confining Phase
• Wilson Loop Effective Action
• Approximate Center Symmetry
• Correlation functions of large Wilson loops
• Summary

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Wilson Loop Distributions and Diffusion

Wilson-loops

\[ W = \frac{1}{2} \text{tr} P \exp \{ ig \oint_C dx^\mu A_\mu(x) \} \]

Vacuum expectation values of Wilson loops

\[ \langle 0|W|0 \rangle = \int d[A] e^{-S[A]} W[A] , \quad S = -\frac{1}{4} \int d^4x F^a_{\mu\nu} F^{a \mu\nu} \]

of sufficiently large size satisfy in the confining phase an area law

\[ \langle 0|W|0 \rangle \sim e^{-\sigma A} \]

with string tension \( \sigma \).
For rectangular loops $\mathcal{A} = RT$

and for sufficiently large times $T \gg R$ Wilson loops determine the interaction energy $V(R)$ of static quarks

$$\langle 0 | W | 0 \rangle \sim e^{-TV(R)}, \quad V(R) \sim g^2 \frac{1}{R}, \quad R \to 0$$

$$V(R) \sim \sigma R, \quad R \to \infty$$
Distribution of Wilson loops

\[ p(\omega) = \langle 0 | \delta (\omega - W) | 0 \rangle \]

**SU(2) lattice gauge theory**

Wilson loop distribution for rectangular loops of different areas

Values of Wilson loops in different planes centered at the origin as function of the area (fm²) for a single configuration
Diffusion of Wilson Loops

When increasing size of loop in 4-dimensional Euclidean space as function of parameter $t$ - ”time ”, value of the (untraced) Wilson loop carries out Brownian motion on the group manifold, $S^3 (SU(2))$. The motion of an ensemble of these degrees of freedom is in turn described by a diffusion equation. Due to gauge invariance of Wilson-loops diffusion is independent of 2-nd polar and of azimuthal angle on $S^3$. With

$$\omega = \cos \vartheta$$

Laplacian given by

$$\Delta = \frac{1}{\sin^2 \vartheta} \frac{d}{d \vartheta} \sin^2 \vartheta \frac{d}{d \vartheta}$$
Diffusion on the sphere

\[ \Delta = \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} \]

\[ = -L^2 \]

Uniform Distribution on the sphere
Diffusion equation on $S^3$

$$\left( \frac{d}{dt} - \Delta \right) G(\vartheta, t) = \frac{1}{\sin^2 \vartheta} \delta(\vartheta) \delta(t)$$

Eigenfunctions of Laplace operator

$$\psi_n = \sqrt{\frac{2}{\pi}} \frac{\sin n \vartheta}{\sin \theta}, \quad -\Delta \psi_n = (n^2 - 1) \psi_n$$

Solution of diffusion equation

$$G(\vartheta, t) = \frac{2}{\pi \sin \vartheta} \theta(t) \sum_{n=1}^{\infty} n \sin n \vartheta e^{-(n^2-1)t}$$

$$t^{-3/2} \exp -\frac{\vartheta^2}{4t}, \quad t \to 0$$

$$G'(\vartheta, t) \sim$$

$$\frac{2}{\pi}, \quad t \to \infty$$
QED-behavior for small $t$, uniform distribution on $S^3$ asymptotically.

Relation between "time" $t$ and Wilson-loop size $R$ not determined. "Time" $t$ obtained from fit of expectation value of Wilson loop, 

$$\langle 0|W(R)|0 \rangle = \exp(-3t)$$
Casimir Scaling and Screening in Confining Phase
Higher representation Wilson loops

Expectation value of an observable $\mathcal{O}(\vartheta)$ in a diffusion process on the $S^3$

$$
\langle \mathcal{O}(\vartheta) \rangle = \frac{2}{\pi} \int_{0}^{\pi} \sin \vartheta \, d\vartheta \, \mathcal{O}(\vartheta) \sum_{n=1}^{\infty} n \sin n \vartheta \, e^{-(n^2-1)t}
$$

Wilson loops in the $2j+1$ representation of SU(2)

$$
W_j(\vartheta) = \frac{1}{2j+1} \text{tr} \exp 2i\vartheta \begin{pmatrix} -j \\ \vdots \\ j \end{pmatrix} = \frac{1}{2j+1} \frac{\sin(2j+1)\vartheta}{\sin \vartheta} \sim \psi_{2j+1}(\vartheta)
$$

$$
\langle W_j \rangle = e^{-4j(j+1)t}
$$

Laplace-Beltrami = Quadratic Casimir
Approximate Casimir Scaling SU(3) Lattice Gauge Theories

Potentials and their ratios in different representations of SU(3), G.S. Bali
Casimir Scaling ⇔ Wilson Loop Diffusion

Two regimes in confining phase

- **Regime of Casimir Scaling**
  Confinement of fundamental and adjoint charges
  Wilson loop distribution result of diffusion

- **Screening Regime**
  Confinement of half-integer (fundamental) charges
  and screening of integer (adjoint) charges
Strong coupling implies screening

Planar tiling of fundamental loop

3-d tiling of adjoint loop

3-d tiling suppressed in regime of Casimir scaling
Screening

Diffusion of Wilson loops in the presence of forces described by potential $V(\vartheta)$

Vanishing of fundamental Wilson loop for size $\to \infty$ guaranteed if $V(\vartheta)$ symmetric around equator of $S^3$

