Finite volume effects for masses and decay constants

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Exploration of Hadron Structure and Spectroscopy using Lattice QCD

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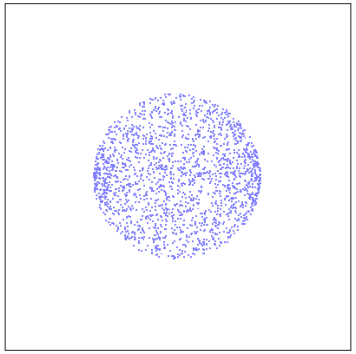
- Introduction/Motivation
- Finite volume effects in Chiral Perturbation Theory
- Asymptotic formulae
- Two-loop calculation of $M_\pi(L)$
- Summary

Work done in collaboration with G. Colangelo, S. Dürr

Motivation to study finite volume effects



$$
\begin{aligned}
L &= 2\text{fm} \\
M_\pi &= 200\text{MeV} \qquad (1)
\end{aligned}
$$

- Finite volume effect of the pion mass:

$$\frac{M_\pi(L) - M_\pi}{M_\pi} \stackrel{(1)}{=} 4\%$$

  - Finite volume effect is small
  - But needs to be taken into account for very precise lattice calculations.
  - Extrapolation $L \rightarrow \infty$ is by no means straightforward
  - ChPT provides the proper analytic framework

Introduction: ChPT in finite volume

- Expansion in m_q/Λ and p/Λ
- Quantized momenta in finite volume: $p = \frac{2\pi}{L} n$
- Condition of applicability for ChPT:

$$m_q \ll \Lambda \qquad \text{and} \qquad \frac{2\pi}{L} \ll \Lambda$$

$$\Lambda \sim 4\pi F_\pi \qquad \Rightarrow \qquad 2LF_\pi \gg 1$$

Introduction: ChPT in finite volume

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$$\Lambda \sim 4\pi F_\pi \qquad \Rightarrow \qquad 2LF_\pi \gg 1$$

- Once this condition is respected, we still have two different physical situations:

$$LM_\pi \gg 1 \Rightarrow p\text{−regime}: \qquad M_\pi \sim \frac{1}{L} \sim \mathcal{O}(p)$$

$$LM_\pi \lesssim 1 \Rightarrow \varepsilon\text{−regime}: \qquad M_\pi \sim \frac{1}{L^2} \sim \mathcal{O}(\varepsilon^2)$$

p-regime: $M_\pi L \gg 1$

- Calculational rule in ChPT for isotropic finite box with periodic boundary conditions:

$$\text{Lagrangian}: \qquad \mathcal{L}_{\text{eff}}^{\text{L}} = \mathcal{L}_{\text{eff}}^{\infty}$$

$$\text{Propagator}: \qquad G_{\text{L}}(x^0, \vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^3} G(x^0, \vec{x} + \vec{n}L)$$

J.Gasser, H.Leutwyler 87

- Implication for perturbative calculation:

$$\int \frac{d^4p}{(2\pi)^4} f(p) \overset{L<\infty}{\Longrightarrow} \int \frac{dp_0}{2\pi} \frac{1}{L^3} \sum_{\vec{p}} f(p) \overset{(*)}{=} \int \frac{d^4p}{(2\pi)^4} f(p) \sum_{\vec{n} \in \mathbb{Z}^3} e^{i\vec{p}\vec{n}L}$$

$(*)$: Poisson summation formula

p-regime: $M_\pi L \gg 1$

- Calculational rule in ChPT for isotropic finite box with periodic boundary conditions:

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$$\text{Propagator :} \qquad G_{\rm L}(x^0,\vec{x}) = \sum_{\vec{n}\in\mathbb{Z}^3} G(x^0,\vec{x}+\vec{n}L)$$

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- Examples:

$$M_\pi(L) = M_\pi \left(1 + \tfrac{1}{4}\xi\, g(\lambda) + \mathcal{O}(\xi^2)\right) \ ,$$
$$F_\pi(L) = F_\pi \left(1 - \xi\, g(\lambda) + \mathcal{O}(\xi^2)\right) \ ,$$

$$\xi = \frac{M_\pi^2}{(4\pi F_\pi)^2}\,, \qquad \lambda = M_\pi\, L\,, \qquad g(\lambda) = \sum_{\vec{n}\backslash\{0\}} \int_0^\infty dx\, e^{-\frac{1}{x} - \frac{x}{4}\vec{n}^2\lambda^2}\,.$$

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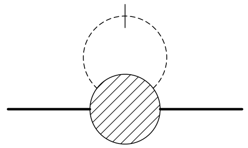
Lüscher's formula (applied for M_K)

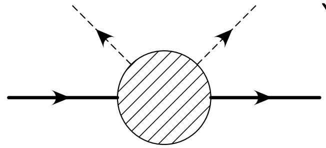
- Leading corrections for $M_\pi L \gg 1$:

$$M_K(L) - M_K \quad = \quad$$



$$+ \mathcal{O}(e^{-\sqrt{2}M_\pi L})$$

$$= \sum_{|\vec{n}|=1} \int d^4\ell \left(\frac{e^{i\vec{\ell}\vec{n}L}}{M_\pi^2 + \ell^2} \right.$$



$$\left. \vphantom{\frac{1}{1}} \right) + \mathcal{O}(e^{-\sqrt{2}M_\pi L}) \ ,$$

