Staggered Chiral Perturbation Theory and the Fourth-Root Trick

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Outline

- What is \( SxPT \) and is it valid with \( \sqrt[4]{\text{Det}} \)?
- Replica trick & notation
- Four flavors
  - Degenerate case
  - Expanding around degenerate case
  - Assumptions on analyticity & consequences
- Fewer than four flavors
  - Decoupling in chiral theory and in lattice QCD
  - Assumptions needed to connect the above
- One-flavor & three-flavor paradoxes, and their resolution
- Consequences if schpt is valid:
  - Implications for health of rooted theory
  - Is rooted staggered a mixed theory?
- Conclusions, remarks, speculations
Chiral perturbation theory ($\chi$PT) provides a nice framework for thinking about the fourth root

Much simpler than lattice QCD itself

- Low energy constants (LECs) are taken as unknowns (“mod them out” from the corresponding QCD theory)
- Most info in $\chi$PT is in the order by order chiral expansion (perturbative)
- But gives nonperturbative info about QCD
- Lowest energy, longest distance sector: any problems from rooting (unitarity violations, nonlocality) ought to show up here
**SXPT**

- **Lee & Sharpe** found the LO chiral theory for a single unrooted staggered field including $a^2$ taste violations. *Nomenclature: 1 staggered field = 1 flavor (4 tastes if unrooted; 1 taste if rooted)*

- **Aubin & C.B.:**
  - Generalized Lee-Sharpe to many flavors
  - Proposed taking into account fourth root by locating sea-quark loops and multiplying each by $\frac{1}{4}$
  - Sea-quark loops found by quark-flow approach [Sharpe]
  - Replica trick is equivalent, but systematic and algebraic — better here
  - Staggered chiral perturbation theory (SXPT) is defined as this chiral theory for staggered quarks, including discretization errors and the above procedure for taking $\sqrt[4]{\text{Det}}$ into account

- **Question:** is SXPT the correct chiral theory?
Overview

- We know (trivially) how fourth root works when we have 4 degenerate flavors: \( \left( \sqrt[4]{\text{Det}} \right)^4 = \text{Det} \)
  - Get 1-flavor, unrooted theory
  - Local lattice action & known chiral theory [Lee & Sharpe]
- To get non-degenerate 4-flavor theory, expand around degenerate point
  - Need non-trivial assumptions about mass dependence (analyticity, absence of phase transition)
- To get theory with 3 flavors, decouple a quark (“charm”)
  - Need another assumption about how decoupling works
- I claim assumptions are “plausible.”
  - Plausibility is in eye of beholder!
  - Assumptions at least not obviously wrong
  - These assumptions are necessary for $S\chi PT$ to work
Replica trick

- Systematic & algebraic way to find sea-quark loops and multiply by $1/4$
- Introduced for partially quenched theory by Damgaard and Splittorff
- First used for $S\chi$PT by Aubin & CB, Lattice ’03
- Replicate the sea-quark flavors, replacing each field by $n_R$ identical copies ($n_R =$ positive integer)
- Calculate order by order in corresponding (unrooted) chiral theory
- Take $n_R \to 1/4$ at end
  - Dependence on $n_R$ is polynomial at any finite order in $S\chi$PT, so $n_R \to 1/4$ is well-defined
  - Treat LECs as free parameters for each $n_R$ — LECs are taken independent of $n_R$ in this procedure
Replica trick

- Difficult to give meaning to replica trick at QCD level:
  - Beyond weak-coupling perturbation theory, dependence on $n_R$ almost certainly non-polynomial
  - Analytic continuation from integers not unique
  - $\exists$ ideas by Shamir for defining a version of replica trick for QCD, but not used here
- In $\mathcal{S}x\mathcal{P}T$ replica trick also only meaningful order by order
  - Will assume no phase change as we move away from degenerate point, where phase of chiral theory is known
(n_F, n_T, n_R) notation

- \( (n_F, n_T, n_R)_{LQCD} \) is generating functional for lattice QCD theory with:
  - \( n_F \) flavors
  - \( n_T \) tastes
  - \( n_R \) replicas of each flavor

- \( (n_F, n_T, n_R)_\chi \) is corresponding generating functional for chiral theory

- Omit \( n_R \) if it is trivially equal to 1 (because replica trick not relevant)

- Sources for generating functionals to be discussed later
$$(n_F, n_T, n_R)$$ notation

