Finite volume effects for masses and decay constants

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Exploration of Hadron Structure and Spectroscopy using Lattice QCD
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- Introduction/Motivation
- Finite volume effects in Chiral Perturbation Theory
- Asymptotic formulae
- Two-loop calculation of $M_\pi(L)$
- Summary

Work done in collaboration with G. Colangelo, S. Dürr
Motivation to study finite volume effects

$L = 2\text{fm}$

$M_{\pi} = 200\text{MeV}$ (1)

- Finite volume effect of the pion mass:
  
  \[ \frac{M_{\pi}(L) - M_{\pi}}{M_{\pi}} \equiv 4\% \]  

- Finite volume effect is small
- But needs to be taken into account for very precise lattice calculations.
- Extrapolation $L \to \infty$ is by no means straightforward
- ChPT provides the proper analytic framework
Introduction: ChPT in finite volume

- Expansion in $m_q/\Lambda$ and $p/\Lambda$
- Quantized momenta in finite volume: $p = \frac{2\pi}{L} n$
- Condition of applicability for ChPT:

\[
m_q \ll \Lambda \quad \text{and} \quad \frac{2\pi}{L} \ll \Lambda
\]

\[
\Lambda \sim 4\pi F_\pi \quad \Rightarrow \quad 2LF_\pi \gg 1
\]
Introduction: ChPT in finite volume

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  \[ \Lambda \sim 4\pi F_\pi \quad \Rightarrow \quad 2LF_\pi \gg 1 \]

- Once this condition is respected, we still have two different physical situations:
  \[ LM_\pi \gg 1 \Rightarrow p-\text{regime} : \quad M_\pi \sim \frac{1}{L} \sim \mathcal{O}(p) \]
  \[ LM_\pi \lesssim 1 \Rightarrow \varepsilon-\text{regime} : \quad M_\pi \sim \frac{1}{L^2} \sim \mathcal{O}(\varepsilon^2) \]
\( p\text{-regime: } M_\pi L \gg 1 \)

- Calculational rule in ChPT for isotropic finite box with periodic boundary conditions:

  \[
  \text{Lagrangian:} \quad L^L_{\text{eff}} = L^\infty_{\text{eff}} \\
  \text{Propagator:} \quad G_L(x^0, \vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^3} G(x^0, \vec{x} + \vec{n}L)
  \]

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- Implication for perturbative calculation:

  \[
  \int \frac{d^4 p}{(2\pi)^4} f(p) \overset{L \ll \infty}{\rightarrow} \int \frac{dp_0}{2\pi} \frac{1}{L^3} \sum_{\vec{p}} f(p) \overset{(*)}{=} \int \frac{d^4 p}{(2\pi)^4} f(p) \sum_{\vec{n} \in \mathbb{Z}^3} e^{i\vec{p}\vec{n}L}
  \]

  \((*)\): Poisson summation formula
\( p\)-regime: \( M_\pi L \gg 1 \)

- Calculational rule in ChPT for isotropic finite box with periodic boundary conditions:

  \[
  \text{Lagrangian:} \quad \mathcal{L}^L_{\text{eff}} = \mathcal{L}^\infty_{\text{eff}} \\
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  \]

- Examples:

  \[
  M_\pi(L) = M_\pi \left(1 + \frac{1}{4} \xi g(\lambda) + \mathcal{O}(\xi^2)\right), \\
  F_\pi(L) = F_\pi \left(1 - \xi g(\lambda) + \mathcal{O}(\xi^2)\right),
  \]

  \[
  \xi = \frac{M_\pi^2}{(4\pi F_\pi)^2}, \quad \lambda = M_\pi L, \quad g(\lambda) = \sum_{\vec{n} \setminus \{0\}} \int_0^\infty dx \, e^{-\frac{x}{4} - \frac{x}{4} \vec{n}^2 \lambda^2}.
  \]

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Outline

- Introduction/Motivation
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Lüscher’s formula (applied for $M_K$)