$\vartheta = \frac{\pi}{2}$

$$V(\pi/2 - \vartheta) = V(\vartheta - \pi/2)$$
Symmetric potentials admix $j = 1/2$ to half-integer states, $j = 0$ to integer Laplace-eigenfunctions

Confinement $\rightarrow$ Symmetric $V$

Casimir scaling $\rightarrow$ Vanishing $V$
Wilson Loop Effective Action

Connecting to QCD

Effective action of Wilson loops from diffusion (=Euclidean Quantum-Mechanics of point particle)

\[ S \sim \int dt \dot{\vartheta}^2 \]

In the confinement regime \( t \sim \rho^2 \)

\[ S_{\text{eff}}[\vartheta] = \frac{3}{8\pi\sigma} \int \frac{d\rho}{\rho} \left( \frac{d\vartheta}{d\rho} \right)^2 \]

Yang-Mills action in gauge

\[ A_{\varphi}^a = \delta_{a,3} \frac{1}{g\pi \rho} \left( \frac{\pi}{2} - a(t, \rho, z) \right) \]

where (circular) Wilson loops become elementary de-
degrees of freedom

\[ W(t, \rho, z) = \sin(a(t, \rho, z)) \]

\[ S_{YM} = \int \rho \, d\rho \, d\varphi \, dz \, dt \]

\[ \times \left\{ \frac{1}{2\pi^2 g^2 \rho^2} [(\partial_{\rho} a)^2 + (\partial_z a)^2 + (\partial_t a)^2] + \mathcal{L}' \right\}, \]

Assume

• coupling of Wilson loops at different \( t, z \) to each other to be negligible beyond a range \( t^2 + z^2 \leq \lambda^2 \)

• within this range no dependence of the Wilson loops on \( t, z \)

• negligible coupling to other degrees of freedom
\[ S_{YM} \approx \frac{\lambda^2}{g^2} \int \frac{d\rho}{\rho} (\partial_\rho a)^2 \]

From comparison with effective action

\[ \lambda^2 = \frac{3g^2}{8\pi\sigma} \]

Approximate Center Symmetry

As suggested by the diffusion model Lagrangian is symmetric under reflections of the Wilson loops at the equator of \( S^3 \)

\[ \tilde{Z} : a(t, \rho, z) \rightarrow -a(t, \rho, z) \]

accompanied by changes in the other fields. Residual gauge transformation with twist - in analogy with center symmetry if a space time dimension is compact.
Unless for vanishing $\rho$, $\rho A_\varphi \to 0$, a singular magnetic field $\sim \delta(\rho^2)$ of infinite action is generated and therefore

$$\rho \to 0 : a(t, \rho, z) \to \frac{\pi}{2}$$

Boundary condition at $\rho = 0$ prevents symmetry $\tilde{Z}$ to be realized and gives rise to distributions of Wilson-loops for small $\rho$ concentrated around $a = \pi/2$. With increasing loop size, the effect of the ”boundary condition” at $\rho = 0$ decreases and the symmetry is approximatively restored as in diffusion of Wilson loops. The vacuum expectation value of the Wilson loop is a measure of the restoration of the symmetry

$$\langle 0|W(t, \rho, z)|0 \rangle \sim \exp(-\sigma \pi \rho^2)$$

If exact the symmetry ($\tilde{Z}$) would require $\langle 0|W|0 \rangle = 0$. 
Approximate center symmetry makes the dynamics of Polyakov and Wilson loops of large size similar. Symmetries protect gauge strings from decaying.

In particular from

$$\langle 0| P(d) P(0)|0 \rangle \sim \exp(-\sigma \beta d)$$

one expects

$$\langle 0| W(d, \rho) W(0, \rho)|0 \rangle \sim \exp(-\sigma 2\pi \rho d)$$

Accounting for $\exp(-\pi \rho^2)$ suppressed contributions from the violation of the $\tilde{Z}$ symmetry

$$\langle 0| W(d, \rho) W(0, \rho)|0 \rangle \approx e^{-2\sigma \pi \rho^2} (c_0 + c_1 e^{-m_{gbd}} + ..) + d_1 e^{-\sigma 2\pi \rho d}$$
• The first class of contribution arises from the intermediate ground-state and glueballs. Not present for Polyakov loops - forbidden by exact center symmetry $\mathbb{Z}$

• Second class of contributions - common to Wilson and Polyakov loops - arises from intermediate states which are odd under $\mathbb{Z}$ and $\hat{\mathbb{Z}}$ respectively
For sufficiently large Wilson loops we expect

\[
\left[ \langle 0 | W(d, R) W(0, R) | 0 \rangle - \langle 0 | W(0, R) | 0 \rangle^2 \right] / \langle 0 | W(0, R) | 0 \rangle^2 \\
\approx c_1 e^{-m_g b \frac{\xi + R^2}{2R}} + d_1 e^{-2\sigma \xi} + \ldots
\]
For large loops, up to the glueball correction, the surface of minimal area dominates. Change in configuration around $d \approx R$ - Gross-Ooguri transition.

Wilson loop operators applied to the vacuum generate approximative stationary states of energy

$$E[W(R)] \approx \sigma 2\pi R$$

Single (gauge) string states are approximative eigenstates of the Hamiltonian.
Summary

• Wilson loop distribution - result of diffusion process

• Casimir scaling: a new regime in the confinement phase, different from strong coupling limit

• Approximate center symmetry associated with confinement

• Sufficiently large Wilson loops generate approximate stationary single string states