The rooting trick for staggered fermions: legal or not ?

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Workshop on the Fourth Root of the Staggered Fermion Determinant

INT Seattle, March 20-21, 2006

Setup: three questions

- Naive fermions yield 2^d degenerate species in the continuum limit in d dimensions.
- The Kogut-Susskind thinning procedure reduces them by a factor $2^{d/2}$, hence staggered fermions have a remnant $2^{d/2}$-fold multiplicity.
- There is no exact symmetry to further reduce it, hence continuum-motivated "tricks" must be invoked to produce effective undoubled flavors.

1) Is the **taste-projection** trick legal for observables involving only valence quarks ?
2) Is the **rooted-determinant** trick legal for observables involving only sea quarks ?
3) Can the two coexist – does the **combination** yield a legal discretization of QCD ?

For this talk: "legal" = "correct in the continuum limit"

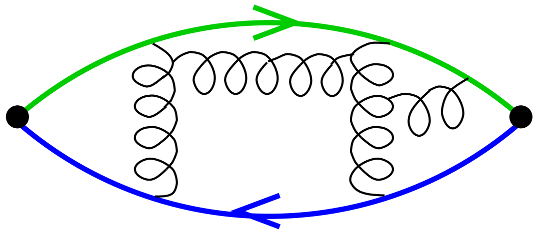
For this talk: $\mu_u = \mu_d = \mu_s = 0$ and $T = 0$

Introduction (1): QCD spectroscopy

hadronic correlator in QCD with $N_f \geq 2$ quarks: $C(t) = \int d^4x\, C(t,\mathbf{x})\, e^{\mathrm{i}\mathbf{px}}$ where

$$C(x) = \langle O(x)\, O(0)^\dagger \rangle \;=\; \frac{1}{Z}\int DU\, D\bar{q} Dq\;\; O(x)\, O(0)^\dagger\;\; e^{-S_G - S_F}$$

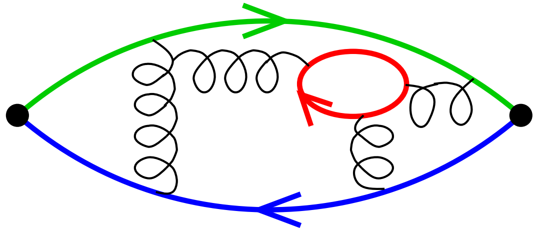
with $O(x) = \bar{d}(x)\Gamma u(x)$ and $\Gamma = \gamma_5, \gamma_4\gamma_5$ for $\pi^\pm$ and
$S_G = \beta \sum (1 - \frac{1}{3}\mathrm{Re}\,\mathrm{Tr}\, U_{\mu\nu}(x))$, $S_F = \sum \bar{q}(D+m)q$



(A) Quenched QCD: quark loops neglected

$$\langle \bar{d}(x)\Gamma_1 u(x)\; \bar{u}(0)\Gamma_2 d(0)\rangle \;=\; \frac{1}{Z}\int DU\;\; \det(D+m)^{N_f}\; e^{-S_G}$$

$$\times\;\; \mathrm{Tr}\Big\{\Gamma_1 (D+m)^{-1}_{x0}\Gamma_2\; \underbrace{(D+m)^{-1}_{0x}}_{\gamma_5[(D+m)^{-1}_{x0}]^\dagger\gamma_5}\Big\}$$



(B) Full QCD

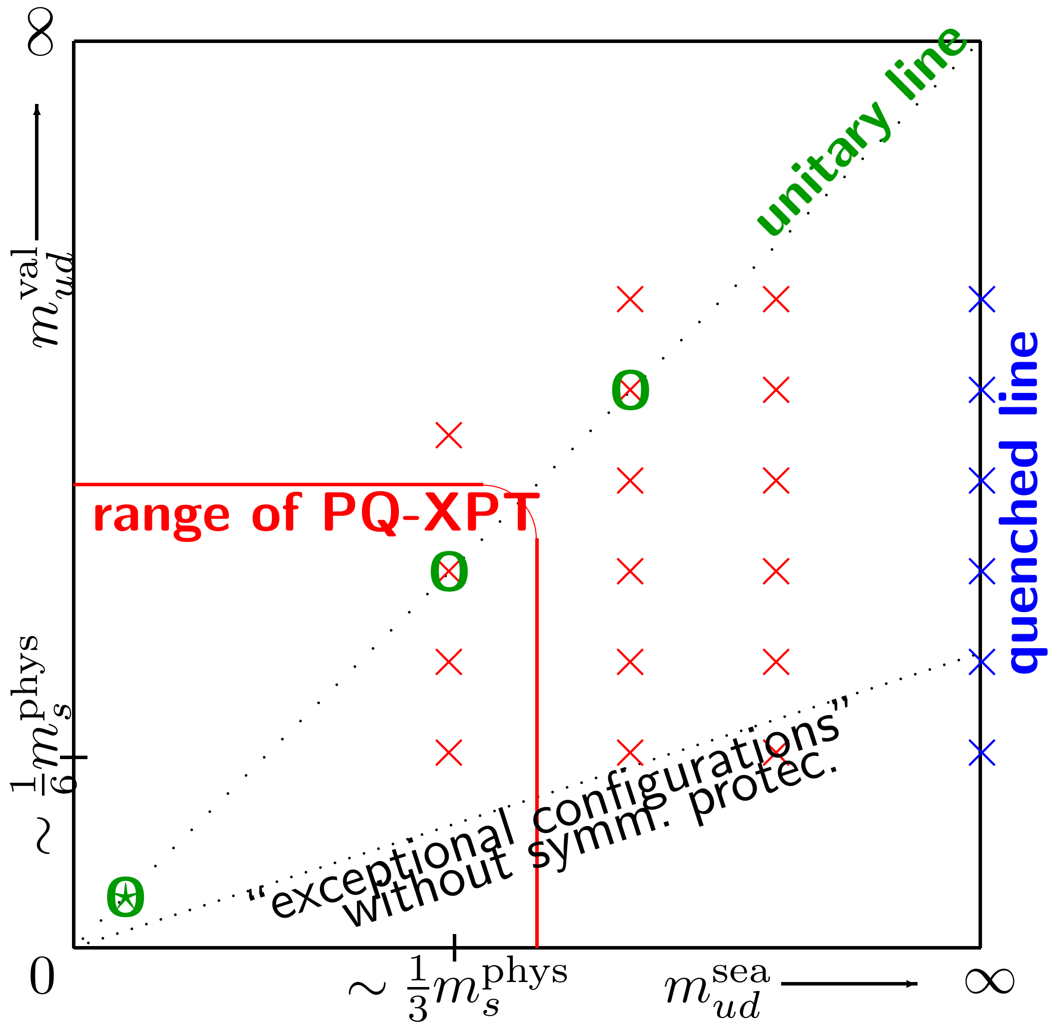
- usually $m_u = m_d$ [isospin $SU(2)$ symmetry good: $m_{ud} \equiv \frac{m_u + m_d}{2}$]
- usually $m_{\mathrm{valence}} \neq m_{\mathrm{sea}}$ ("parametric solver slow-down")

Introduction (2): strategy of PQ data taking

- PQ-QCD is a useful *extension* of QCD, *same* low-energy constants as (full) QCD [unlike Q-QCD]
  [Sharpe, Shoresh, Bernard, Golterman]

- performing $a \to 0$ first and $m \to 0$ in a second step is safe but requires (lots of) precise data

- (crucial) practical issues:
  - **renormalization**
  - **scale setting**
  - **regime of applicability of XPT**



unitary line: well-defined Hamiltonian in the continuum and with certain discretized actions (sea and valence matched) at $a > 0$