- Integrations over spatial momenta $\vec{\ell}$ can be performed within claimed accuracy. It remains the integral over ℓ_0.

$$M_K(L) - M_K = -\frac{3}{16\pi^2 M_K L} \int_{-\infty}^{\infty} d\ell_0 \; e^{-\sqrt{M_\pi^2 + \ell_0^2}L} T_{\pi K}^{I=0}(i\ell_0) + \mathcal{O}(e^{-\sqrt{2}\, M_\pi L}) \, ,$$

M. Lüscher 86

Lüscher's formula:

$P = \pi, K, \eta$:

$$M_P(L) - M_P = -\frac{3}{16\pi^2 M_P L} \int_{-\infty}^{\infty} dy \; e^{-\sqrt{M_\pi^2+y^2}\,L} T_{\pi P}^{I=0}(iy) + \mathcal{O}(e^{-\sqrt{2}\,M_\pi L}) \, .$$

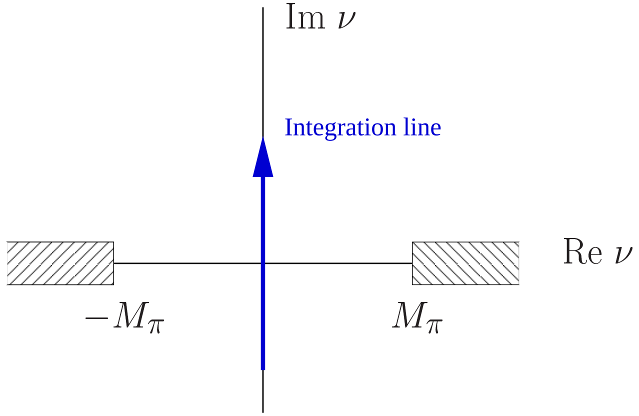
- The coupling to the ligthest particle matters, i.e. the pions.
- Leading corrections of order $\exp(-M_\pi L)$.

Lüscher's formula:

$P = \pi, K, \eta$:

$$M_P(L) - M_P = -\frac{3}{16\pi^2 M_P L} \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}\,L}\, T_{\pi P}^{I=0}(iy) + \mathcal{O}(e^{-\sqrt{2}\,M_\pi L})\,.$$

- The coupling to the ligthest particle matters, i.e. the pions.
- Leading corrections of order $\exp(-M_\pi L)$.
- The formula expresses the corrections over a (analytically continued) physical amplitude.
  - Analyticity properties of $T_{\pi P}^{I=0}(\nu)$:

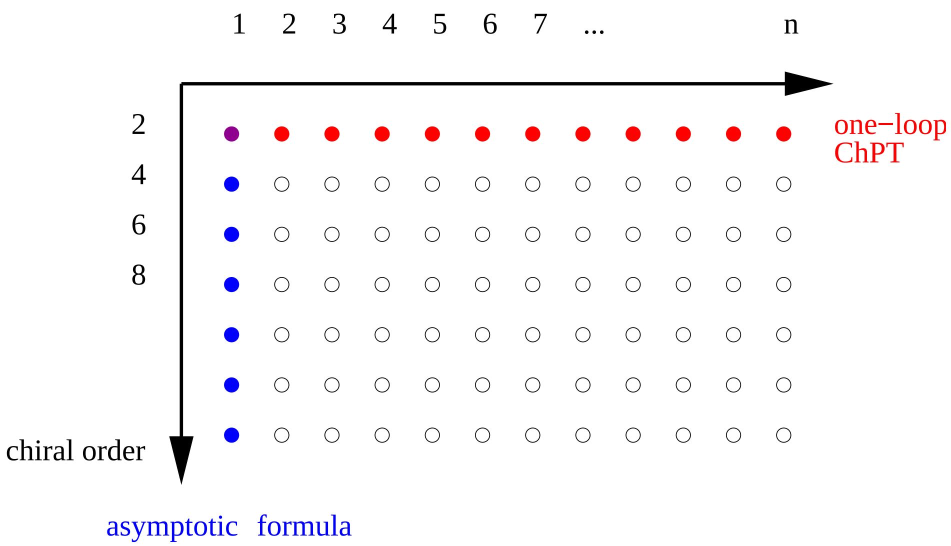


Lüscher's formula or ChPT?:

$$\Delta M_{\pi,\text{Lüscher}} = -\frac{3}{16\pi^2 M_\pi L}\int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}L}\, T_{\pi\pi}^{I=0}(iy) + \mathcal{O}(e^{-\sqrt{2}M_\pi L})\,,$$

$$\Delta M_{\pi,\text{ChPT}} = \xi\frac{M_\pi}{4}\, g(M_\pi L) + \mathcal{O}(\xi^2)\,.$$

- The two formulae give the leading term in two different expansions.

