Deep inelastic scattering and the OPE in lattice QCD

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[WD & CJD Lin PRD 73, 014501 (2006)]
DIS structure of hadrons

- Deep-inelastic scattering process critical to development of QCD

- Hadron structure parameterised through structure functions

\[ x = - \frac{q^2}{2p \cdot q} \]

Momentum fraction carried by parton
Deep inelastic scattering

• Cross-section:

\[ \frac{d\sigma}{d\Omega dE'} = \frac{e^4}{16\pi^2 Q^4} \ell_{\mu\nu} W^{\mu\nu}(p, q) \]

• Lepton tensor \( \ell_{\mu\nu} \) well known

• Hadron tensor related to forward Compton scattering by optical theorem

\[ W^{\mu\nu} = \frac{1}{2\pi} \text{Im}[T^{\mu\nu}] \]

• Contains information on target structure
DIS on spin-$\frac{1}{2}$ targets

\[ W^{\mu\nu} = -F_1 g^{\mu\nu} + F_2 \frac{p^\mu p^\nu}{\nu} + \frac{i}{\nu} g_1 \epsilon^{\mu\nu\lambda\sigma} q_\lambda s_\sigma + \frac{i}{\nu^2} g_2 \epsilon^{\mu\nu\lambda\sigma} q_\lambda (p \cdot q s_\sigma - s \cdot q p_\sigma) \]

- Four structure functions: $F_{1,2}, g_{1,2}$
- Depend on Bjorken $x$ and $Q^2$
- $g_{1,2}$ require polarised beam and target
- Terms $\sim q^\mu$ dropped since $q^\mu \ell_{\mu\nu} = q^\nu \ell_{\mu\nu} = 0$
Parton distributions

- Factorisation theorem:

\[
F(x, Q^2) = \sum_i \int dz \ C_i \left( \frac{x}{z}, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) f_i(z, \alpha_s(\mu^2))
\]

- Parton distributions: \( u(x), \Delta \bar{s}(x), \delta d(x), g(x) \)

- Universal: DIS, Drell-Yan, W\(^{+/-}\) production

- Can be fitted from experimental data (MRST, GRV, CTEQ,...)

- Lattice: insight, information on unmeasured PDFs
Operator product expansion

• Wilsonian (light-cone) operator product expansion:

• Compton tensor expanded in matrix elements of local operators

• Classify by twist = dimension - spin

• Near Bjorken limit, twist-2 operators dominate

• Higher twists suppressed by powers of $Q^2$
Hadron structure in Lattice QCD

- Lattice techniques can investigate light-cone hadron structure from QCD

- Two difficulties:
  - Euclidean space
  - Reduced symmetry: $O(4) \rightarrow H(4)$

- Focused on calculating matrix elements of twist-2 operators \[ \langle x^n \rangle_q = \int_{-1}^{1} dx \, x^n q(x) \] ⇒ Mellin moments of PDFs
Local matrix element

- Matrix elements of twist-two operators

\[ \langle p | \bar{q} \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_n} q | p \rangle = \langle x^n \rangle_q p_{\mu_1} \cdots p_{\mu_n} \]

- Lattice calculations via ratio of correlators

\[ \langle x^n \rangle_q \sim \]

\[ x_n \]
Local matrix element

- Matrix elements of twist-two operators

\[ \langle p | \bar{q} \gamma_{\{\mu_1} D_{\mu_2} \cdots D_{\mu_n}\} q | p \rangle = \langle x^n \rangle_q \ p_{\mu_1} \cdots p_{\mu_n} \]

- Lattice calculations via ratio of correlators

- Limited to low moments by power-divergent operator mixing
  - Choose “safe irreps” of H(4) only for \( n \leq 4 \)
  - Renormalisation becomes extremely complex for large \( n \) (perturbative and non-perturbative)
Summary of lattice results

- QCDSF, LHP and RBCK,... collaborations
- Matrix elements in proton, pion, rho
- Lowest three moments of $q(x)$, $\Delta q(x)$, $\delta q(x)$
- Also calculate moments of GPDs
- Chiral and infinite volume extrapolations
- Only isovector moments accessible - isosinglet MEs have disconnected contractions: all-to-all propagators or external fields [hep-lat/0410011]
Example: axial charge

- \( g_A = \langle x^0 \rangle \Delta u - \Delta d \) simplest twist-two operator
Moments ⇒ PDFs

• Well defined problem: inverse Mellin transform
  • Rigourously requires all integer moments!

