Staggered Chiral Perturbation Theory and the Fourth-Root Trick

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Outline

- What is SχPT and is it valid with $\sqrt[4]{\mathrm{Det}}$?
- Replica trick & notation
- Four flavors
  - Degenerate case
  - Expanding around degenerate case
  - Assumptions on analyticity & consequences
- Fewer than four flavors
  - Decoupling in chiral theory and in lattice QCD
  - Assumptions needed to connect the above
- One-flavor & three-flavor paradoxes, and their resolution
- Consequences if schpt is valid:
  - Implications for health of rooted theory
  - Is rooted staggered a mixed theory?
- Conclusions, remarks, speculations

χPT

- Chiral perturbation theory (χPT) provides a nice framework for thinking about the fourth root
- Much simpler than lattice QCD itself
  - Low energy constants (LECs) are taken as unknowns ("mod them out" from the corresponding QCD theory)
  - Most info in χPT is in the order by order chiral expansion (perturbative)
  - But gives nonperturbative info about QCD
  - Lowest energy, longest distance sector: any problems from rooting (unitarity violations, nonlocality) ought to show up here

SχPT

- Lee & Sharpe found the LO chiral theory for a single unrooted staggered field including a^2 taste violations
  nomenclature: 1 staggered field = 1 flavor (4 tastes if unrooted; 1 taste if rooted)
- Aubin & C.B.:
  - Generalized Lee-Sharpe to many flavors
  - Proposed taking into account fourth root by locating sea-quark loops and multiplying each by $\frac{1}{4}$
  - Sea-quark loops found by quark-flow approach [Sharpe]
  - Replica trick is equivalent, but systematic and algebraic — better here
  - Staggered chiral perturbation theory (SχPT) is defined as this chiral theory for staggered quarks, including discretization errors and the above procedure for taking $\sqrt[4]{\mathrm{Det}}$ into account
- Question: is SχPT the correct chiral theory?

Overview

- We know (trivially) how fourth root works when we have 4 degenerate flavors: $\left(\sqrt[4]{\mathrm{Det}}\right)^4 = \mathrm{Det}$
  - Get 1-flavor, unrooted theory
  - Local lattice action & known chiral theory [Lee & Sharpe]
- To get non-degenerate 4-flavor theory, expand around degenerate point
  - Need non-trivial assumptions about mass dependence (analyticity, absence of phase transition)
- To get theory with 3 flavors, decouple a quark ("charm")
  - Need another assumption about how decoupling works
- I claim assumptions are "plausible."
  - Plausibility is in eye of beholder!
  - Assumptions at least not obviously wrong
  - These assumptions are necessary for SχPT to work

Replica trick

- Systematic & algebraic way to find sea-quark loops and multiply by $1/4$
- Introduced for partially quenched theory by Damgaard and Splittorff
- First used for SχPT by Aubin & CB, Lattice '03
- Replicate the sea-quark flavors, replacing each field by n_R identical copies ($n_R =$ positive integer)
- Calculate order by order in corresponding (unrooted) chiral theory
- Take $n_R \to 1/4$ at end
  - Dependence on n_R is polynomial at any finite order in SχPT, so $n_R \to 1/4$ is well-defined
  - Treat LECs as free parameters for each n_R — LECs are taken independent of n_R in this procedure

Replica trick

- Difficult to give meaning to replica trick at QCD level:
  - Beyond weak-coupling perturbation theory, dependence on n_R almost certainly non-polynomial
  - Analytic continuation from integers not unique
  - $\exists$ ideas by Shamir for defining a version of replica trick for QCD, but not used here
- In SχPT replica trick also only meaningful order by order
  - Will assume no phase change as we move away from degenerate point, where phase of chiral theory is known

(n_F, n_T, n_R) notation

- $(n_F, n_T, n_R)_{LQCD}$ is generating functional for lattice QCD theory with:
  - n_F flavors
  - n_T tastes
  - n_R replicas of each flavor
- $(n_F, n_T, n_R)_\chi$ is corresponding generating functional for chiral theory
- Omit n_R if it is trivially equal to 1 (because replica trick not relevant)
- Sources for generating functionals to be discussed later