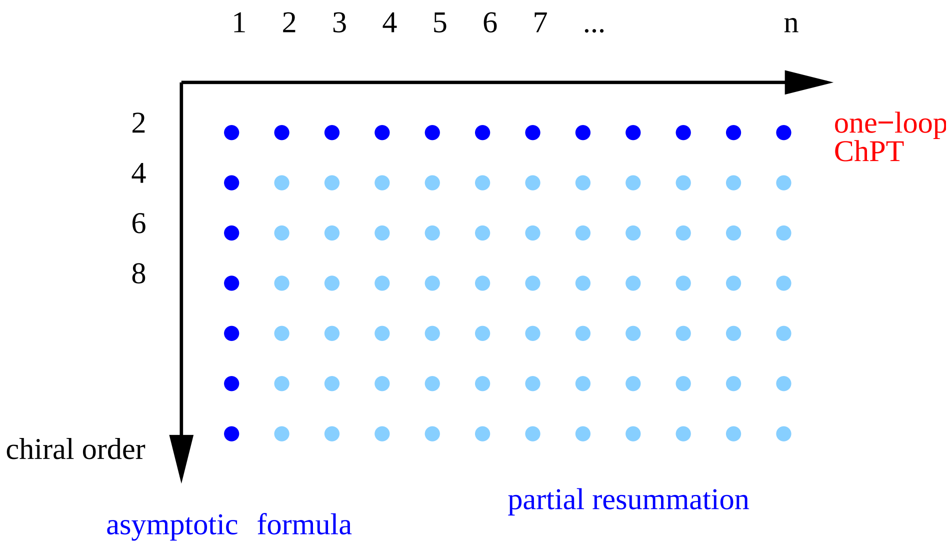


Extension of Lüscher's formula

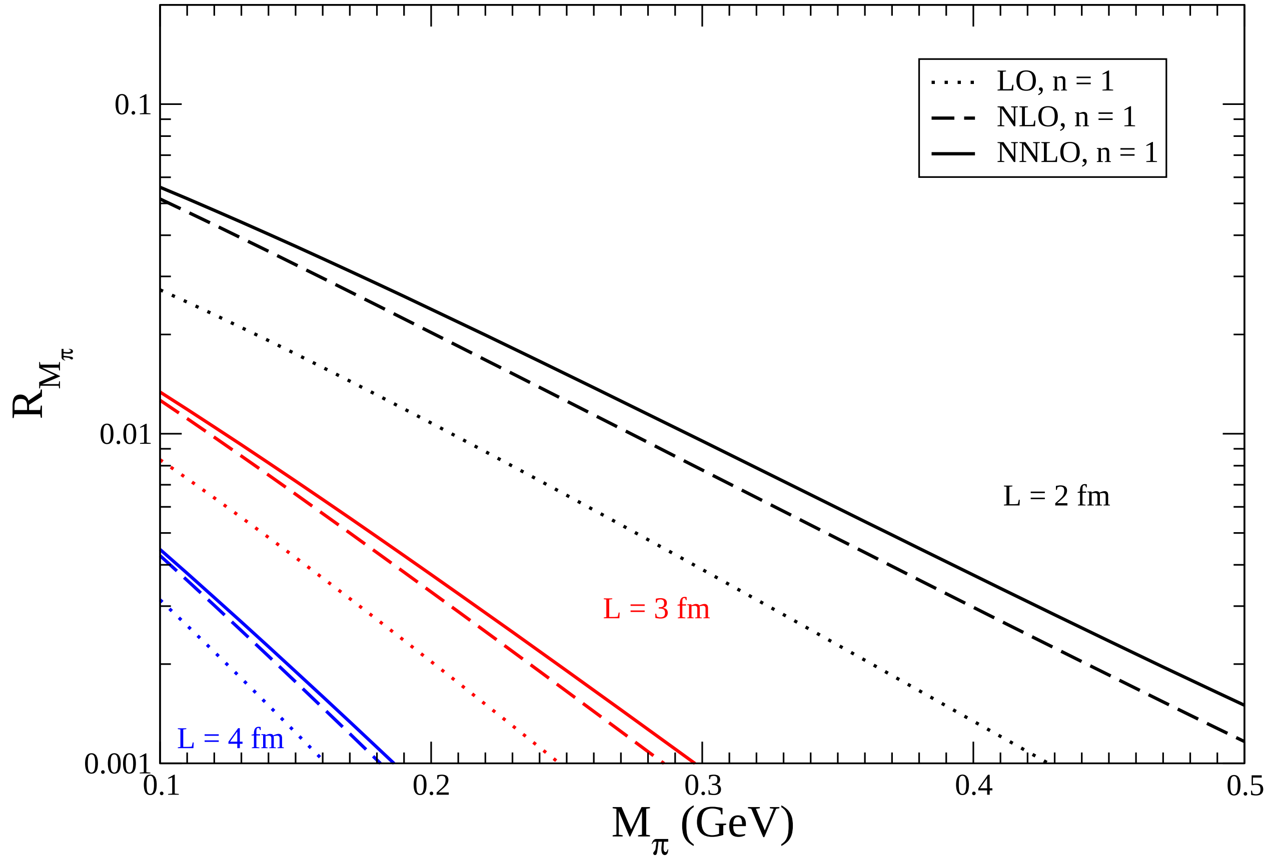
- One can extend the formula so that it contains contributions from all $|\vec{n}|$ of a single propagator:

$$\Delta M_\pi = -\frac{1}{32\pi^2 M_\pi L} \sum_{|\vec{n}|=1}^{\infty} \frac{m(|\vec{n}|)}{|\vec{n}|} \int_{-\infty}^{\infty} dy e^{-\sqrt{M_\pi^2+y^2}|\vec{n}|L} T_{\pi\pi}^{I=0}(iy) + \mathcal{O}(e^{-2M_\pi L})$$

- The extension does not provide all exponentially subleading terms!