• How useful are just 3 moments?
  • Fit parametric form for PDF
  • But: standard PDF parameterisations have 6 or more parameters

• A different approach is needed
OPE in Euclidean space

- OPE of Compton tensor can be studied directly in Euclidean space
  1. Calculate current-current commutator
  2. Extrapolate to continuum - restore $O(4)$
  3. Match to Euclidean OPE to extract matrix elements of local operators
- Determines the same moments as in Minkowski space.
- Analytic continuation is in Wilson coefficients
- Still a very difficult calculation [Schierholz ‘99]
Fictitious heavy quarks

- Consider Compton tensor for currents that couple light quarks to heavy fictitious quark $\Psi$
- No disconnected contractions $\Rightarrow$ practical
- Heavy quark mass acts like photon virtuality to suppress higher twists
- Heavy quark integrated out after OPE
  - Determine same PDF moments
  - Effects in perturbative Wilson coefficients
Lattice correlators

- Quark contractions in Compton correlator

- Greatly simplified with heavy quark: no disconnected contributions in isovector case
Heavy quark Compton tensor

- Leading twist contributions

- Higher twist contributions

- Significantly reduced by heavy quark
Heavy quark DIS

- Compton tensor with heavy quark

\[ T^{\mu\nu}(p, q) = \sum_S \int d^4 x \ e^{iq \cdot x} \langle p, S|T \left[ J^{\mu}_{\Psi,\psi}(x) J^{\nu}_{\Psi,\psi}(0) \right]|p, S\rangle \]

- Heavy-light vector current \( J^{\mu}_{\Psi,\psi} = \overline{\psi} \Gamma^\mu \Psi + \overline{\Psi} \Gamma^\mu \psi \)

- Operator product expansion: heavy propagator

\[ \overline{\psi} \frac{-i (i\slashed{D} + q) + m_\Psi}{(i\slashed{D} + q)^2 + m^2_\Psi} \psi = -\overline{\psi} \frac{-i (i\slashed{D} + q) + m_\Psi}{Q^2 + \slashed{D}^2 - m^2_\Psi} \sum_{n=0}^{\infty} \left( \frac{-2i q \cdot \slashed{D}}{Q^2 + \slashed{D}^2 - m^2_\Psi} \right)^n \psi \]

- Denominator ⇒ \( \tilde{Q}^2 = Q^2 - M^2_\Psi + \alpha M_\Psi + \beta_n \)

- Higher twists?

Heavy-light meson mass

Binding energy \( \sim \Lambda \)

Higher twists \( \sim \Lambda^2 \)
Higher twists in OPE

• Derivative squared generates towers of higher twist operators

\[ \sum_{n=0}^{\infty} \overline{\psi} \left( \frac{-2i \cdot q \cdot \vec{D} + \vec{D}^2}{Q^2 - m_\psi^2} \right)^n \psi \rightarrow \ldots q_{\mu_1} \ldots q_{\mu_3} \overline{\psi} \vec{D}^{\mu_1} \vec{D}^4 \vec{D}^{\mu_2} \vec{D}^{\mu_3} \vec{D}^2 \psi \ldots \]

• Corrections should be \( \mathcal{O} \left[ \left( \frac{\Lambda_{\text{QCD}}^2}{Q^2} \right)^j \right] \)

• Depend on \( n \) since non-Abelian

• Can be include in lattice analysis

• Possible to use a free fermion field: no HT
Heavy quark DIS

- Usual twist-two operators

\[ \mathcal{O}^{\mu_1 \ldots \mu_n}_{\psi} = \bar{\psi} \gamma^{\{\mu_1} (i \vec{D}^{\mu_2}) \cdots (i \vec{D}^{\mu_n})^{\mu_1 \ldots \mu_n} \psi - \text{traces} \]

- Twist-3 scalar operators also contribute: \( e(x) \)

\[ \hat{\mathcal{O}}^{\mu_1 \ldots \mu_n}_{\psi} = \bar{\psi} (i \vec{D}^{\mu_1}) \cdots (i \vec{D}^{\mu_n}) \psi - \text{traces} \]

- Matrix elements

\[
\sum_S \langle p, S | \mathcal{O}^{\mu_1 \ldots \mu_n}_{\psi} | p, S \rangle = A^n_{\psi}(\mu^2) [p^{\mu_1} \ldots p^{\mu_n} - \text{traces}]
\]

\[
\sum_S \langle p, S | \hat{\mathcal{O}}^{\mu_1 \ldots \mu_n}_{\psi} | p, S \rangle = i M \hat{A}^n_{\psi}(\mu^2) [p^{\mu_1} \ldots p^{\mu_n} - \text{traces}]
\]
Heavy quark DIS