Relevant theories:

- $$(1, 4)_{LQCD}$$ and $$(1, 4)_{\chi}$$
  - Single unrooted staggered field
  - $$(1, 4)_\chi$$ is S\chi PT of Lee & Sharpe.
  - No replica trick necessary

- $$(n_F, 4, n_R)_{LQCD}$$ and $$(n_F, 4, n_R)_\chi$$
  - $$n_F$$ staggered fields,
  - $$n_R$$ indicated explicitly $$\Rightarrow$$ integer only
  - $$(n_F, 4, n_R)_\chi$$ is S\chi PT of Aubin & CB for $$n_R \cdot n_F$$ sea-quark flavors (still no rooting)
Relevant theories (continued):

- $(n_F, \"1\")_{LQCD}$ and $(n_F, \"1\")_\chi
  - $n_F$ staggered fields with $\sqrt[4]{\text{Det}}$ taken
  - Quotes on “1” taste $\Rightarrow$ don’t assume fourth root works
  - $(n_F, \"1\")_\chi$ is by definition the chiral theory generated by $(n_F, \"1\")_{LQCD}$
  - Want to find $(n_F, \"1\")_\chi$ unambiguously

- $(n_F, 4, \frac{1}{4})_\chi$
  - Chiral theory $(n_F, 4, n_R)_\chi$ with the replica trick $n_R \rightarrow 1/4$
  - Defines $\chi$PT for rooted theory
  - Does $(n_F, \"1\")_\chi = (n_F, 4, \frac{1}{4})_\chi$?
  - Avoid “$(n_F, 4, \frac{1}{4})_{LQCD}$” because replica trick ambiguous at QCD level
Remarks

• Chiral theories \((n_F, 4, n_R)_\chi\) are key objects

• \((n_F, 4, n_R)_{LQCD}\), in particular \((4, 4, n_R)_{LQCD}\), introduced for convenience
  
  • Used formally; help keep track of \(n_R\) factors relating valence- to sea-quark matrix elements
  
  • Almost certainly can be eliminated at the expense of less intuitive argument at the chiral level
  
  • Unnecessary that the standard, broken realization of chiral symmetry assumed in \((4, 4, n_R)_\chi\) actually occurs in \((4, 4, n_R)_{LQCD}\)
  
  • Unpleasant fact that asymptotic freedom & spontaneous chiral symmetry breaking (?) is lost for \(n_R > 1\) in \((4, 4, n_R)_{LQCD}\) is irrelevant

• Worried? — just increase \(n_c\) (number of colors) [Heller]
\( n_F = 4 \) basics

- Want to show:
  \[(4, "1")_\chi \equiv (4, 4, \frac{1}{4})_\chi \]

- Use "\( \equiv \)" to compare two chiral theories: same functions of the LECs
- True equality only if adjust LECs to be the same

- Start with degenerate 4-flavor theory: \( \mathcal{M} = \bar{m} I \), where \( I \) is identity matrix in flavor space:
  \[
  \left. (4, "1")_{LQCD} \right|_{\mathcal{M}=\bar{m}I} = \left. (1, 4)_{LQCD} \right|_{\bar{m}} \\
  \left. (4, "1")_\chi \right|_{\mathcal{M}=\bar{m}I} \equiv \left. (1, 4)_\chi \right|_{\bar{m}} \equiv \left. (4, 4, \frac{1}{4})_\chi \right|_{\mathcal{M}=\bar{m}I}
  \]

- Last equivalence manifest order by order in \( SxPT \)
  - Taking \( 4n_R \) degenerate flavors and then putting \( n_R = 1/4 \)
    \[\iff\] one-flavor theory
\( n_F = 4: \) expansion around degenerate point

- To move away from degenerate limit, add taste-singlet scalar sources for sea-quark fields:

\[
\mathcal{L}_{(4, \text{"1"})} = \cdots + \bar{m} \bar{\Psi}_i(x) \Psi_i(x) + \bar{\Psi}_i(x) s^{ij}(x) \Psi_j(x) + \ldots
\]

\[
\mathcal{L}_{(4,4,n_R)} = \cdots + \bar{m} \bar{\Psi}_i^r(x) \Psi_i^r(x) + \bar{\Psi}_i^r(x) s^{ij}(x) \Psi_j^r(x) + \ldots
\]

[sum over \( i, j \) (flavor indices) and \( r \) (replica index)]