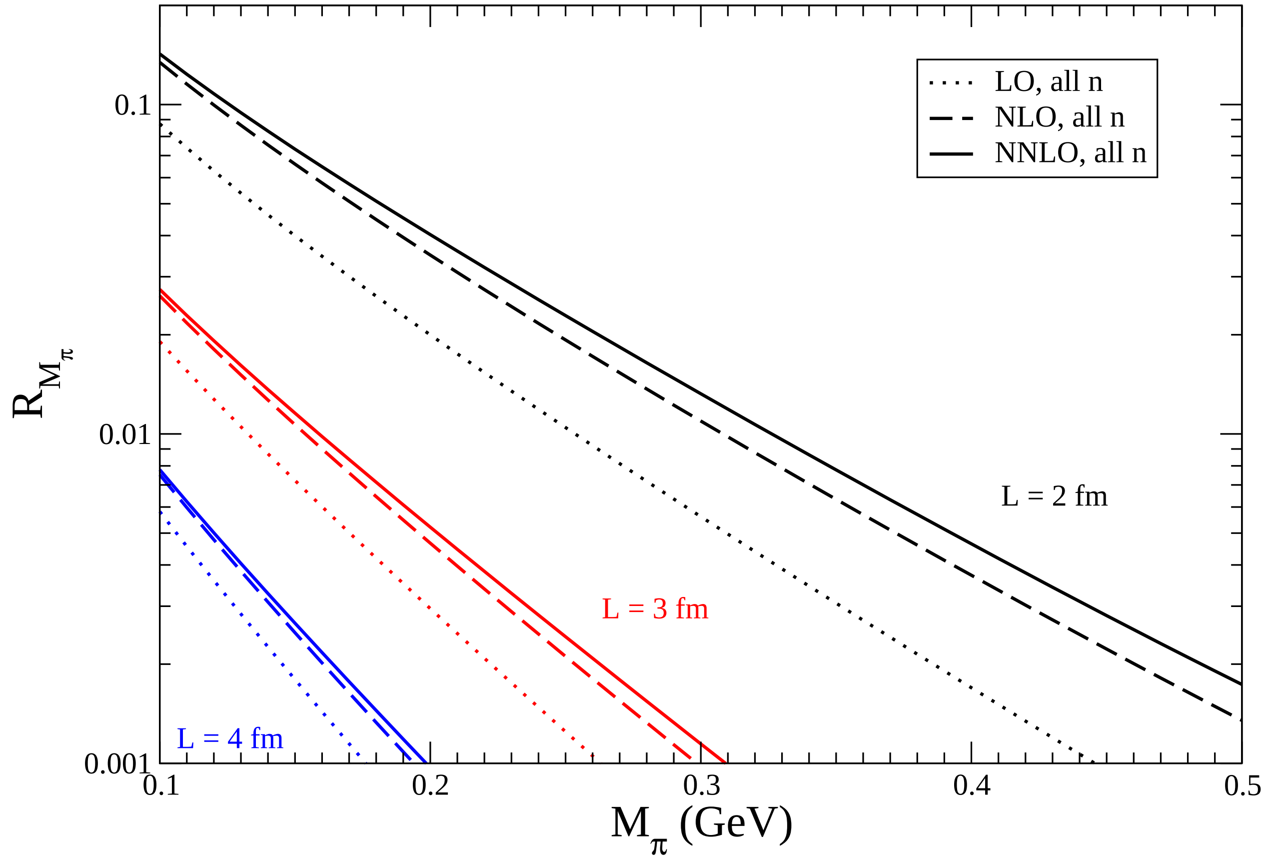


Non-leading exponential terms in $M_\pi(L)$



$$R_{M_\pi} = \frac{M_\pi(L) - M_\pi}{M_\pi}$$

Non-leading exponential terms in $M_\pi(L)$



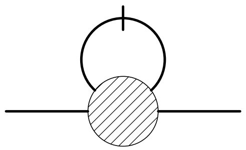
$$R_{M_\pi} = \frac{M_\pi(L) - M_\pi}{M_\pi}$$

Asymptotic formula for decay constants

- In Euclidean space: $\langle \pi(p) | A_\mu(0) | 0 \rangle_L = i p_\mu F_\pi(L) \quad .$

$$\Delta M_\pi = -\frac{3}{16\pi^2 M_\pi L} \int_{-\infty}^{\infty} dy \, e^{-\sqrt{M_\pi^2+y^2}\,L} T_{\pi\pi}(iy) + \mathcal{O}(e^{-\sqrt{2}\,M_\pi L}) \,,$$

$$=$$



$$+ \mathcal{O}(e^{-\sqrt{2} M_\pi L}) \,,$$

$$T_{\pi\pi}(\nu) \longleftrightarrow \langle \pi\pi | \pi\pi \rangle \quad .$$

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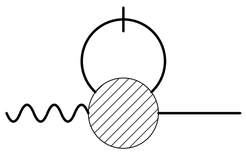
$$\Delta F_\pi = \frac{3}{8\pi^2 M_\pi L} \int_{-\infty}^{\infty} dy \, e^{-\sqrt{M_\pi^2+y^2}\,L} N_F(iy) + \mathcal{O}(e^{-\sqrt{2}\,M_\pi L}) \,,$$

$$=$$



$$+ \mathcal{O}(e^{-\sqrt{2} M_\pi L}) \,,$$

$$N_F(\nu) \longleftrightarrow \langle 3\pi | A_\mu | 0 \rangle \sim A(\tau \to 3\pi \nu_\tau) \quad .$$