- Leads to

\[ T_{\Psi,\psi}^{\{\mu\nu\}} = i \sum_{n=2,\text{even}}^{\infty} A_{\Psi}^n (\mu) \zeta^n \left\{ \delta^{\mu\nu} \left[ C_n \frac{\widetilde{Q}^2}{q^2} n C_n^{(1)}(\eta) - 2 \eta C_n^{(2)}(\eta) \right] + C'_n C_n^{(1)}(\eta) \right\} + \frac{p^\mu p^\nu \widetilde{Q}^2}{(p \cdot q)^2} \frac{8 \eta^2 C_{n-2}^{(3)}(\eta)}{n(n-1)} \]

\[ + \frac{4 \eta \delta^{\mu\nu}}{p \cdot q} \left[ C_n \frac{\widetilde{Q}^2}{q^2} (n-1)^2 \eta^2 - 4 \eta \frac{C_{n-2}^{(3)}(\eta)}{n(n-1)} \right] + \frac{q^\mu q^\nu}{q^2} \left[ C_n \frac{\widetilde{Q}^2}{q^2} n(n-2) C_n^{(1)}(\eta) - 2 \eta (2n-3) C_n^{(2)}(\eta) \right] + \frac{8 \eta^2 C_{n-2}^{(3)}(\eta)}{n(n-1)} \]

\[ - 2i \frac{M(m_{\Psi} - m)}{\widetilde{Q}^2} \delta^{\mu\nu} \sum_{n=0,\text{even}}^{\infty} \hat{C}_n \hat{A}_{\psi}^n (\mu) \zeta^n C_n^{(1)}(\eta) \]

- where

\[ \zeta = \frac{\sqrt{p^2 q^2}}{\tilde{Q}^2}, \quad \eta = \frac{p \cdot q}{\sqrt{p^2 q^2}} \]

- Gegenbauer polynomials from target mass corrections: powers of \([p^2 / q^2]^j\)

Wilson coefficients

Gegenbauer polynomials

PDF moments

Not small
Heavy quark DIS

... or more comprehensibly (target rest frame)

\[ T^{\{34\}}_{\Psi, \psi}(p, q) = \sum_{n=2,\text{even}}^{\infty} A^n_{\Psi}(\mu) f(n) \]

where \( f(n) \) is completely known:

\[ f(n) = -\sqrt{q_0^2 - Q^2}\zeta^n \left\{ \frac{2}{q_0} \left[ C_n \frac{\tilde{Q}^2}{Q^2} \frac{(n-1)\eta C^{(2)}_{n-1}(\eta) - 4\eta^2 C^{(3)}_{n-2}(\eta)}{n(n-1)} + \ldots \right] \right\} \]

- Kinematic variables & Wilson coefficients

- Parameters \( \alpha, \beta \)

- Fits to calculations of LHS \( \Rightarrow A^n_{\Psi}(\mu), \alpha, \beta \)
Lattice details

- Scale hierarchy: $\Lambda_{QCD} \ll |Q|$, $m_\Psi \ll a^{-1}$
  - Need fine lattice spacings: $a^{-1} \sim 5$ GeV
- Heavy quark quenched: very cheap
- Use variety of masses
- Different $q^\mu$ also easy
- Correlator analysis is straightforward
- Lattice renormalisation and matching is simple
\[ G_{(4)}^{\mu\nu}(p, q, t, \tau; \Gamma) = \sum_{x, z} \sum_{y} \sum_{N, N'} \sum_{s, s'} e^{i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{q} \cdot \mathbf{y}} \Gamma_{\beta\alpha} \langle 0|\chi_\alpha(\mathbf{x}, t)|\bar{\Psi}_\psi(y + z, \tau + \frac{\tau}{2})J_\mu^{\Psi,\psi}(z, \frac{\tau}{2})\bar{\chi}_\beta(0, 0)|0\rangle \]

\[ = \sum_{x, z} \sum_{y} \sum_{N, N'} \sum_{s, s'} e^{i(p - p_N) \cdot x} e^{i(p_N - p_{N'}) \cdot z} e^{i\mathbf{q} \cdot \mathbf{y}} e^{-(E_N + E_{N'}) \frac{\tau}{2}} \Gamma_{\beta\alpha} \]

\[ \times \langle 0|\chi_\alpha(0)|E_N, p_N, s\rangle \langle E_N, p_N, s|\bar{J}_\mu^{\Psi,\psi}(y, \tau)\bar{J}_\nu^{\Psi,\psi}(0)|E_{N'}, p_{N'}, s'\rangle \langle E_{N'}, p_{N'}, s'|\bar{\chi}_\beta(0)|0\rangle \]

\[ \xrightarrow{t \to \infty} e^{-E_0 t} \sum_{y} \sum_{s, s'} e^{i\mathbf{q} \cdot \mathbf{y}} \sum_{N, N'} \sum_{x, z} \sum_{p, s'} \Gamma_{\beta\alpha} \langle 0|\chi_\alpha(0)|E_0, p, s\rangle \langle E_0, p, s|\bar{J}_\mu^{\Psi,\psi}(y, \tau)\bar{J}_\nu^{\Psi,\psi}(0)|E_0, p, s'|\bar{\chi}_\beta(0)|0\rangle \]