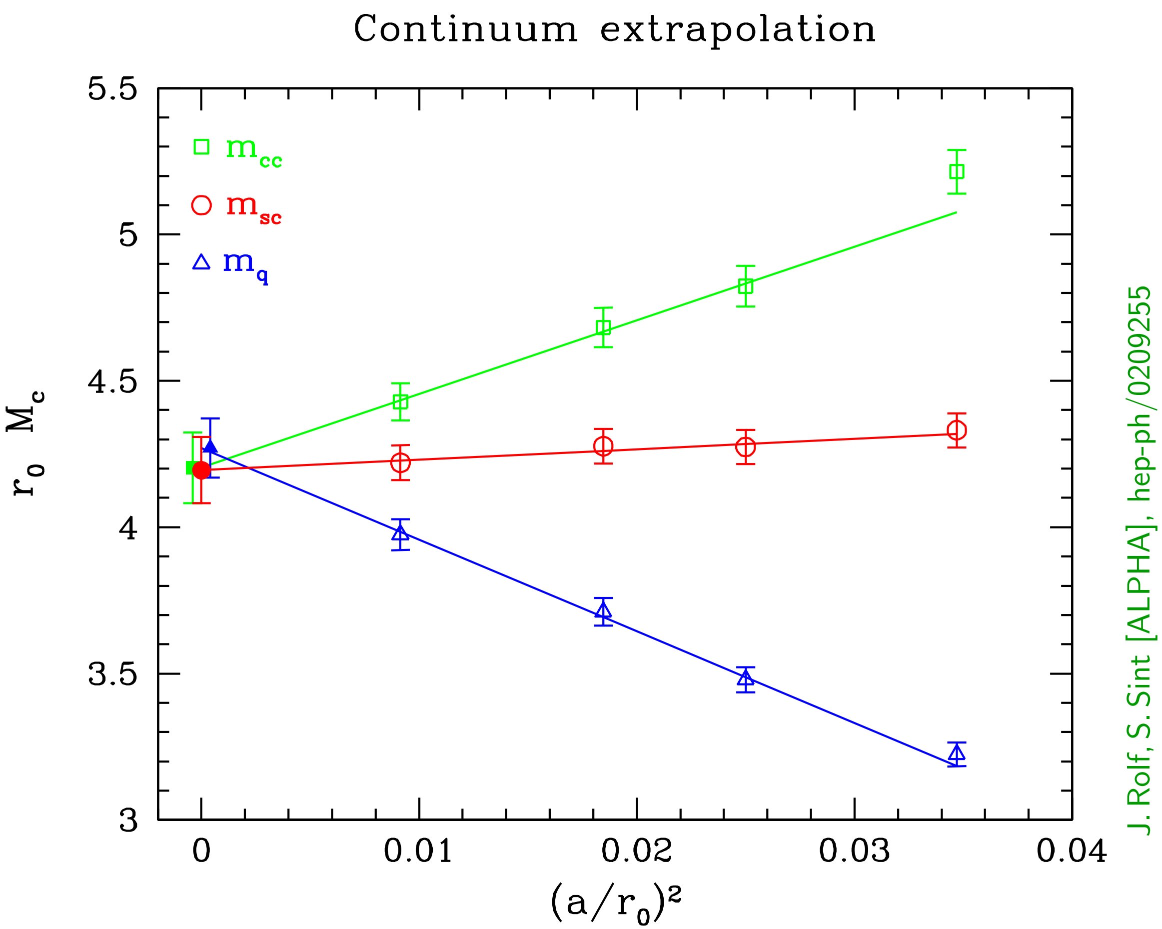
Introduction (3): universality/locality/unitarity

- **universality**
  the continuum limit is the same with *any* legal action [RG-FP]
- **scaling**
  lattice artefacts scale away: [Symanzik]

$$\frac{F_\pi}{M_n}\Big|_{\text{latt}} = \frac{F_\pi}{M_n}\Big|_{\text{cont}} + \text{const}\,(a/r_0)^n$$

$$n = \begin{cases} 1 & \text{for Wilson fermions} \\ 2 & \text{for clover/staggered/overlap} \end{cases}$$

  caveat: "const" may be large !
- **locality**
  starting with a legal action and adding a local irrelevant piece yields another legal action



- **unitarity**
  Q-QCD in the continuum is sick due to a lack of unitarity [Bernard, Golterman, Sharpe]
  F-QCD (on the "unitary line") is *exactly* unitary with pure Wilson fermions
  F-QCD (on the "unitary line") is *not proven* unitary with clover/overlap fermions
  F-QCD (on the "unitary line") is *likely not* unitary with rooted staggered fermions

1) Is the **taste-projection** trick legal for observables involving only valence quarks ?

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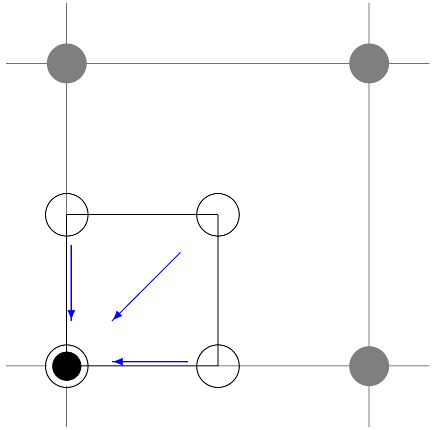
Valence sector (1): staggered tastology

N_f staggered fields $\chi = u,d,s,...$ with 4 tastes each; decomposition $\chi(x, x+a\hat{1}, ..., x+a\hat{1}+a\hat{2}+a\hat{3}+a\hat{4}) \rightarrow q(X)$ collects $2^{d/2}$ tastes with $2^{d/2}$ components each in one "blocked node"



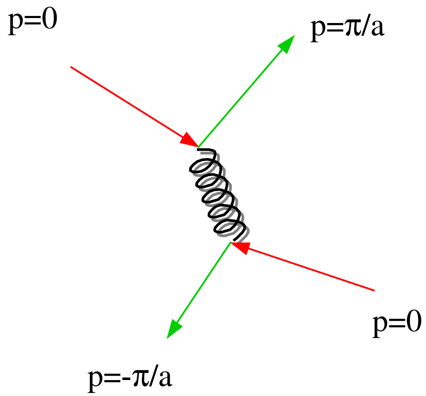
$\{X\} = \{N\}b$ with $b = 2a$, then the free action takes the form

$$S_{\mathrm{st}} = b^4 \sum_{X,\mu} \bar{q}(X)\,[\nabla_\mu(\gamma_\mu \otimes I) - \frac{b}{2}\triangle_\mu(\gamma_5 \otimes \tau_\mu\tau_5)]\;q(X)$$

with (spinor$\otimes$taste), $\tau_\mu = \gamma_\mu^*, \tau_5 = \gamma_5$ and the blocked derivatives