- Leading corrections for $M_\pi L \gg 1$:

$$M_K(L) - M_K = \sum_{|\vec{n}|=1} \int d^4\ell \left( \frac{e^{i\vec{\ell}\vec{n}L}}{M_\pi^2 + \ell^2} \right) + O(e^{-\sqrt{2}M_\pi L})$$

- Integrations over spatial momenta $\vec{\ell}$ can be performed within claimed accuracy. It remains the integral over $\ell_0$.

$$M_K(L) - M_K = -\frac{3}{16\pi^2 M_K L} \int_{-\infty}^{\infty} d\ell_0 \ e^{-\sqrt{M_\pi^2 + \ell_0^2} L} T_{\pi K}^{I=0}(i\ell_0) + O(e^{-\sqrt{2}M_\pi L})$$

M. Lüscher 86
Lüsher’s formula:

\[ P = \pi, K, \eta: \]

\[ M_P(L) - M_P = -\frac{3}{16\pi^2 M_P L} \int_{-\infty}^{\infty} dy \ e^{-\sqrt{M^2 + y^2} L} T_{\pi P}^{I=0}(iy) + O(e^{-\sqrt{2} M_P L}). \]

- The coupling to the lightest particle matters, i.e. the pions.
- Leading corrections of order \( \exp(-M_\pi L) \).
Lüscher’s formula:

\[ P = \pi, K, \eta: \]

\[
M_P(L) - M_P = -\frac{3}{16\pi^2 M_P L} \int_{-\infty}^{\infty} dy \, e^{-\sqrt{M_\pi^2 + y^2} L} T_{\pi P}^{I=0}(i y) + O(e^{-\sqrt{2} M_\pi L}).
\]

- The coupling to the lightest particle matters, i.e. the pions.
- Leading corrections of order \( \exp(-M_\pi L) \).
- The formula expresses the corrections over a (analytically continued) physical amplitude.
- Analyticity properties of \( T_{\pi P}^{I=0}(\nu) \):
Lüscher’s formula or ChPT?:

\[ \Delta M_{\pi,\text{Lüsher}} = -\frac{3}{16\pi^2 M_\pi L} \int_{-\infty}^{\infty} dy \ e^{-\sqrt{M_\pi^2 + y^2} L} T_{\pi\pi}^{I=0}(iy) + \mathcal{O}(e^{-\sqrt{2}M_\pi L}), \]

\[ \Delta M_{\pi,\text{ChPT}} = \xi \frac{M_\pi}{4} g(M_\pi L) + \mathcal{O}(\xi^2). \]

- The two formulae give the leading term in two different expansions.

![Graph showing the asymptotic formula and one-loop ChPT](attachment:graph.png)
One can extend the formula so that it contains contributions from all $|\vec{n}|$ of a single propagator:

$$\Delta M_\pi = -\frac{1}{32\pi^2 M_\pi L} \sum_{|\vec{n}|=1}^{\infty} \frac{m(|\vec{n}|)}{|\vec{n}|} \int_{-\infty}^{\infty} dy e^{-\sqrt{M_\pi^2+y^2} |\vec{n}| L} T_{\pi \pi}^{I=0} (iy) + O(e^{-2M_\pi L})$$

The extension does not provide all exponentially subleading terms!
Non-leading exponential terms in $M_\pi(L)$

$$R_{M_\pi} = \frac{M_\pi(L) - M_\pi}{M_\pi}$$

C.Haeferl - Seattle, p.11
Non-leading exponential terms in $M_\pi(L)$

\[ R_{M_\pi} = \frac{M_\pi(L) - M_\pi}{M_\pi} \]
Asymptotic formula for decay constants