\[ G_{(2)}(p, t; \Gamma) = \sum_{x} e^{i\mathbf{p} \cdot \mathbf{x}} \Gamma_{\beta\alpha} \langle 0|\chi_\alpha(0)\bar{\chi}_\beta(\mathbf{x}, t)|0\rangle \]

\[ T_{\Psi,\psi}^{\{\mu\nu\}}(p, q) = 4M a \sum_{\tau} e^{iq_4 \tau} \left[ \lim_{t \to \infty} \frac{G_{(4)}^{\{\mu\nu\}}(p, q, t, \tau; \Gamma_4)}{G_{(2)}(p, t; \Gamma_4)} \right] \]
Moment extraction

- Want to extract 6-8 moments
- Two scenarios for $n$ dependence

$M_\Psi = 3.5 \text{ GeV}, \; Q^2 = 1.5 \text{ GeV}^2, \; q_0 = 2.8 \text{ GeV} \quad M_\Psi = 2.1 \text{ GeV}, \; Q^2 = -3.9 \text{ GeV}^2, \; q_0 = 2.0 \text{ GeV}$

$\alpha = 0.4, 1.2 \text{ GeV}, \quad \beta = 0, \quad M_N = 1.2 \text{ GeV}$
Other observables

- Can consider correlator of vector/axial-vector currents (neutrino scattering): odd moments
- Can use unphysical currents
  - Eg: scalar/vector correlator \( \Rightarrow \) moments of transversity distribution
- Moments of GPDs also accessible
- Moments of meson distribution amplitudes
Pion distribution amplitude

- Distribution amplitudes (light-cone WFs) important for QCDF/SCET in hard processes
- Lattice can determine moments of meson distribution amplitudes: e.g. $\phi_\pi(x)$

$$\langle \xi^n \rangle_\pi = \int_0^1 \xi^n \phi_\pi(\xi) d\xi$$

- Local ME method limited to one moment!
- OPE method can determine higher moments
- Not related to a physical process
Pion distribution amplitude

• Lattice calculations of the tensor

\[ S_{\Psi,\psi}^{\mu\nu}(p, q) = \int d^4 x \, e^{i q \cdot x} \langle \pi^+(p) | T[V_{\Psi,\psi}^\mu(x) A_{\Psi,\psi}^\nu(0)] | 0 \rangle \]

• After OPE determine same matrix elements as for distribution amplitude:

\[ \langle \pi^+(p) | \bar{\psi} \gamma^{\mu_1} \gamma_5 (i D)^{\mu_2} \ldots (i D)^{\mu_n} \psi | 0 \rangle = f_\pi \langle \xi^{n-1} \rangle_\pi \left[ p^{\mu_1} \ldots p^{\mu_n} - \text{traces} \right] \]

• Numerical work under construction

• Computationally similar to pion FF
OPE on the lattice

- Improve understanding of hadron structure from Euclidean lattice calculations
  - Moments of parton distributions
  - Moments of generalised parton distributions
  - Moments of distribution amplitudes
- Use of “OPE without OPE” made practical for isovector distributions by heavy quark
- Exploratory studies feasible today
Supplementary Slides
Gegenbauer polynomials

\[ C_n^{(\lambda)}(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k \Gamma(n - k + \lambda)}{k!(n - 2k)!\Gamma(\lambda)} (2z)^{n-2k} \]

\[ \frac{\partial}{\partial z} C_n^{(\lambda)}(z) = 2\lambda C_{n-1}^{(\lambda+1)}(z) \]

\[ C_n^{(\lambda)}(z) = \frac{2(\lambda + n + 1)z}{2\lambda + n} C_{n+1}^{(\lambda)}(z) - \frac{n + 2}{2\lambda + n} C_{n+2}^{(\lambda)}(z) \]
Target mass corrections

• Result from deviation of bound-state from light cone

• OPE basis constructed from operators of definite spin \( \Rightarrow \) trace subtractions

• Eg: \[ p^\mu_1 p^\mu_2 - \text{tr} = p^\mu_1 p^\mu_2 - p^2 g^{\mu_1 \mu_2} \]

\[ p^\mu_1 \ldots p^\mu_n - \text{tr} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{(-1)^j (n-j)!}{2^j n!} \right) M^{2j} g^{\{\mu_1 \mu_2 \ldots \mu_{2j-1} \mu_{2j} p^\mu_{2j+1} \ldots p^\mu_n\}} \]

• Contraction with \( q^\mu \)'s leads to Gegenbauer's