$$(\nabla_\mu q)(X) = \frac{q(X+b\hat{\mu}) - q(X-b\hat{\mu})}{2b}$$

$$(\triangle_\mu q)(X) = \frac{q(X+b\hat{\mu}) - 2q(X) + q(X-b\hat{\mu})}{b^2}$$



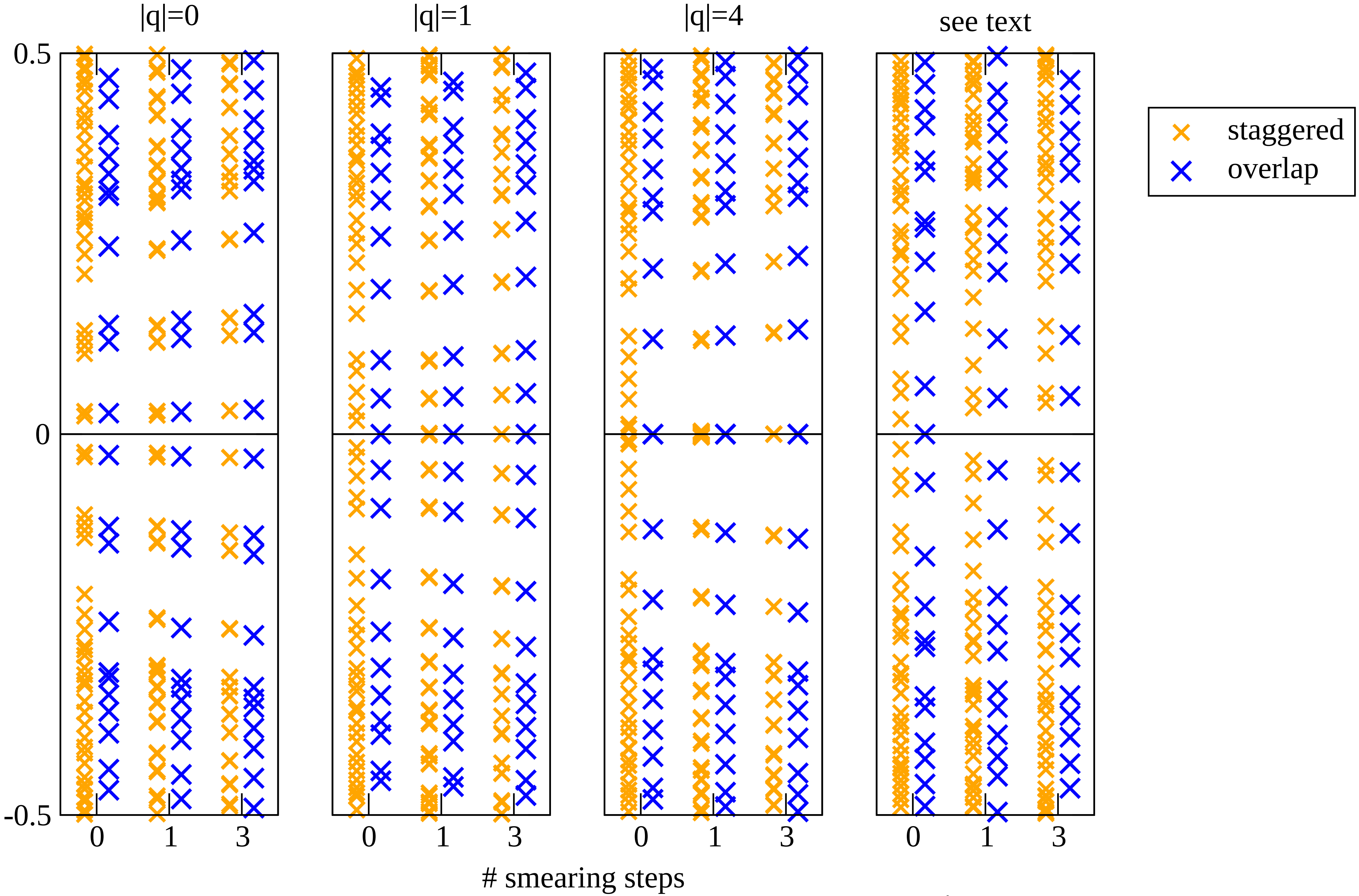
$\Longrightarrow$ in taste basis taste interactions stem from dim=5 Wilson-like term

$\longrightarrow$ do they go away with $a \to 0$ without any trace ? (order of limits?)

x-space: different tastes (and individ. components) see slightly different local gauge field

p-space: $p \sim \pi/a$ gluons kick field from one taste to another (flavor exact with N_f fields)

$\Longrightarrow$ identification staggered=physical flavor requires taste interactions minimized/absent

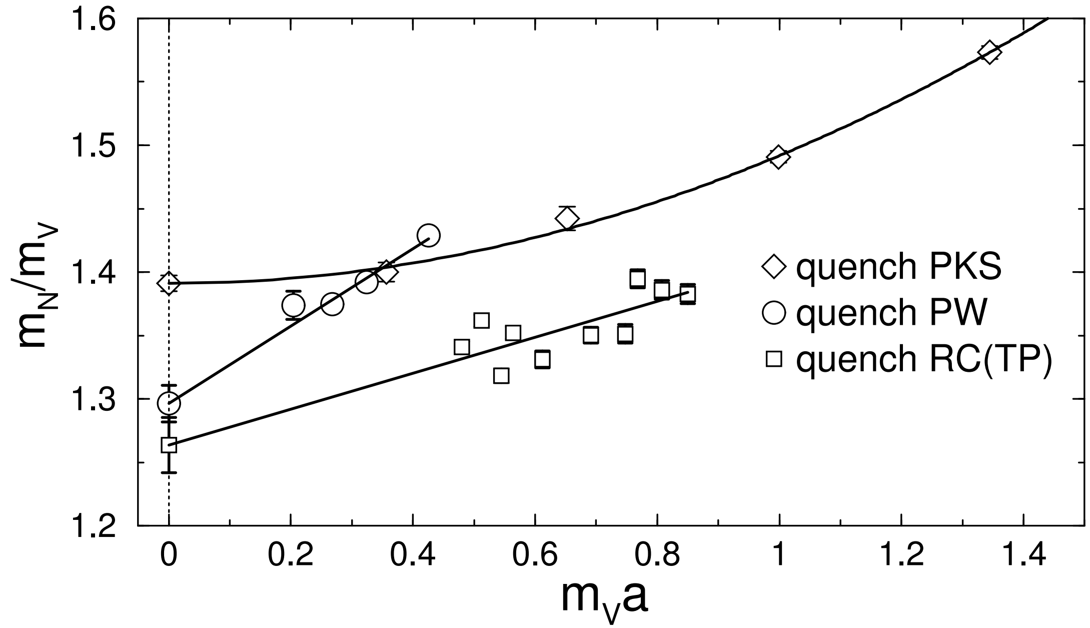


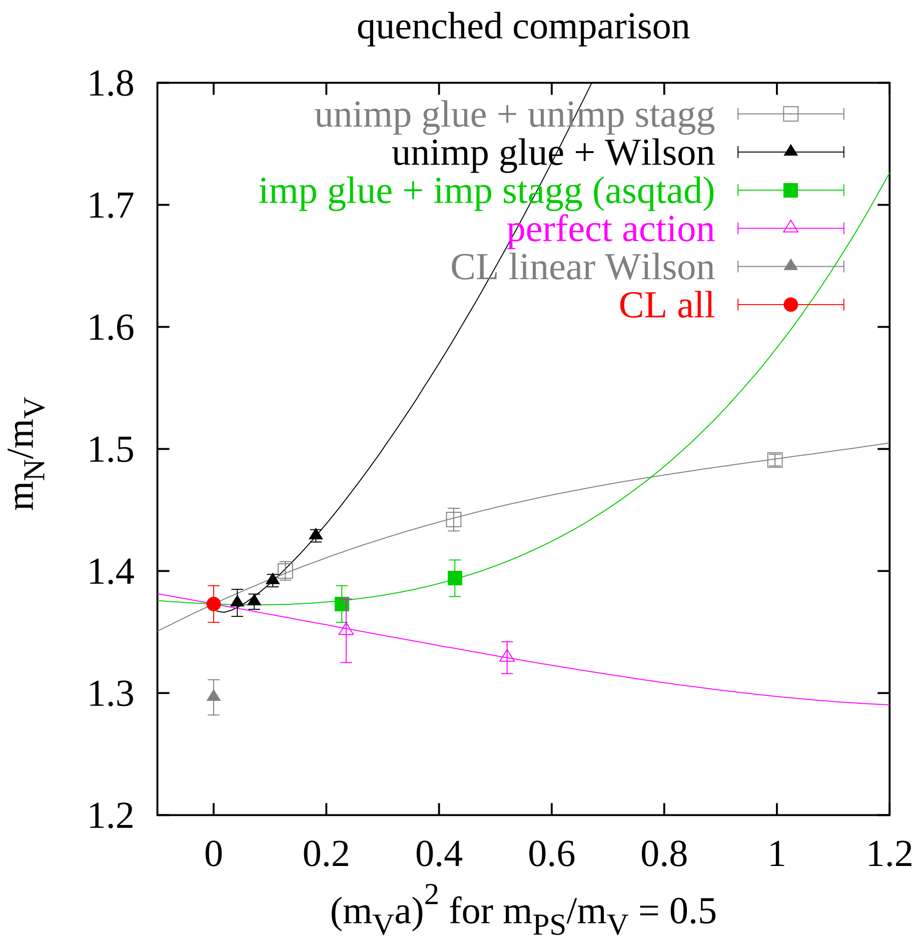
SD, C.Hoelbling, PRD 71, 054501 (2005) [hep-lat/0411022]

(Schwinger model, $\beta = 7.2, 24^2$)

Valence sector (2): historic universality test

S. Aoki, Lat'00 [hep-lat/0011074]; C.T.H. Davies, Lat'04 [hep-lat/0409039]





$\Longrightarrow$ fake non-universal behavior of staggered action in pure valence sector is gone

1) Is the **taste-projection** trick legal for observables involving only valence quarks ?