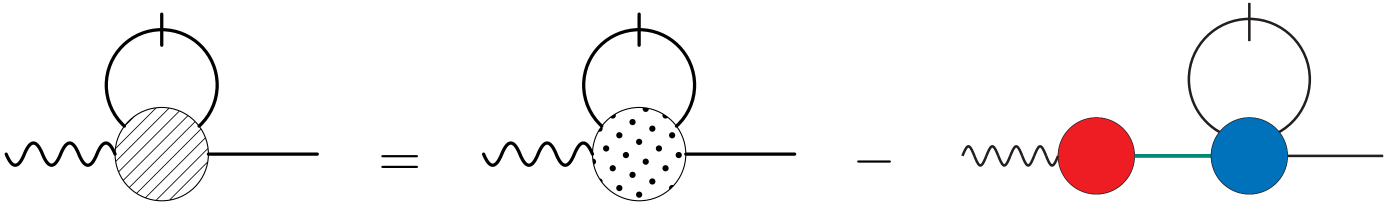
G.Colangelo, C.H. 04

Definition of the amplitude $N_F(\nu)$

$$\Delta F_\pi \quad = \quad \frac{3}{8\pi^2 M_\pi L} \int_{-\infty}^{\infty} dy\; e^{-\sqrt{M_\pi^2+y^2}L} N_F(iy) + \mathcal{O}(e^{-\sqrt{2}\,M_\pi L})\,,$$

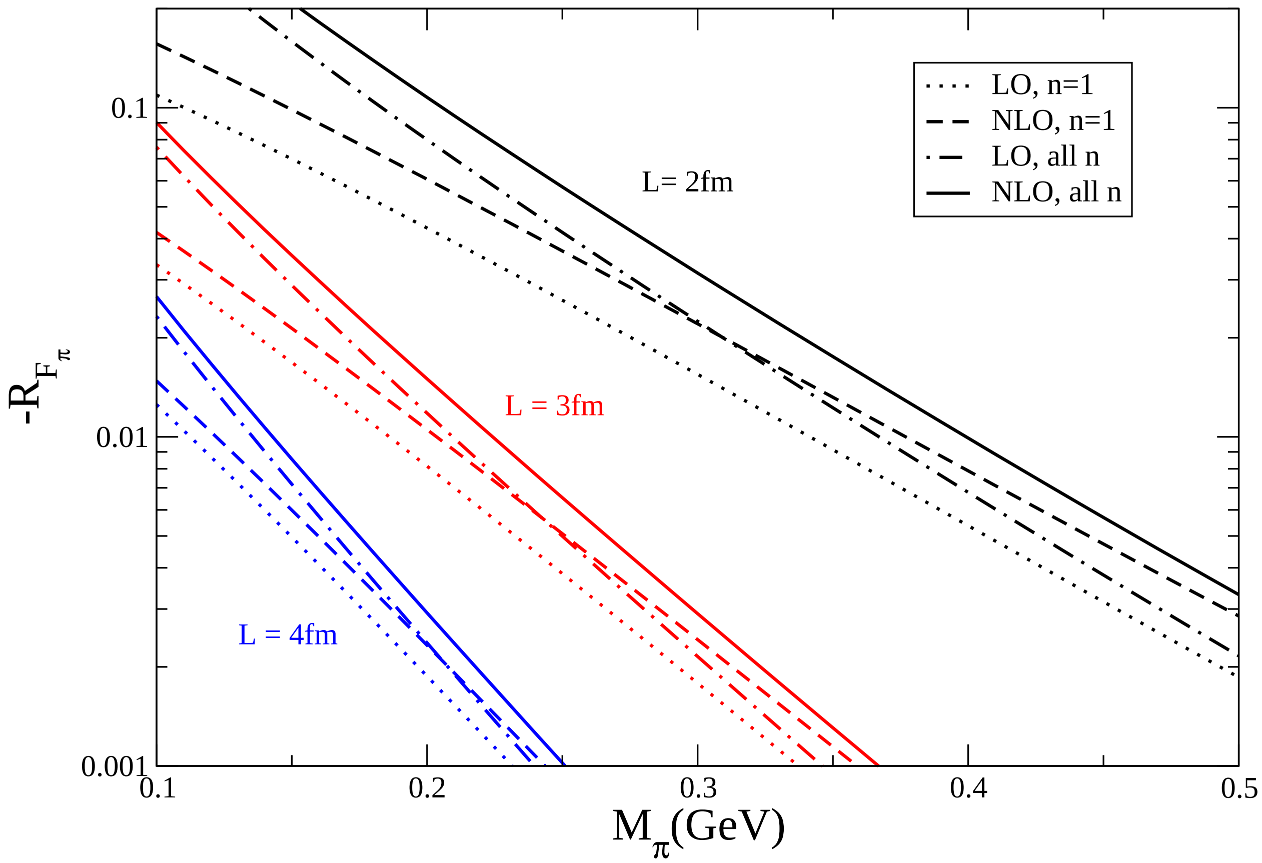
- $N_F(\nu)$ is the subtracted $\langle 3\pi|A_\mu|0\rangle$ matrix element in the forward kinematic region.