- When \( s \neq 0 \), we don’t yet know that \( (4, 4, \frac{1}{4})_\chi \) is right chiral theory

- Define \( V[s] \) as amount of mismatch:

\[
(4, \text{"1"}; s)_\chi \doteq (4, 4, \frac{1}{4}; s)_\chi + V[s]
\]

- \( V[s] = 0 \) when \( s = 0 \) or whenever flavor symmetry is exact
\( n_F = 4 \): expansion around degenerate point

- Example of possible term in \( V[s] \):

\[
V_1 = \bar{m}^2 \int d^4x \, d^4y \left( \frac{1}{\Box + M^2} \right)_{x,y} \left( \text{Tr} \left[ s(x)s(y) \right] - \frac{1}{4} \text{Tr} \left[ s(x) \right] \text{Tr} \left[ s(y) \right] \right)
\]

*with \( 1/M \) a distance scale that might not vanish when \( \alpha \to 0 \)*

- Claim:

\[
\prod_n \left. \frac{\partial}{\partial s_{injn}(x_n)} \right|_{s=0} (4, \text{"1"}; s) \chi = \prod_n \left. \frac{\partial}{\partial s_{injn}(x_n)} \right|_{s=0} (4, 4, \frac{1}{4}; s) \chi
\]

\[
\Rightarrow \prod_n \left. \left( \frac{\partial}{\partial s_{injn}(x_n)} V[s] \right) \right|_{s=0} = 0
\]

- Prove by relating sea Green’s functions to valence Green’s functions in partially quenched theory

- Then can keep \( s = 0 \), where equivalence is known
\( n_F = 4: \) partial quenching argument

- Add \( n_V \) staggered valence fields with sources \( \sigma^{\alpha\beta} \) to all LQCD theories

\[
cL = \cdots + \bar{m} \bar{q}_\alpha(x) q_\alpha(x) + \bar{q}_\alpha(x) \sigma^{\alpha\beta}(x) q_\beta(x) + \cdots
\]

- \( n_V \) ghost fields also added, but not coupled to \( \sigma^{\alpha\beta} \): cancel valence Det when \( \sigma = 0 \)

\[
(4, \text{"1"}; s=0, \sigma)_{LQCD} = (1, 4; s=0, \sigma)_{LQCD}
\]

\[
\Rightarrow (4, \text{"1"}; s=0, \sigma)_\chi = (1, 4; s=0, \sigma)_\chi \overset{*}{=} (4, 4, \frac{1}{4}; s=0, \sigma)_\chi
\]

- Last equivalence again manifest order by order in \( SxPT \)
  - Should be safe from any subtlety of type discussed by Golterman, Sharpe & Singleton
  - e.g. non-trivial saddle point for ghost mesons
\( n_F = 4 \): partial quenching argument

- Relate derivatives w.r.t. \( s \) to derivatives w.r.t. \( \sigma \)
- Derivatives w.r.t. \( s \) in rooted theory bring down factors of \( 1/4 \) from
  \[
  \sqrt[4]{\text{Det}(D + \tilde{m} + s)} = \exp \frac{1}{4} \text{tr} \ln(D + \tilde{m} + s)
  \]
- Different terms (≡ different contractions) associated with different powers of \( 1/4 \)
  - Power of \( 1/4 \) is just the number of quark loops implied by corresponding contractions
- Derivatives w.r.t. \( s \) in replicated theory produce corresponding powers of \( n_R \) from sea-quark counting
- But with arbitrary \( n_V \), can always adjust valence flavor indices on \( \sigma \) derivatives so only one contraction possible
\( n_F = 4: \text{ partial quenching argument} \)

- Examples \((i \neq j, \alpha \neq \beta, \text{no sums})\):

\[
\left. \frac{\partial}{\partial s^{ij}(x)} \frac{\partial}{\partial s^{ji}(y)} \right|_{s=0} (4, \text{“1”}; s, \sigma = 0)_{LQCD} = \frac{1}{4} \langle \text{tr} \left( G_j(x, y)G_i(y, x) \right) \rangle
\]

\[
= \frac{1}{4} \left( \frac{\partial}{\partial \sigma^{\alpha \beta}(x)} \frac{\partial}{\partial \sigma^{\beta \alpha}(y)} \right) (4, \text{“1”}; s = 0, \sigma)_{LQCD} \bigg|_{\sigma=0}
\]