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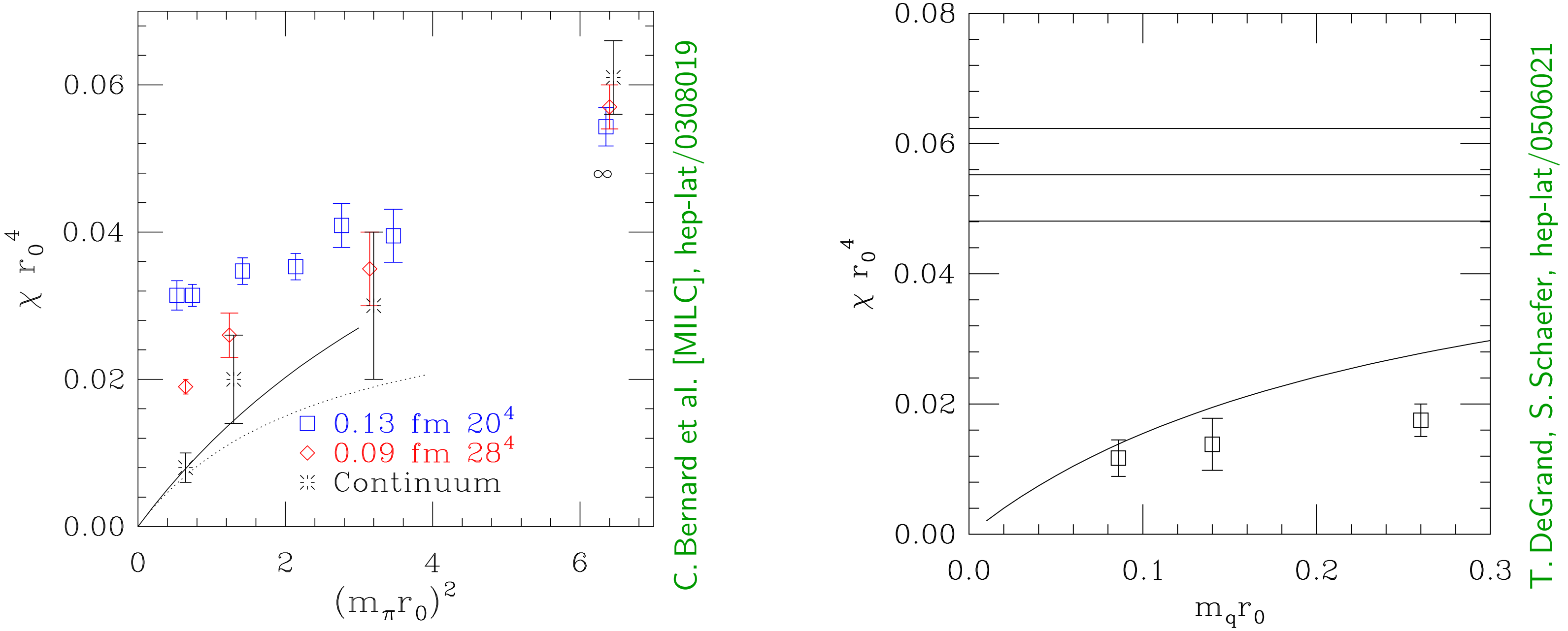
Sea sector (1): topological susceptibility in 4D

idea: $\chi_{\rm top} = \frac{\langle \nu^2 \rangle}{V}$ probes vacuum structure, depends exclusively on $m \equiv m^{\rm sea}$

$$\chi_{\rm top} \to \begin{cases} \frac{\Sigma}{2/m_{ud}+1/m_s+\ldots} & \text{for } m\to 0 \quad \text{with } \Sigma = -\lim_{m_q\to 0} \langle \bar{q}q \rangle \Big|_{m_{ud},m_s,\ldots} \\ \chi_{\rm top}^{\rm qu} & \text{for } m \to \infty \quad \text{with } \chi_{\rm top}^{\rm qu} = \frac{F^2}{2N_f}(M_{\eta'}^2 + M_\eta^2 - 2M_K^2) \end{cases}$$

interpolation: $\frac{1}{\chi_{\rm top}} = \frac{2/m_{ud}+1/m_s+\ldots}{\Sigma} + \frac{1}{\chi_{\rm top}^{\rm qu}}$ [SD, NPB 611, 281 (2001)]