(n_F, n_T, n_R) notation

Relevant theories:

- $(1,4)_{LQCD}$ and $(1,4)_\chi$
  - Single unrooted staggered field
  - $(1,4)_\chi$ is SχPT of Lee & Sharpe.
  - No replica trick necessary

- $(n_F, 4, n_R)_{LQCD}$ and $(n_F, 4, n_R)_\chi$
  - n_F staggered fields,
  - n_R indicated explicitly $\Rightarrow$ integer only
  - $(n_F, 4, n_R)_\chi$ is SχPT of Aubin & CB for $n_R \cdot n_F$ sea-quark flavors (still no rooting)

(n_F, n_T, n_R) notation

Relevant theories (continued):

- $(n_F, \text{“1”})_{LQCD}$ and $(n_F, \text{“1”})_\chi$
  - n_F staggered fields with $\sqrt[4]{\mathrm{Det}}$ taken
  - Quotes on “1” taste $\Rightarrow$ don’t assume fourth root works
  - $(n_F, \text{“1”})_\chi$ is by definition the chiral theory generated by $(n_F, \text{“1”})_{LQCD}$
  - Want to find $(n_F, \text{“1”})_\chi$ unambiguously

- $(n_F, 4, \frac{1}{4})_\chi$
  - Chiral theory $(n_F, 4, n_R)_\chi$ with the replica trick $n_R \to 1/4$
  - Defines SχPT for rooted theory
  - Does $(n_F, \text{“1”})_\chi$ = $(n_F, 4, \frac{1}{4})_\chi$?
  - Avoid “$(n_F, 4, \frac{1}{4})_{LQCD}$” because replica trick ambiguous at QCD level

Remarks

- Chiral theories $(n_F, 4, n_R)_\chi$ are key objects
- $(n_F, 4, n_R)_{LQCD}$, in particular $(4, 4, n_R)_{LQCD}$, introduced for convenience
  - Used formally; help keep track of n_R factors relating valence- to sea-quark matrix elements
  - Almost certainly can be eliminated at the expense of less intuitive argument at the chiral level
  - Unnecessary that the standard, broken realization of chiral symmetry assumed in $(4, 4, n_R)_\chi$ actually occurs in $(4, 4, n_R)_{LQCD}$
  - Unpleasant fact that asymptotic freedom & spontaneous chiral symmetry breaking(?) is lost for $n_R > 1$ in $(4, 4, n_R)_{LQCD}$ is irrelevant
  - Worried? — just increase n_c (number of colors) [Heller]

$n_F = 4$ basics

- Want to show:

$$(4, \text{“1”})_\chi \doteq (4, 4, \tfrac{1}{4})_\chi$$

  - Use “$\doteq$” to compare two chiral theories: same functions of the LECs
  - True equality only if adjust LECs to be the same

- Start with degenerate 4-flavor theory: $\mathcal{M} = \bar{m}I$, where I is identity matrix in flavor space:

$$\begin{aligned} (4, \text{“1”})_{LQCD}\Big|_{\mathcal{M}=\bar{m}I} &= (1,4)_{LQCD}\Big|_{\bar{m}} \\ (4, \text{“1”})_{\chi}\Big|_{\mathcal{M}=\bar{m}I} &\doteq (1,4)_{\chi}\Big|_{\bar{m}} \doteq (4, 4, \tfrac{1}{4})_\chi\Big|_{\mathcal{M}=\bar{m}I} \end{aligned}$$

- Last equivalence manifest order by order in SχPT
  - Taking $4n_R$ degenerate flavors and then putting $n_R = 1/4$ $\Longleftrightarrow$ one-flavor theory