\[
\left. \frac{\partial}{\partial s^{ii}(x)} \frac{\partial}{\partial s^{ii}(x)} \right|_{s=0} (4, \text{“1”}; s, \sigma = 0)_{LQCD} =
\]

\[
= \frac{1}{4} \langle \text{tr} \left( G_i(x, y)G_i(y, x) \right) \rangle + \left( \frac{1}{4} \right)^2 \langle \text{tr} \left( G_i(x, x) \right) \text{tr} \left( G_i(y, y) \right) \rangle
\]

\[
= \left[ \frac{1}{4} \left( \frac{\partial}{\partial \sigma^{\alpha \beta}(x)} \frac{\partial}{\partial \sigma^{\beta \alpha}(y)} \right) + \left( \frac{1}{4} \right)^2 \frac{\partial}{\partial \sigma^{\alpha \alpha}(x)} \frac{\partial}{\partial \sigma^{\beta \beta}(y)} \right] (4, \text{“1”}; s = 0, \sigma)_{LQCD} \bigg|_{\sigma=0}
\]

- For \((4, 4, n_R)\) theory, just replace \(1/4 \rightarrow n_R\)
\( n_F = 4: \) partial quenching argument

- Can therefore write:

\[
\prod_n \frac{\partial}{\partial s^{i_n j_n}(x_n)} \left( 4, \text{"1"}; s, \sigma = 0 \right)_{LQCD} \bigg|_{s=0} = \\
= \sum_C \left( \frac{1}{4} \right)^{L_C} \prod_n \frac{\partial}{\partial \sigma^{\alpha_n^C \beta_n^C}(x_n)} \left( 4, \text{"1"}; s = 0, \sigma \right)_{LQCD} \bigg|_{\sigma=0}
\]

\[
\prod_n \frac{\partial}{\partial s^{i_n j_n}(x_n)} \left( 4, 4, n_R; s, \sigma = 0 \right)_{LQCD} \bigg|_{s=0} = \\
= \sum_C (n_R)^{L_C} \prod_n \frac{\partial}{\partial \sigma^{\alpha_n^C \beta_n^C}(x_n)} \left( 4, 4, n_R; s = 0, \sigma \right)_{LQCD} \bigg|_{\sigma=0}
\]

- \( C \) labels a contraction with \( L_C \) valence quark loops
- Valence indices \( \alpha_n^C, \beta_n^C \) adjusted so only one contraction
- Same arrangements of valence flavor indices & powers \( L_C \) work in both cases
\( n_F = 4 \): partial quenching argument

- Pass to corresponding chiral theories:

\[
\prod_n \frac{\partial}{\partial s_{i_n j_n}(x_n)} (4, "1"; s, \sigma = 0) \chi \bigg|_{s=0} =
\]

\[
= \sum_C \left( \frac{1}{4} \right)^{L_C} \prod_n \frac{\partial}{\partial \sigma^{\alpha_n \beta_n}(x_n)} (4, "1"; s = 0, \sigma) \chi \bigg|_{\sigma=0}
\]

\[
\prod_n \frac{\partial}{\partial s_{i_n j_n}(x_n)} (4, 4, n_R; s, \sigma = 0) \chi \bigg|_{s=0} =
\]

\[
= \sum_C (n_R)^{L_C} \prod_n \frac{\partial}{\partial \sigma^{\alpha_n \beta_n}(x_n)} (4, 4, n_R; s = 0, \sigma) \chi \bigg|_{\sigma=0}
\]

- At any finite order in chiral perturbation theory both sides of last eqn are polynomial in \( n_R \). Can take \( n_R \to 1/4 \)
\( n_F = 4 \): partial quenching argument

- After \( n_R \rightarrow 1/4 \) in second eqn:

\[
\prod_n \left. \frac{\partial}{\partial s_i n_j(x_n)} \right|_{s=0} (4, "1"; s, \sigma = 0) \chi = \sum_C \left( \frac{1}{4} \right)^{L_C} \prod_n \left. \frac{\partial}{\partial \sigma \alpha_n^C \beta_n^C(x_n)} \right|_{\sigma=0} (4, "1"; s = 0, \sigma) \chi
\]

\[
\prod_n \left. \frac{\partial}{\partial s_i n_j(x_n)} \right|_{s=0} (4, 4, \frac{1}{4}; s, \sigma = 0) \chi = \sum_C \left( \frac{1}{4} \right)^{L_C} \prod_n \left. \frac{\partial}{\partial \sigma \alpha_n^C \beta_n^C(x_n)} \right|_{\sigma=0} (4, 4, \frac{1}{4}; s = 0, \sigma) \chi
\]