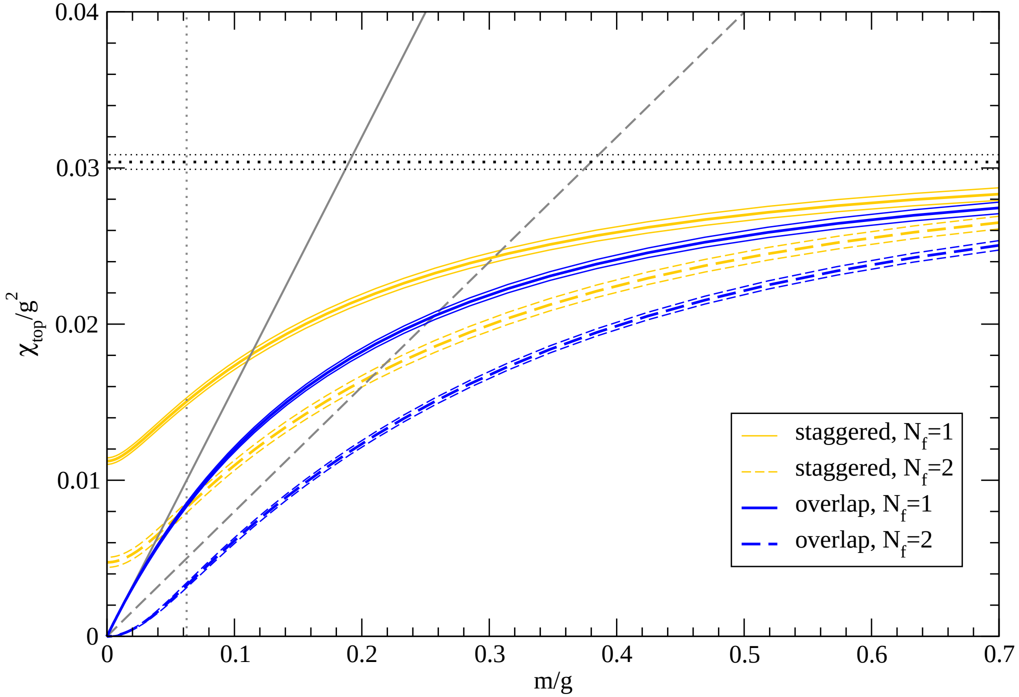


$\Longrightarrow$ cut-off effects may hide underlying continuum behavior

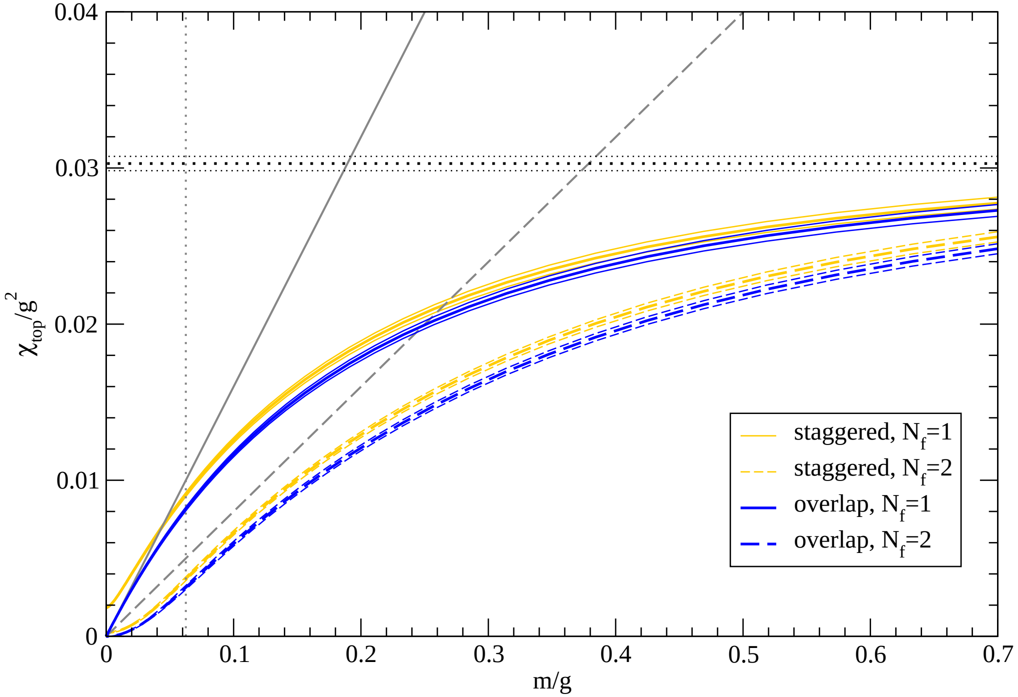
Sea sector (2): topological susceptibility in 2D

$\chi_\text{top} = \frac{\beta}{V}\langle q^2\rangle$ where $\langle . \rangle$ refers to $\begin{cases} \det^{N_f}(D_m^\text{ov}) \\ \det^{N_f/2}(D_m^\text{st}) \end{cases}$ $\quad q = \begin{cases} \text{ind} = -\frac{1}{2}\text{tr}(\gamma_5 D_\text{ov}) \\ \frac{1}{2\pi}\int F_{12}\, d^2x \end{cases}$

nsmear=0 ("thin link")



nsmear=1 ("thick link")

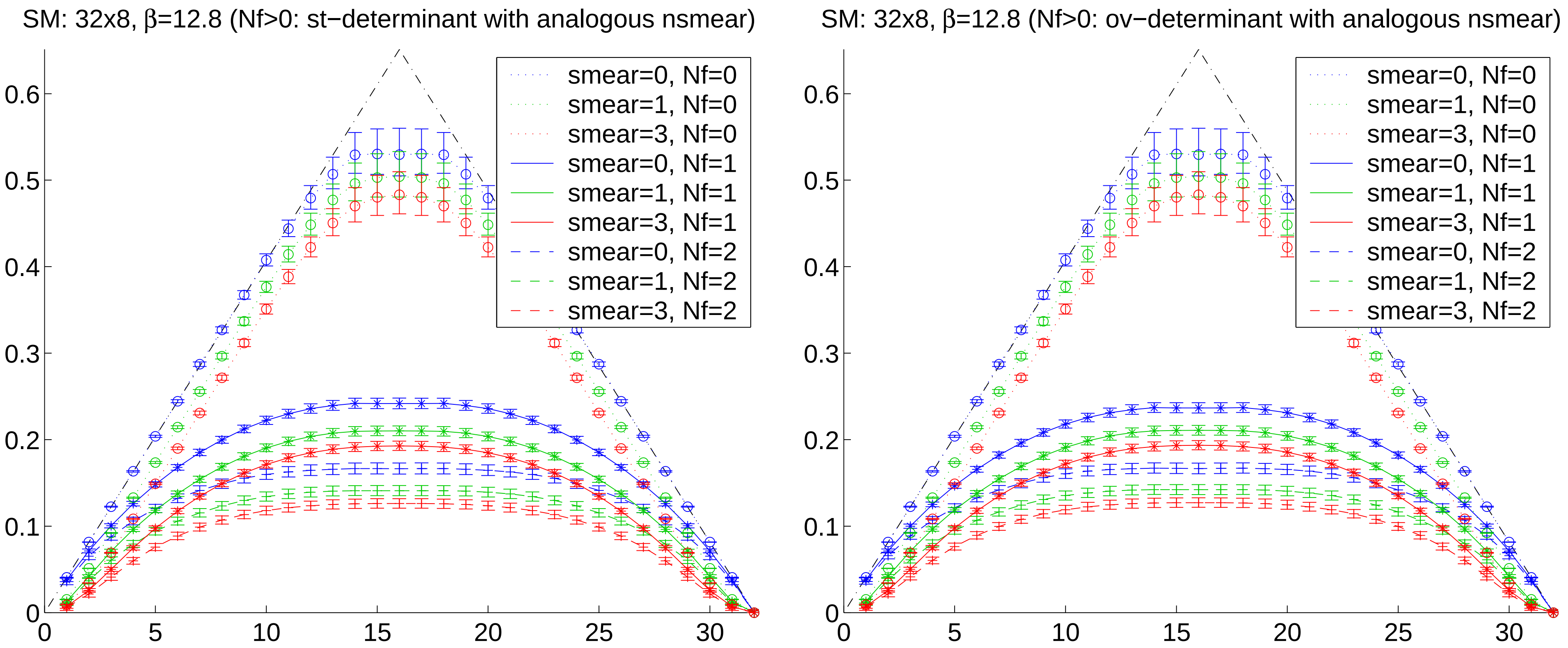


overlap: small $O(a^2)$ effects for all m, fairly insensitive to filtering

staggered: huge $O(a^2)$ effects for $m \to 0$, sizably reduced by filtering

Sea sector (3): heavy quark force in 2D

SM: screening effect on HQ potential by determinant rather pronounced



SD, C.Hoelbling, PRD 71, 054501 (2005) [hep-lat/0411022]

$\longrightarrow$ in 2D determinant rooting issue exists for $N_f = 1$, but not for $N_f = 2$

$\Longrightarrow$ numerical evidence for universal CL in HQ-force, both for $N_f = 1$ and $N_f = 2$