$$N_F(\nu) = -i\frac{p^\mu}{M_\pi}\left(\langle(2\pi)_{I=0}\pi|A_\mu|0\rangle - iQ_\mu F_\pi \frac{T_{\pi\pi}^{I=0}}{M_\pi^2 - Q^2}\right)$$



G.Colangelo and C.H. 04

Finite volume corrections for F_π



$$R_{F_\pi} = \frac{F_\pi(L) - F_\pi}{F_\pi}$$

Universality of asymptotic formula

Masses (Lüscher):

ΔM	scattering	
M_π	$\pi\pi \to \pi\pi$	✓
M_K	$\pi K \to \pi K$	✓
M_η	$\pi\eta \to \pi\eta$	✓
M_B	$\pi B \to \pi B$	★
M_N	$\pi N \to \pi N$	★

Decay constants:

ΔF	decays	
F_π	$\tau \to 3\pi$	✓
F_K	$K_{\ell 4}$	✓
F_η	η_{ℓ_4}	✓
F_B	$B_{\ell 4}$	★

✓ G. Colangelo, C.H. 04; G. Colangelo, S. Dürr, C.H. 05

★ G. Colangelo, A. Fuhrer, S. Lanz, work in progress

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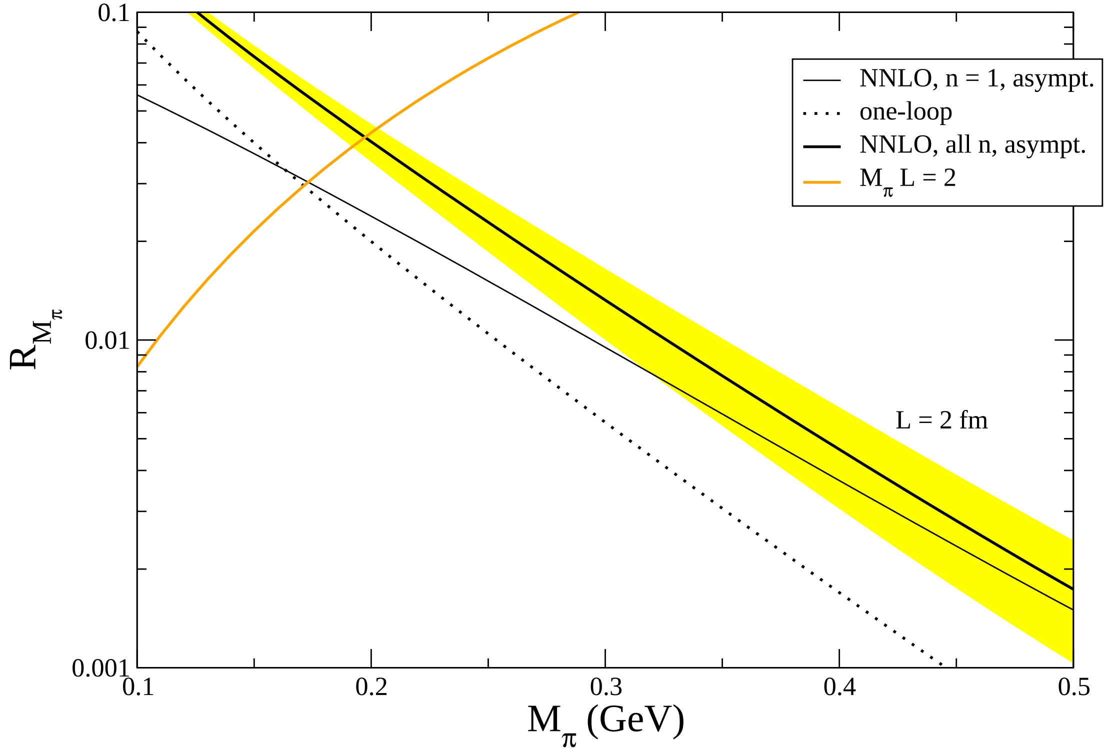
Motivation for two-loop calculation



$$R_{M_\pi} = \frac{M_\pi(L) - M_\pi}{M_\pi}$$

- The “best estimate” goes beyond the one-loop level, but misses terms at the two-loop level
- Only a full two-loop calculation can clarify, how good the “best estimate” is

Two-loop calculation

$$\mathcal{L}_{\text{eff}}^{\text{L}} = \mathcal{L}_{\text{eff}}^{\infty} \quad , \qquad G_{\text{L}}(x^0, \vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^3} G(x^0, \vec{x} + \vec{n}L)$$