- Right sides equal since \((4, "1"; s = 0, \sigma) \chi \equiv (4, 4, \frac{1}{4}; s = 0, \sigma) \chi\)

- So left sides equal, which is what we wanted to show
\( n_F = 4: \text{ analyticity assumptions} \)

- So all derivatives of \( V[s] \) vanish at \( s = 0 \)
- If \( V[s] \) analytic in \( s \) — up to possible isolated singularities — it vanishes everywhere
- Strong assumption; is it obviously too strong?
  - “Don’t expect convergent expansions in QFT”
  - Factorial growth of large orders in perturbation theory: expansion at best asymptotic
  - But here every order is zero!
- How could analyticity go wrong?
  - Line of singularities, domain boundary
    - Ground state for \( (4, "1")_\chi \) changes discontinuously from state assumed by \( (4, 4, \frac{1}{4})_\chi \)
  - Inside the range of \( m \) & \( a \) studied by MILC, such a singularity would have probably been detected
  - No evidence outside MILC range, though
$n_F = 4$: analyticity assumptions

- How could analyticity go wrong? (continued)
  - Essential singularity at $s = 0$
    - Term like $\exp(-1/V_1^2)$ is logically possible
    - Best I can say right now is there’s no reason to expect it (no obvious IR problem; expanding around massive theory)
  - Speculations later

- NB: Not assuming that $(4, "1")_\chi$ and $(4, 4, \frac{1}{4})_\chi$ are separately analytic, only that difference is

- If $V[s]$ not analytic, then $S\chi PT$ is wrong
$n_F = 3$: decoupling

- Try to get to $n_F = 3$ by taking one mass ("charm") large
- Take $m_c$ large as possible w/o leaving region where $\chi_{PT}$ applies
  - Nominally, say $m_c \sim 2m_s^{\text{phys}}$
- Take other masses small for clean separation ($m_s \ll m_s^{\text{phys}}$)
- Integrate out $m_c$ from $(4, 4, \frac{1}{4})_{\chi}$
  - Should get $(3, 4, \frac{1}{4})_{\chi}$
  - Since perturbative, there is little doubt here
  - Explicit check is planned (CB & X. Du)
- So charm has decoupled from low energy physics when $m_c \sim 2m_s^{\text{phys}}$
- Assume it remains decoupled from low energy physics as $m_c$ increases to $\gg 1/a$
\( n_F = 3: \) decoupling

- When \( m_c \gg 1/a \), it is much larger than all eigenvalues of \( D \)
  - \( \sqrt[4]{\text{Det}(D + m_c)} \) independent of gauge field
  - charm decouples from \((4, "1")_{LQCD}\), leaving \((3, "1")_{LQCD}\)

\[
(3, "1")_{\chi} \doteq (3, 4, \frac{1}{4})_{\chi}
\]

- If true for small \( u, d, s \) masses, then analyticity assumption implies still true for physical ones

- Can repeat to argue \((2, "1")_{\chi} \doteq (2, 4, \frac{1}{4})_{\chi}\) and \((1, "1")_{\chi} \doteq (1, 4, \frac{1}{4})_{\chi}\)

- Decoupling assumption not only sufficient but also necessary for \( n_F = 3 \) \( \chi \)PT:
  - Any new physical effects entering for \( 2m_{s,\text{phys}} \lesssim m_c \lesssim 1/a \) automatically violate chiral theory
One-flavor paradox

- Theory with 1 flavor should have only heavy pseudoscalar, $\eta'$, no light pseudo-Goldstone bosons
- SxPT for 1 rooted-staggered flavor has 16 pseudoscalars ("pions"); only the taste-singlet is heavy
- Different weightings (factors of $1/4$) in rooted case compared to unrooted case, but otherwise similar — all pions contribute at $a \neq 0$
- For consistency, light pions must decouple from pure-glue correlation functions when $a \to 0$
- Work by CB, DeTar, Fu, Prelovsek; more details in DeTar’s talk tomorrow
One-flavor paradox