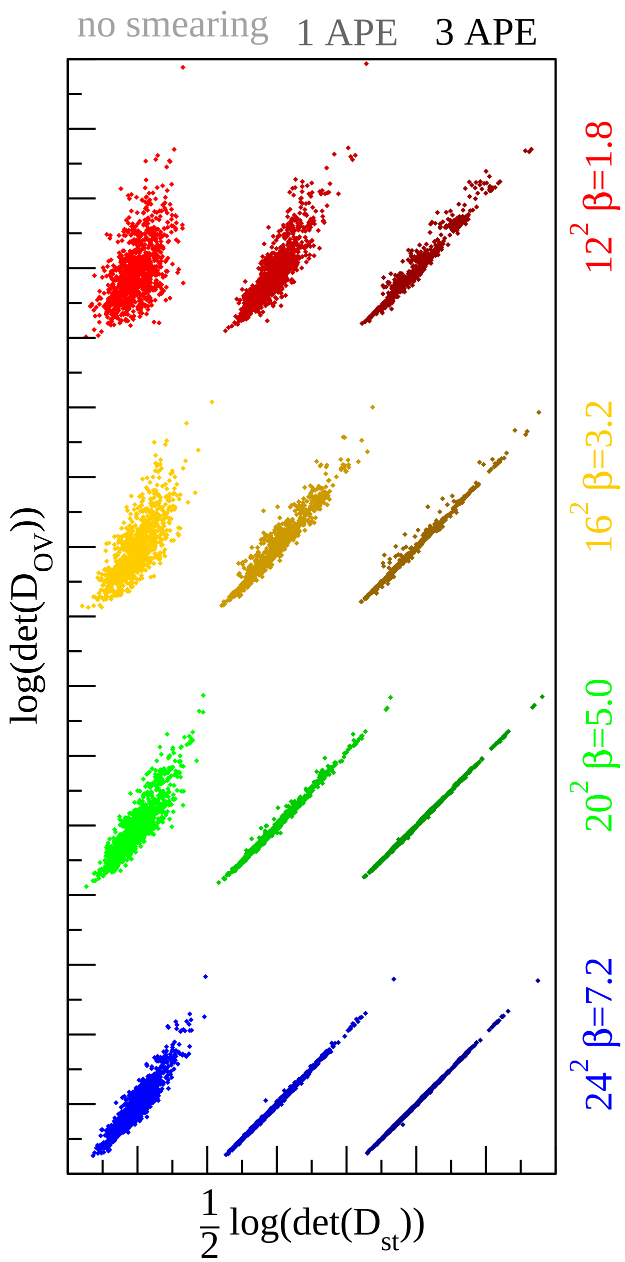
Sea sector (4): correlation of logarithmic determinants

key question is whether rooted staggered and standard overlap determinant generate ensembles which differ by cut-off effects only, i.e. whether (for fixed $m>0$)

$$\frac{\det(D_{\rm ov}(U))}{\det(D_{\rm ov}(U'))} = \sqrt[2^{d/2}]{\frac{\det(D_{\rm st}(U))}{\det(D_{\rm st}(U'))}}\,\Big(1+O(a^2)\Big)$$

SD, C.Hoelbling, PRD 71, 054501 (2005) [hep-lat/0411022]

$\Longrightarrow$ numerical evidence favors the idea of a "universal" ensemble being generated for $a\to 0$



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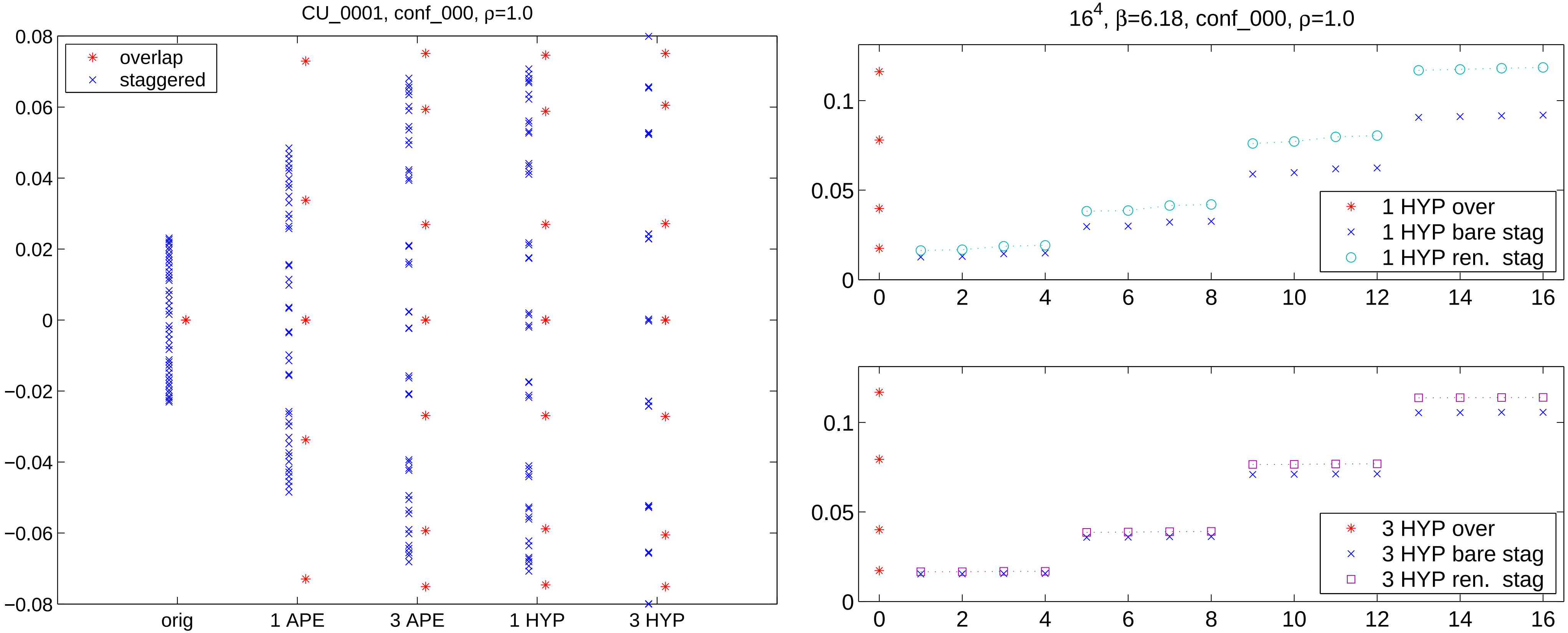
Full theory (1): rationale for hybrid action QCD

"hybrid action" = "mixed action" [rooted staggered sea and overlap/domain-wall valence]

idea: consider $\mathrm{spec}(D_{\mathrm{st}}, D_{\mathrm{ov}})$ on *same* configuration, both dynamical (l) and quenched (r)



⟶ correspondence of low-energy part of $\mathrm{spec}(D_{\mathrm{st}})$ and $\mathrm{spec}(D_{\mathrm{ov}})$ at $a \simeq 0.1\,\mathrm{fm}$ feasible

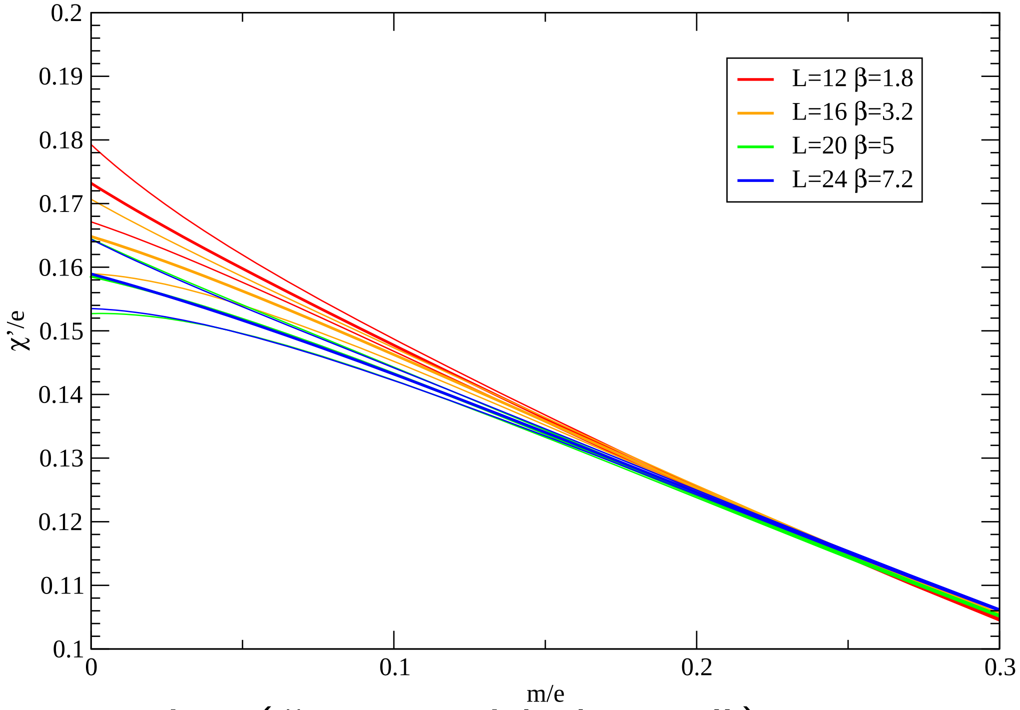
⟹ quantitative agreement after rescaling with $Z_S^{\mathrm{ov}}/Z_S^{\mathrm{st}}$ [*not* conf-specific] yields strong argument for "mixed action" studies