- $M_\pi(L)$ defined as pole of the connected correlation function

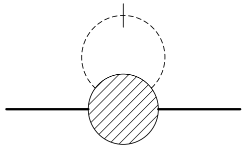
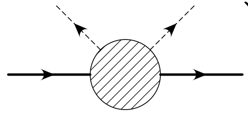
$$G_L(p)^{-1} \quad = \quad M^2 + p^2 - \Sigma_L(p^2) \quad ,$$

$\Sigma_L(p^2)$: self-energy in finite volume

- Motivated from asymptotic formula: split self-energy in terms of number of propagators in finite volume

$$M_\pi(L)^2 = M_\pi^2 - \Sigma^{(1)} - \Sigma^{(2)}$$

Two-loop calculation

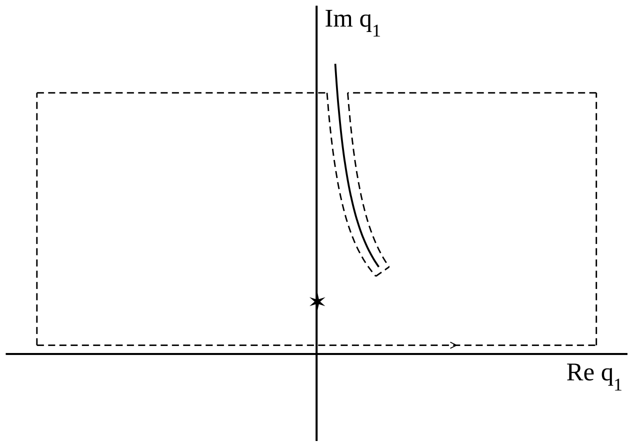
$$\Sigma^{(1)} =$$

$$= \sum_{\vec{n}} \int d^4q \left(\frac{e^{i\vec{q}\vec{n}L}}{M_\pi^2 + q^2} \right.$$

$$\left. \right)$$

$$\Sigma^{(1)} = I_p + I_c + \mathcal{O}(\xi^3)$$

I_p = residue of contour integration

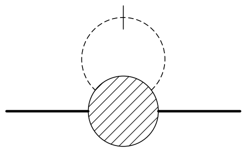
$\Rightarrow$ asymptotic formula

I_c = from integration along the cut

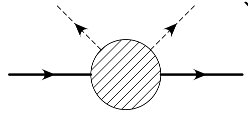


Two-loop calculation

$$\Sigma^{(1)} =$$



$$= \sum_{\vec{n}} \int d^4q \left(\frac{e^{i\vec{q}\vec{n}L}}{M_\pi^2 + q^2} \right.$$



$$\left. \right)$$

$$\Sigma^{(1)} = I_p + I_c + \mathcal{O}(\xi^3)$$

$$I_c = -\frac{\mathrm{i} M_\pi^2}{32\pi^3 M_\pi L} \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}} \int_{-\infty}^{\infty} dy \int_4^{\infty} d\tilde{s}\, \frac{e^{-\sqrt{n(\tilde{s}+y^2)} M_\pi L}}{\tilde{s} + 2\mathrm{i}y} \,\mathrm{disc}\left[T_{\pi\pi}\right]$$

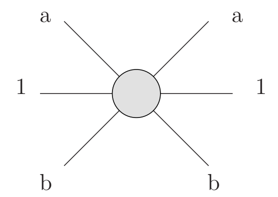
G.Colangelo and C.H. 06

- Numerically, the contributions from I_c are very small

Two-loop calculation

- $\Sigma^{(2)}$ can be expressed in terms of a three particle scattering amplitude . . .

$$\Sigma^{(2)} = \frac{1}{8}\sum_{\substack{\mathbf{n},\mathbf{r}\neq \mathbf{0}\\ \mathbf{n}\neq\mathbf{r}}} \int \frac{d^4q}{(2\pi)^4}\,\frac{d^4k}{(2\pi)^4}\,\frac{e^{i\mathbf{qn}L}}{(M_\pi^2+q^2)}\,\frac{e^{i\mathbf{kr}L}}{(M_\pi^2+k^2)}$$



$\hat{T}_{\pi\pi\pi}$

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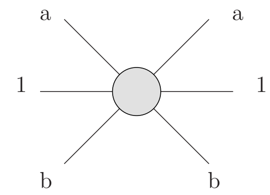
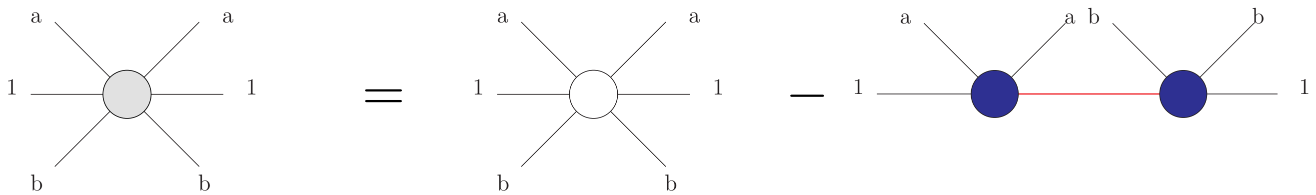