- Mock up the kind of pure-glue correlation function that can persist in continuum limit: add taste-singlet scalar source to rooted one-flavor theory:

\[ \mathcal{L}_{\text{source}} = s(z) \bar{\Psi}(z) \Psi(z) \]

- To show factors resulting from rooting, take the \( R^{th} \) power of the determinant; set \( R = \frac{1}{4} \) at end

\[
(1, "1")_{LQCD} = \frac{\int \mathcal{D}A \exp\{-S_G(A) + R \text{tr} (\ln(D + m + s))\}}{\int \mathcal{D}A \exp\{-S_G(A) + R \text{tr} (\ln(D + m))\}}
\]

- Look at connected part of

\[
G(x-y) = \left( \frac{\partial}{\partial s(x)} \frac{\partial}{\partial s(y)} (1, "1")_{LQCD} \right)_{s=0}
\]

[“connected” ⇒ subtract \( \langle \bar{\Psi} \Psi \rangle^2 \)]
One-flavor paradox

- Calculate $G(x-y)$ for large $|x - y|$ in LO $\chi$PT
- First rewrite in terms of valence Green’s functions

$$G(x-y) = R\left(\frac{\partial}{\partial \sigma^\alpha \beta(x)} \frac{\partial}{\partial \sigma^\beta \alpha(y)} (1, "1")_{LQCD}\right)_{\sigma=0}$$

$$+ R^2 \left(\frac{\partial}{\partial \sigma^\alpha \alpha(x)} \frac{\partial}{\partial \sigma^\beta \beta(y)} (1, "1")_{LQCD}\right)_{\sigma=0}$$
One-flavor paradox

- Calculate $G(x-y)$ for large $|x - y|$ in LO $S\chi$PT
- First rewrite in terms of valence Green’s functions

\[
G(x-y) = R \left( \frac{\partial}{\partial \sigma^{\alpha\beta}(x)} \frac{\partial}{\partial \sigma^{\beta\alpha}(y)} (1, "1")_{LQCD} \right)_{\sigma=0} \\
+ R^2 \left( \frac{\partial}{\partial \sigma^{\alpha\alpha}(x)} \frac{\partial}{\partial \sigma^{\beta\beta}(y)} (1, "1")_{LQCD} \right)_{\sigma=0}
\]

\[\text{R term} \quad \text{R}^2 \text{ term}\]
One-flavor paradox: diagrams

QCD valence contraction
(term proportional to $R$)
One-flavor paradox: diagrams

QCD valence contraction (term proportional to $R$)

chiral quark flow (note hairpins)
One-flavor paradox: diagrams

QCD valence contraction
(term proportional to $R^2$)
One-flavor paradox: diagrams

QCD valence contraction (term proportional to $R^2$)  

chiral quark flow (note hairpins)
One-flavor paradox: resolution

\[ \tilde{G}(q) = \mu^2 \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{2R^2}{n_R^2} \left( \frac{1}{(p^2 + M_{\eta'}^2)} \right) - \left( \frac{4R}{n_R} - \frac{2R^2}{n_R^2} \right) \frac{1}{(p^2 + M_I^2)} \right\} + 
\]

\[ + (Rn_R + R^2) \sum_{\Xi} \frac{1}{(p^2 + M_{\Xi}^2)} \left( \frac{1}{(p + q)^2 + M_{\Xi}^2} \right) - \left( \frac{4R}{n_R} - \frac{2R^2}{n_R^2} \right) \frac{1}{(p^2 + M_I^2)} \]

- \( M_{\eta'} \) heavy
- \( M_\Xi \) light (for all \( \Xi \); including \( M_I \))
- Setting \( R = 1/4 = n_R \), red terms vanish
- When \( a \to 0 \), all 16 of \( M_\Xi \) degenerate \( \Rightarrow \) blue terms vanish
- So only \( \eta' \) left in intermediate state in continuum \( \sqrt{\text{ }} \)
Three-flavor paradox

- Creutz: Continuum QCD with $n_F = 3$ (or any odd $n_F$) is not even under $m \to -m$, but rooted staggered determinant is even
  - staggered $D$ is anti-hermitian, eigenvalues of $D + m$ come in pairs $m \pm i\lambda$, so $\text{Det}(D + m)$ is function of $m^2$