SD, C.Hoelbling, U.Wenger, PRD 70, 094502 (2004) [hep-lat/0406027]

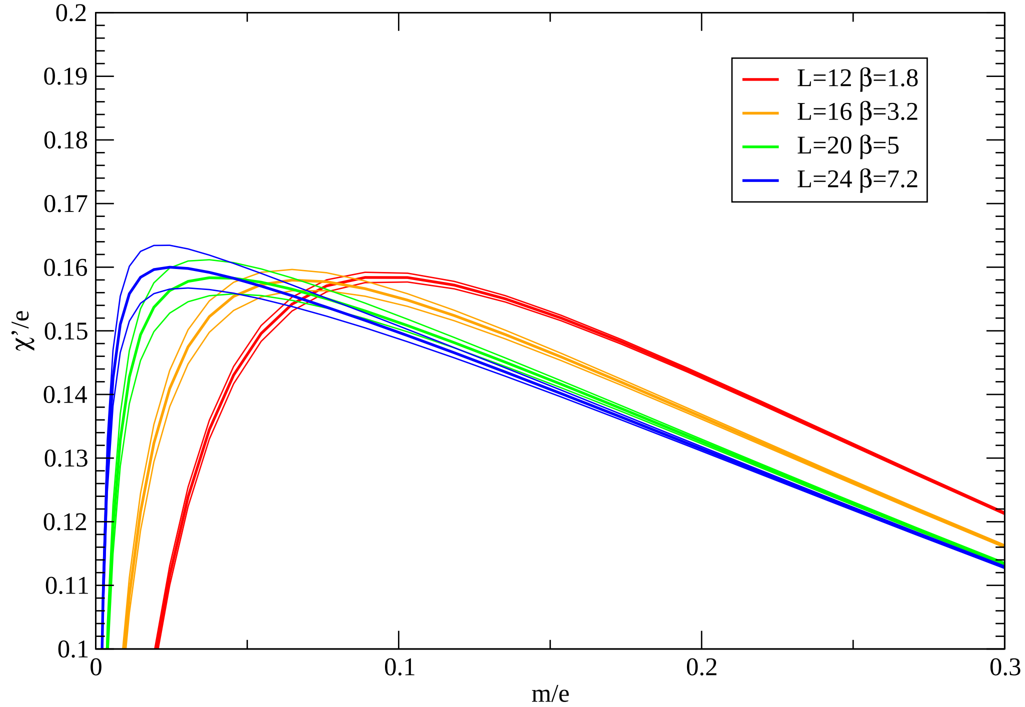
Full theory (2): 1-flavor condensate [in 2D]

SM: condensate in 1-flavor theory reflects anomaly: $\lim_{m\to 0}\langle\bar\psi\psi\rangle/e = e^\gamma/(2\pi^{3/2})$

overlap (nsmear=1)



staggered (nsmear=1)



- overlap ("universal behavior"):

$$\lim_{a\to 0}\lim_{m\to 0}\frac{\chi'^{\,\mathrm{ov}}_{\mathrm{sca}}(m/e,a^2)}{e} = \frac{e^\gamma}{2\pi^{3/2}}$$

$$\lim_{m\to 0}\lim_{a\to 0}\frac{\chi'^{\,\mathrm{ov}}_{\mathrm{sca}}(m/e,a^2)}{e} = \frac{e^\gamma}{2\pi^{3/2}}$$

- staggered ("non-commutativity"):

$$\lim_{a\to 0}\lim_{m\to 0}\frac{\chi'^{\,\mathrm{st}}_{\mathrm{sca}}(m/e,a^2)}{e} = 0\ (!!!)$$

$$\lim_{m\to 0}\lim_{a\to 0}\frac{\chi'^{\,\mathrm{st}}_{\mathrm{sca}}(m/e,a^2)}{e} = \frac{e^\gamma}{2\pi^{3/2}}$$

$\Longrightarrow$ staggered cut-off effects get progressively worse towards chiral limit (cheapness ?)

$\Longrightarrow$ similar non-commutativity with (partial) quenching SD, PoS LAT2005, 144; C. Bernard, PRD 71, 094020

Full theory (3): legitimacy of GEP

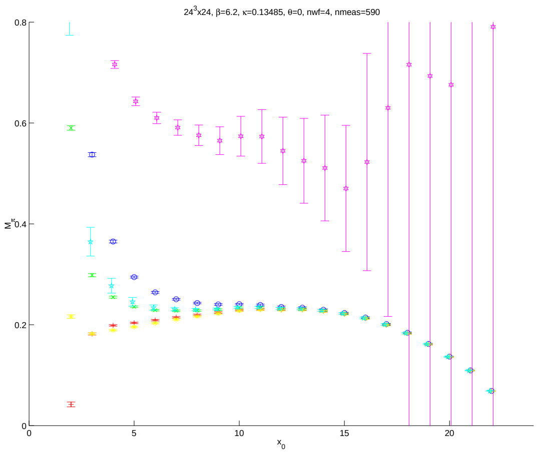
GEP: given A,B find generalized eigenvalues λ_i and eigenvectors v_i such that $Av_i = \lambda_i B v_i$

GEP: standard technique for noise reduction and excited states suppression/enhancement, if cross-correlator with several interpolating fields available [Michael, Lüscher, Wolff]

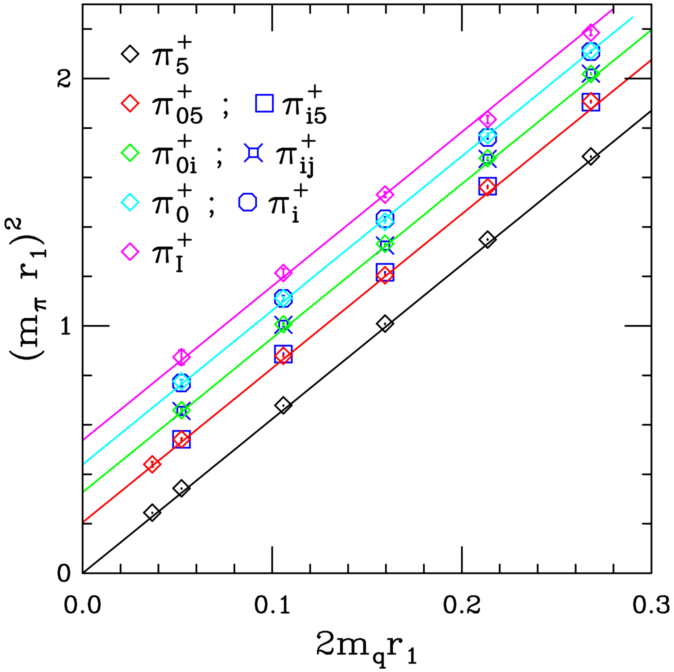
- want to measure HQ potential $V(r)$ on dynamical backgrounds
- for each r construct N different interpolating fields and use them at source and sink, i.e measure $N \times N$ correlator $C_{ij}(t)$ with $i, j = 1 ... N$
- solve GEP with $A = C(t_{\mathrm{maj}}), B = C(t_{\mathrm{min}})$ for two time-slices $t_{\mathrm{min}} < t_{\mathrm{maj}}$ to achieve [in N-state approximation] $\mathrm{const} = 0$ in eigenvalue formula

$$\lambda_i = \exp(-E_i(t_{\mathrm{maj}} - t_{\mathrm{min}})) \left[1 + \mathrm{const}\, e^{-\min\{E_i - E_{i-1}, E_{i+1} - E_i\}(t_{\mathrm{maj}} - t_{\mathrm{min}})}\right]$$