- In Euclidean space:
  \[
  \langle \pi(p) | A_\mu(0) |0 \rangle_L = ip_\mu F_\pi(L).
  \]

  \[
  \Delta M_\pi = - \frac{3}{16\pi^2 M_\pi L} \int_{-\infty}^{\infty} dy \ e^{-\sqrt{M_\pi^2 + y^2} L} T_{\pi\pi}(iy) + O(e^{-\sqrt{2} M_\pi L}),
  \]

  \[
  = + O(e^{-\sqrt{2} M_\pi L}),
  \]

  \[
  T_{\pi\pi}(\nu) \longleftrightarrow \langle \pi\pi | \pi\pi \rangle.
  \]

  \[
  \Delta F_\pi = \frac{3}{8\pi^2 M_\pi L} \int_{-\infty}^{\infty} dy \ e^{-\sqrt{M_\pi^2 + y^2} L} N_F(iy) + O(e^{-\sqrt{2} M_\pi L}),
  \]

  \[
  = + O(e^{-\sqrt{2} M_\pi L}),
  \]

  \[
  N_F(\nu) \longleftrightarrow \langle 3\pi | A_\mu |0 \rangle \sim A(\tau \rightarrow 3\pi \nu_\tau).
  \]

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G.Colangelo, C.H. 04

C.Haeferli - Seattle, p.13
Definition of the amplitude $N_F(\nu)$

$$\Delta F_\pi = \frac{3}{8\pi^2 M_\pi L} \int_{-\infty}^{\infty} dy \ e^{-\sqrt{M_\pi^2 + y^2} L} N_F(iy) + \mathcal{O}(e^{-\sqrt{2} M_\pi L}),$$

- $N_F(\nu)$ is the subtracted $\langle 3\pi | A_\mu | 0 \rangle$ matrix element in the forward kinematic region.

$$N_F(\nu) = -i \frac{p^\mu}{M_\pi} \left( \langle (2\pi)_{I=0\pi} | A_\mu | 0 \rangle - iQ_\mu F_\pi \frac{T^{I=0}_{\pi\pi}}{M_\pi^2 - Q^2} \right)$$

- $G. Colangelo and C.H. 04$
Finite volume corrections for $F_\pi$

\[ R_{F_\pi} = \frac{F_\pi(L) - F_\pi}{F_\pi} \]

$L = 2\text{fm}$

$L = 3\text{fm}$

$L = 4\text{fm}$
Universality of asymptotic formula

Masses (Lüscher):

<table>
<thead>
<tr>
<th>$\Delta M$</th>
<th>scattering</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_\pi$</td>
<td>$\pi\pi \rightarrow \pi\pi$</td>
</tr>
<tr>
<td>$M_K$</td>
<td>$\pi K \rightarrow \pi K$</td>
</tr>
<tr>
<td>$M_\eta$</td>
<td>$\pi \eta \rightarrow \pi \eta$</td>
</tr>
<tr>
<td>$M_B$</td>
<td>$\pi B \rightarrow \pi B$</td>
</tr>
<tr>
<td>$M_N$</td>
<td>$\pi N \rightarrow \pi N$</td>
</tr>
</tbody>
</table>

Decay constants:

<table>
<thead>
<tr>
<th>$\Delta F$</th>
<th>decays</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_\pi$</td>
<td>$\tau \rightarrow 3\pi$</td>
</tr>
<tr>
<td>$F_K$</td>
<td>$K_{\ell 4}$</td>
</tr>
<tr>
<td>$F_\eta$</td>
<td>$\eta_{\ell 4}$</td>
</tr>
<tr>
<td>$F_B$</td>
<td>$B_{\ell 4}$</td>
</tr>
</tbody>
</table>

★ G. Colangelo, A. Fuhrer, S. Lanz, work in progress
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**Motivation for two-loop calculation**

- The “best estimate” goes beyond the one-loop level, but misses terms at the two-loop level.
- Only a full two-loop calculation can clarify how good the “best estimate” is.

\[
R_{M_\pi} = \frac{M_\pi(L) - M_\pi}{M_\pi}
\]
Two-loop calculation

\[ \mathcal{L}_{\text{eff}}^L = \mathcal{L}_{\text{eff}}^{\infty}, \quad G_L(x^0, \vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^3} G(x^0, \vec{x} + \vec{n}L) \]

- \( M_\pi(L) \) defined as pole of the connected correlation function

\[ G_L(p)^{-1} = M^2 + p^2 - \Sigma_L(p^2), \]

\[ \Sigma_L(p^2) : \quad \text{self-energy in finite volume} \]