- In standard continuum $\chi$PT, mass term (take degenerate for simplicity) is
  \[-m \text{Tr}(\Sigma + \Sigma^\dagger)\]
  - For $n_F = 3$, $m \to -m$ cannot be rotated away by non-anomalous chiral transformation
  - for $m < 0$ ground state is $\Sigma = \exp(\pm 2\pi i/3)$ instead of $\Sigma = 1$
  - theory with $m < 0$ is physically different from $m > 0$
  - $m < 0$ violates parity

- In finite volume, expansion of QCD level theory around $m = 0$ must have odd powers of $m$ as well as even
Three-flavor paradox: resolution

- In $S\chi$PT for $n_F = 3$, there are an even number of flavors $\times$ tastes for any integer $n_R$
  - Can rotate $-m \rightarrow m$ for each $n_R$
  - $(3, 4, \frac{1}{4})_\chi S\chi$PT is a function of $|m|$ only
  - But, in continuum limit, $(3, 4, \frac{1}{4})_\chi$ reproduces continuum $\chi$PT correctly, as long as $m > 0$

- At LQCD level, $\sqrt[4]{\text{Det}(D + m)}$ means that theory does not have to be analytic function of $m$ around $m = 0$, even in finite volume
  - Can be function of $\sqrt[4]{m^4} = |m|$
  - Can be even under $m \rightarrow -m$, and yet not just depend on even powers of $m$
  - Perfectly possible that gives correct odd powers of $m$ for $m > 0$ (as $S\chi$PT says it does) without getting the $m < 0$ case right $\sqrt{\ }$
Consequences: health of rooted theory

- If $S\chi_{PT}$ is correct, what are the implications for validity of rooted theory itself?
- When $a \to 0$, $(n_F, 4, n_R)_\chi$ becomes ordinary $\chi_{PT}$ for $4n_F \cdot n_R$ “flavors”
  - For given flavor combo, all 16 taste pions become degenerate in continuum (before including anomaly effects)
  - Anomaly affects only taste singlet, flavor singlet meson, as always
- Taking $n_R \to 1/4$ order by order produces standard, continuum $\chi_{PT}$ for $n_F$ flavors
  - NB: assumes vacuum of $(n_F, 4, n_R)_\chi (\Sigma = 1)$ is same as vacuum of continuum $\chi_{PT}$— why $m < 0$ doesn’t work
- Existing $S\chi_{PT}$ calculations all show this behavior explicitly
Consequences: health of rooted theory

- Since $S\chi$PT $\rightarrow \chi$PT in continuum, low energy sector of $n_F$-flavor lattice QCD with rooted staggered quarks becomes indistinguishable in structure from ordinary $n_F$-flavor QCD
  - No violations of unitarity
  - No unphysical nonlocal scales
- Says nothing about sectors not described by $\chi$PT, but
  - can probably extend to heavy-light physics using $S\chi$PT for heavy-lights (Aubin & CB)
  - in $n_F = 4$ case, can probably extend to baryons with heavy-baryon $S\chi$PT (Bailey & CB)
    - baryon mass scale might give difficulties in decoupling to get to $n_F < 4$, though
Consequences: health of rooted theory

- Note: saying $S\chi PT$ is valid doesn’t necessarily $\Rightarrow$ LECs are correct
  - For $n_F = 4$, LECs are correct in degenerate case (locality $\Rightarrow$ universality)
  - LECs mass independent, so also correct for four nondegenerate flavors (if $S\chi PT$ right)
  - For $n_F < 4$, decoupling assumptions not strong enough to guarantee correct LECs
    - Would need universality at the lattice QCD level (hope Shamir succeeds)
    - Agreement of simulations with experiment is nice; agreement between different lattice fermions would be better!
Consequences: mixed theory?

- “Mixed” theories have different lattice actions for sea and valence quarks
  - Sea and valence mass renormalizations different \( \Rightarrow \) no simple way to enforce \( m_S = m_V \)
  - Continuum symmetries that rotate valence and sea quark into each other are violated by discretization effects
  - If quark masses adjusted to make meson masses \( M_{SS} = M_{VV} \), then \( M_{SV} \) still differs by terms \( O(a^n) \)
  - Such terms show up as new operators in mixed theory \( \chi PT \) (Bär, Rupak, Shoresh, \ldots)
Consequences: mixed theory?