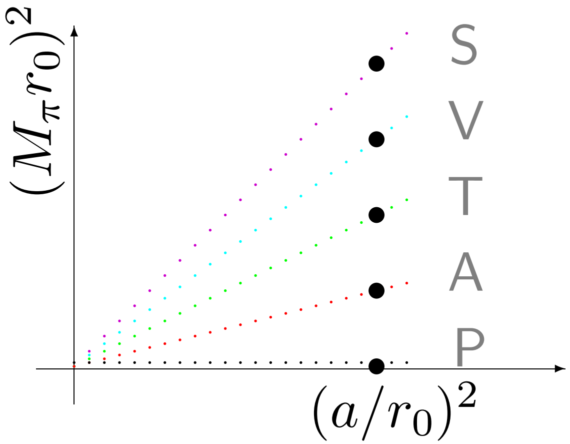
- axioms [bounded $\mathbb{T} = \exp(-a\mathbb{H})$] satisfied:
  - pure Wilson: $\surd$
  - improved Wilson: ?
  - Ginsparg-Wilson: ??
  - rooted staggered: ???



Full theory (4): cuts in staggered amplitudes



taste splitting makes most $\bar{d}(\gamma_5 \otimes T)u$ combinations become non-Goldstone bosons:



W.J.Lee, S.R.Sharpe, PRD 60, 114503 (1999) [hep-lat/9905023]

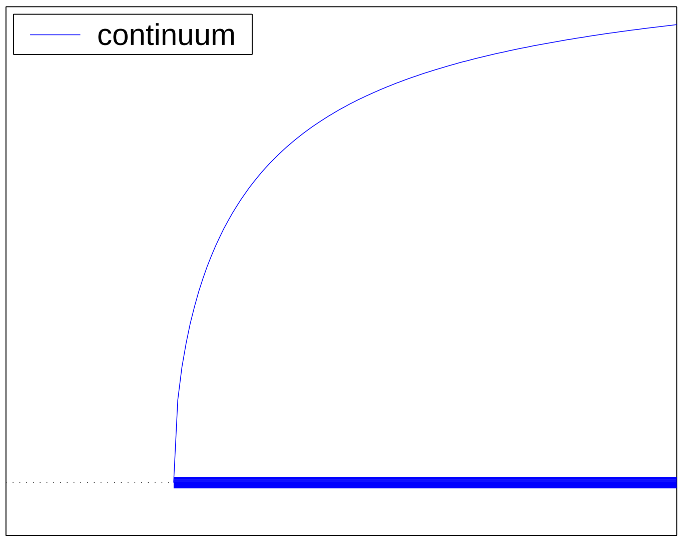
C.Aubin, C.Bernard, PRD 68, 034014 (2003) & PRD 68, 074011 (2003)

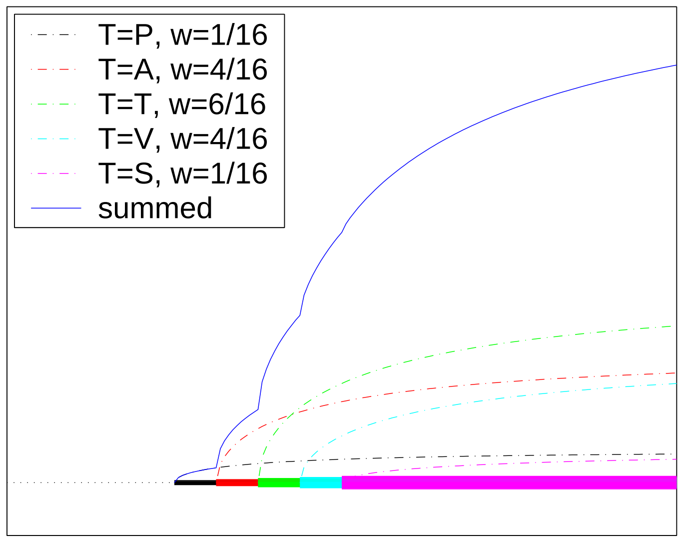
amplitude:
cut gets replaced by 5 cuts

discontinuity:
$\det^{N_f/4}$ brings $w_{P,A,T,V,S}$

vert: $\mathrm{cont} - \mathrm{latt} = O(1)$
hori: $\mathrm{cont} - \mathrm{latt} = O(a^2)$

sufficient for Lellouch Lüscher ???





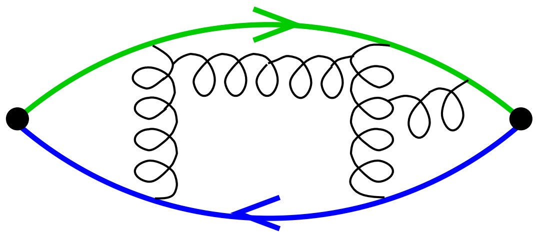
Holy grail: local candidate/ersatz action

- With any undoubled Dirac operator, e.g. $D=D_\mathrm{W}$, one has

$$Z_{N_f} = \int D[U,\bar\psi,\psi]\; e^{-S_G[U]-\sum\int\bar\psi D\psi} = \int D[U]\;\det{}^{N_f}(D)\; e^{-S_G[U]}$$

- With a 4-flavor operator like D_st, one may formally define

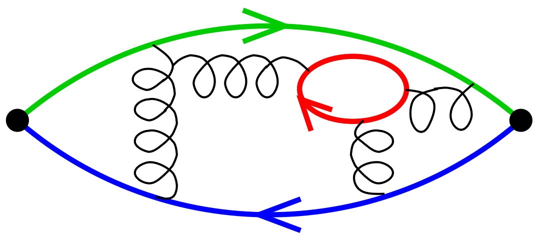
$$Z_{N_f} = \int D[U]\;\det{}^{N_f/4}(D_\mathrm{st})\; e^{-S_G[U]}$$



(A) Quenched QCD: quark loops neglected

but it is not clear whether there is a 1-taste D_ca such that

$$\det{}^{1/4}(D_\mathrm{st}) = \int D[\bar\psi,\psi]\; e^{-\int\bar\psi D_\mathrm{ca}\psi}$$



(B) Full QCD

To guarantee locality/causality of the theory (thus to discuss renormalizability and universality) a "candidate" operator D_ca should exist with K.Jansen, NPPS 129, 3 (2004) [hep-lat/0311039]

(1) $\det(D_\mathrm{ca}) \xrightarrow{a\downarrow 0} \mathrm{const}\cdot\det{}^{1/4}(D_\mathrm{st})$

(2) $||D_\mathrm{ca}(x,y)|| < C\, e^{-\nu|x-y|/a}$ with C,ν independent of U

Summary

- numerical evidence in favor of staggered taste projection in pure valence sector
- numerical evidence in favor of staggered rooted determinant in pure sea sector
- one counter-example in full theory [$d\!=\!2$, $G\!=\!U(1)$, $m\!=\!0$], "practical" issues

It is tempting to "summarize" everything in two (wild) conjectures:

1) The combination $P_{\mathrm{taste}}\{\mathrm{hadroncorrelator}(D_{\mathrm{st},m})\}$ and $\det^{N_f/2^{d/2}}(D_{\mathrm{st},m})$ yields the **correct continuum limit** for $d=2,4$, arbitrary gauge group G and $m>0$.
2) The combination $P_{\mathrm{taste}}\{\mathrm{hadroncorrelator}(D_{\mathrm{st},m})\}$ and $\det^{N_f/2^{d/2}}(D_{\mathrm{st},m})$ is in the **wrong universality class** for $d=2,4$, arbitrary gauge group G and $m=0$.

Lenin: "Dowjerjaj, no prowjerjaj" [trust in the first place, but then go on and check]