- Motivated from asymptotic formula: split self-energy in terms of number of propagators in finite volume

\[ M_\pi(L)^2 = M_\pi^2 - \Sigma^{(1)} - \Sigma^{(2)} \]
Two-loop calculation

\[ \Sigma^{(1)} = \sum_{\vec{n}} = \sum \int d^4 q \left( \frac{e^{i\vec{q}\vec{n}}L}{M_\pi^2 + q^2} \right) \]

\[ \Sigma^{(1)} = I_p + I_c + O(\xi^3) \]

\( I_p = \) residue of contour integration
\[ \Rightarrow \text{asymptotic formula} \]

\( I_c = \) from integration along the cut
Two-loop calculation

\[ \Sigma^{(1)} = \sum \int d^4 q \left( \frac{e^{i\vec{q}\vec{n}L}}{M^2_\pi + q^2} \right) \]

\[ \Sigma^{(1)} = I_p + I_c + \mathcal{O}(\xi^3) \]

\[ I_c = -\frac{iM^2_\pi}{32\pi^3 M_\pi L} \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}} \int_{-\infty}^{\infty} dy \int_{4}^{\infty} d\tilde{s} \frac{e^{-\sqrt{n(\tilde{s}+y^2)}M_\pi L}}{\tilde{s} + 2iy} \text{disc}[T_{\pi\pi}] \]

- Numerically, the contributions from \( I_c \) are very small

G. Colangelo and C. H. 06

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Two-loop calculation

- $\Sigma^{(2)}$ can be expressed in terms of a three particle scattering amplitude . . .

\[
\Sigma^{(2)} = \frac{1}{8} \sum_{n,r \neq 0} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{e^{iqnL}}{(M^2_\pi + q^2)} \frac{e^{ikrL}}{(M^2_\pi + k^2)}
\]

![Diagram showing a three-particle scattering process](image_url)
Two-loop calculation

- $\Sigma^{(2)}$ can be expressed in terms of a three particle scattering amplitude . . .

\[
\Sigma^{(2)} = \frac{1}{8} \sum_{n,r \neq 0, n \neq r} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{e^{i\mathbf{q}\mathbf{n}L}}{(M_\pi^2 + q^2)(M_\pi^2 + k^2)} \frac{e^{i\mathbf{k}\mathbf{r}L}}{M_\pi^2 + Q^2}
\]

. . . that again, needs to be subtracted,

\[
\hat{T}_{\pi\pi\pi} = \langle 3\pi|3\pi \rangle - \frac{R}{M_\pi^2 + Q^2}
\]
Two-loop calculation

- $\Sigma^{(2)}$ can be expressed in terms of a three particle scattering amplitude . . .

$$\Sigma^{(2)} = \frac{1}{8} \sum_{\substack{n,r \neq 0 \\ n \neq r}} \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{e^{i\mathbf{qn}L}}{(M_\pi^2 + q^2)} \frac{e^{i\mathbf{k}rL}}{(M_\pi^2 + k^2)}$$

$$\Sigma^{(2)} = \frac{M_\pi^2 \xi^2}{8} \left[ 9g(\lambda_\pi)^2 - \lambda_\pi g(\lambda_\pi)g'(\lambda_\pi) \right] + M_\pi^2 \xi^2 \Delta + \mathcal{O}(\xi^3)$$

- $\Delta$ includes sunset-type contributions that cannot be represented in terms of $g(\lambda_\pi)$, but may be evaluated numerically
\[ R_{M_\pi} = \frac{M_\pi(L) - M_\pi}{M_\pi} \]

- Resummed asymptotic formula is very accurate for \( M_\pi L > 2 \)
We discussed:

- Role of finite volume effects in lattice calculations
- ChPT as proper framework
- The need to go beyond leading order
- The successful results for $M_{\pi,L}, M_{K,L}, M_{\eta,L}, F_{\pi,L}, F_{K,L}$ when combined with an asymptotic formula à la Lüscher
- First two-loop calculation in $L < \infty$: $M_{\pi,L}$