$\underbrace{\qquad\qquad}_{\hat{T}_{\pi\pi\pi}}$

. . . that again, needs to be subtracted,

$$\hat{T}_{\pi\pi\pi} = \langle 3\pi | 3\pi \rangle - \frac{R}{M_\pi^2 + Q^2}$$



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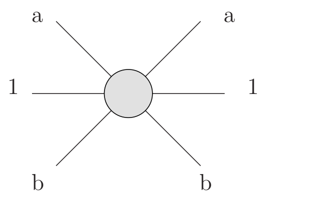

$$\underbrace{\qquad\qquad}_{\hat{T}_{\pi\pi\pi}}$$

$$\Sigma^{(2)} \quad = \quad \frac{M_\pi^2 \xi^2}{8} \left[9g(\lambda_\pi)^2 - \lambda_\pi\, g(\lambda_\pi) g'(\lambda_\pi) \right] + M_\pi^2 \xi^2 \Delta + \mathcal{O}(\xi^3)$$

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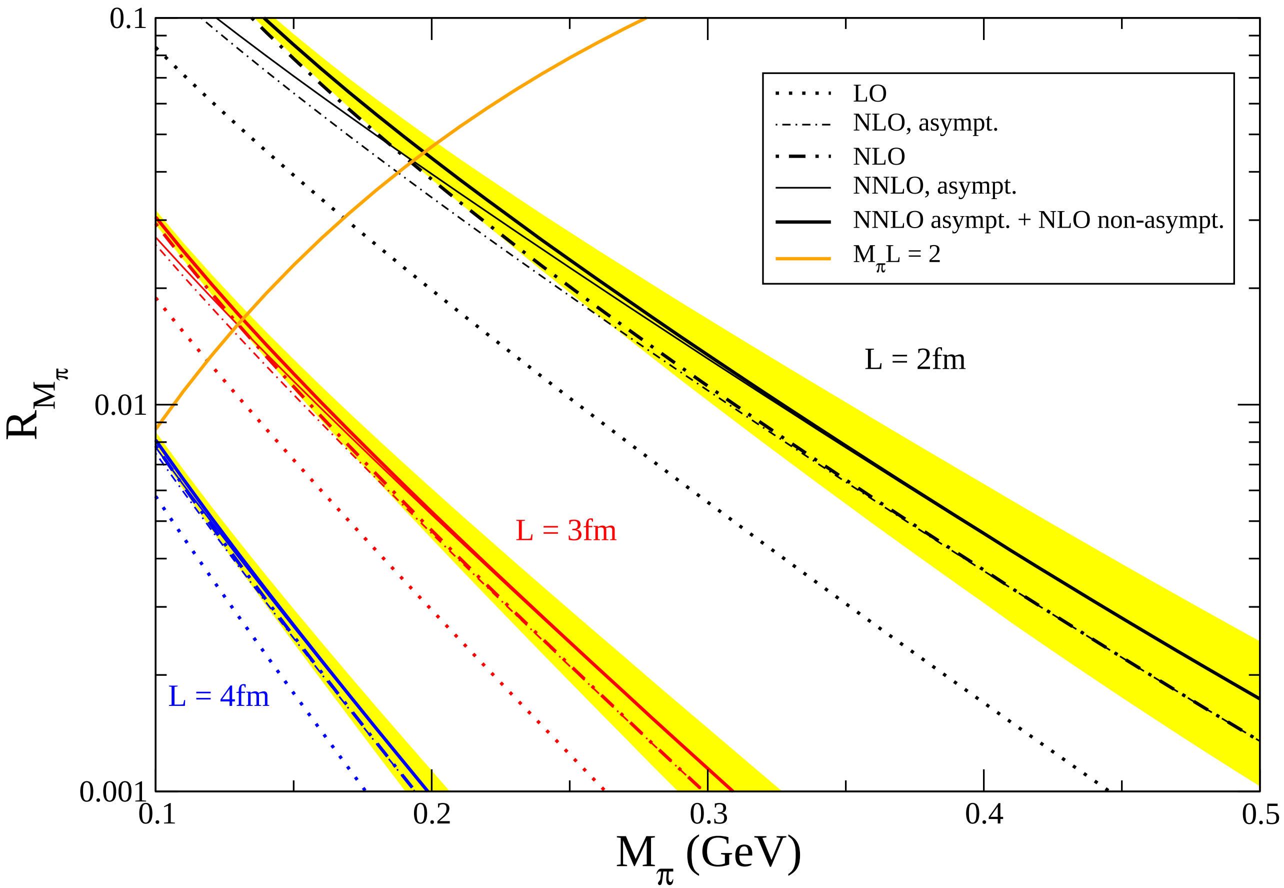
- Δ includes sunset-type contributions that cannot be represented in terms of $g(\lambda_\pi)$, but may be evaluated numerically

Numerical evaluation

$$R_{M_\pi} = \frac{M_\pi(L) - M_\pi}{M_\pi}$$



- Resummed asymptotic formula is very accurate for $M_\pi L > 2$

Summary

We discussed:

- Role of finite volume effects in lattice calculations
- ChPT as proper framework
- The need to go beyond leading order
- The successful results for $M_{\pi,L}$,$M_{K,L}$,$M_{\eta,L}$,$F_{\pi,L}$,$F_{K,L}$ when combined with an asymptotic formula *à la* Lüscher
- First two-loop calculation in $L < \infty$: $M_{\pi,L}$