- Some (e.g. Kennedy) have suggested that rooted staggered sea + staggered valence ("rooted staggered") is a mixed theory
- But not hard to show that perturbative renormalization of sea and valence masses are the same
- Also does not look like a mixed theory non-perturbatively, at least in context of $S\chi PT$
  - $\chi$ obtained order by order from $(n_F, 4, n_R)_{\chi}$
  - $(n_F, 4, n_R)_{\chi}$ have symmetries interchanging valence and sea quarks
  - full symmetry group:
    \[ SU(4n_Rn_F + 4n_V | 4n_V)_{L} \times SU(4n_Rn_F + 4n_V | 4n_V)_{R}. \]
  - Taste symmetries broken on lattice at $O(a^2)$
  - But flavor subgroup ("residual chiral group")
    \[ U(n_Rn_F + n_V | n_V)_{\ell} \times U(n_Rn_F + n_V | n_V)_r \] is exact (up to mass terms)
Consequences: mixed theory?

- Chiral ops that split $M_{SV}$ from $M_{VV}$ & $M_{SV}$ (when $m_V = m_S$) are forbidden by flavor subgroup in $(n_F, 4, n_R)_\chi$

- Corresponding sea-sea, valence-valence, and valence-sea mesons degenerate (when quark masses degenerate) in $(n_F, 4, n_R)_\chi$, and therefore in $(n_F, 4, \frac{1}{4})_\chi$

- Within $S\chi$PT, rooted staggered behaves like partially quenched theory, not like mixed theory

- NB: valence sector “richer” than sea sector
  - Valence sector includes particles in continuum limit whose sea-sector analogues have decoupled from physical theory
  - In normal partially quenched theory, can take more valence quarks than sea quarks & create valence states with no sea-quark analogues
  - Here, there’s no choice: physical sea-quark states are always a proper subspace of valence states
Conclusions, Remarks, Speculations

- Most important assumptions:
  1) Taste symmetry restored in continuum limit of unrooted staggered theory
  2) Difference $V[s]$ between $S\chi PT$ theory $(4, 4, \frac{1}{4})\chi$ and true chiral theory $(4, "1")\chi$ is analytic in $s$ (for space-time independent $s$), up to possible isolated singularities
  3) As “charm” mass increases from $2m_s^{\text{phys}}$, when it has decoupled from chiral theory, to $\gg 1/a$, it remains decoupled from low energy physics

- Assumption 1) unproven but “non-controversial”

- Assumption 2) could be violated by essential singularities at $s = 0$ or by phase boundaries away from $s = 0$
  - Some numerical evidence against phase boundaries in regions of mass (and $a$) investigated by MILC
  - So essential singularity issue seems more pressing
Conclusions, Remarks, Speculations

- Assumption 3) will be tested by MILC in near future (I hope)
  - Simulate $n_F = 4$ theory in region $2m^\text{phys}_s \lesssim m_c \lesssim 1/a$
  - See if describable by $(3, 4, 1/4)^\chi$ at low energy

- Assumptions $\implies S\chi$PT
- But assumptions $\iff S\chi$PT, so testing assumptions tests $S\chi$PT
- One-flavor and three-flavor theories do not provide counter-examples to validity of $S\chi$PT or the fourth-root trick itself
  - But phase with odd number of negative masses not amenable to this approach (luckily QCD is not in that phase)
Conclusions, Remarks, Speculations

• If $S\chi$PT valid, then
  • Rooted theory ok at low energy (pseudoscalar meson sector)
  • Rooted theory not “mixed” (at least as far as $\chi$PT can tell)

• Looks like almost all of my arguments would go through for third root of theory with $n_F \leq 3$!
  • $\Rightarrow (n_F, "4/3")_{\chi} = (n_F, 4, \frac{1}{3})_{\chi}$
  • But that $S\chi$PT has no sensible continuum limit
Conclusions, Remarks, Speculations

- Can the essential singularity be eliminated as a possibility?
- Try to show that all complex derivatives of $V[s]$ vanish at $s = 0$, not just the real derivatives
  - Essential singularity doesn’t have well defined complex derivatives: think of $\exp(-1/z^2)$ when $z = iy$
  - Formally, all arguments from before go through if $s$ is complex
  - But big issue is now that Det is complex — can we choose phase of $\sqrt[4]{\text{Det}}$ consistently and continuously?
- See Golterman, Shamir, & Svetitsky, hep-lat/0602026; Golterman’s talk
Some final thoughts

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Some final thoughts

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2) “There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact.”
   —Mark Twain