$n_F = 4$: expansion around degenerate point

- To move away from degenerate limit, add taste-singlet scalar sources for sea-quark fields:

$$\mathcal{L}_{(4,\text{“1”})} = \cdots + \bar{m}\,\bar{\Psi}_i(x)\Psi_i(x) + \bar{\Psi}_i(x)\,s^{ij}(x)\,\Psi_j(x) + \ldots$$
$$\mathcal{L}_{(4,4,n_R)} = \cdots + \bar{m}\bar{\Psi}^r_i(x)\Psi^r_i(x) + \bar{\Psi}^r_i(x)\,s^{ij}(x)\,\Psi^r_j(x) + \ldots$$

[sum over i, j (flavor indices) and r (replica index)]

- When $s \neq 0$, we don't yet know that $(4, 4, \frac{1}{4})_\chi$ is right chiral theory
- Define $V[s]$ as amount of mismatch:

$$(4, \text{“1”};\, s)_\chi \doteq (4, 4, \tfrac{1}{4};\, s)_\chi + V[s]$$

- $V[s] = 0$ when $s = 0$ or whenever flavor symmetry is exact

$n_F = 4$: expansion around degenerate point

- Example of possible term in $V[s]$:

$$V_1 = \bar{m}^2 \int d^4x\, d^4y \left(\frac{1}{\Box + M^2}\right)_{x,y} \left(\mathrm{Tr}\,[\mathrm{s(x)s(y)}] - \frac{1}{4}\mathrm{Tr}\,[\mathrm{s(x)}]\;\mathrm{Tr}\,[\mathrm{s(y)}]\right)$$

with $1/M$ a distance scale that might not vanish when $a \to 0$

- Claim:

$$\prod_n \frac{\partial}{\partial s^{i_n j_n}(x_n)} (4, \text{“}1\text{”};\, s)_\chi \Big|_{s=0} \doteq \prod_n \frac{\partial}{\partial s^{i_n j_n}(x_n)} (4, 4, \tfrac{1}{4};\, s)_\chi \Big|_{s=0}$$

$$\Longrightarrow \qquad \prod_n \left(\frac{\partial}{\partial s^{i_n j_n}(x_n)} V[s] \right) \Bigg|_{s=0} = 0$$

- Prove by relating sea Green’s functions to valence Green’s functions in partially quenched theory
- Then can keep $s = 0$, where equivalence is known

$n_F = 4$: partial quenching argument

- Add n_V staggered valence fields with sources $\sigma^{\alpha\beta}$ to all LQCD theories

$$cL = \cdots + \bar{m}\bar{q}_\alpha(x)q_\alpha(x) + \bar{q}_\alpha(x)\,\sigma^{\alpha\beta}(x)\,q_\beta(x) + \ldots$$

- n_V ghost fields also added, but not coupled to $\sigma^{\alpha\beta}$: cancel valence Det when $\sigma = 0$

$$(4, \text{“1”};\ s=0, \sigma)_{LQCD} \;=\; (1, 4;\ s=0, \sigma)_{LQCD}$$

$$\Longrightarrow (4, \text{“1”};\ s=0, \sigma)_\chi \;\doteq\; (1, 4;\ s=0, \sigma)_\chi \;\doteq\; (4, 4, \tfrac{1}{4};\ s=0, \sigma)_\chi$$

- Last equivalence again manifest order by order in SχPT
  - Should be safe from any subtlety of type discussed by Golterman, Sharpe & Singleton
  - e.g. non-trivial saddle point for ghost mesons

$n_F = 4$: partial quenching argument

- Relate derivatives w.r.t. s to derivatives w.r.t. σ
- Derivatives w.r.t. s in rooted theory bring down factors of $1/4$ from
$$\sqrt[4]{\mathrm{Det}(D+\bar{m}+s)} = \exp\frac{1}{4}\mathrm{tr}\ln(\mathrm{D}+\bar{\mathrm{m}}+\mathrm{s})$$
- Different terms ($\equiv$ different contractions) associated with different powers of $1/4$
  - power of $1/4$ is just the number of quark loops implied by corresponding contractions
- Derivatives w.r.t. s in replicated theory produce corresponding powers of n_R from sea-quark counting
- But with arbitrary n_V, can always adjust valence flavor indices on σ derivatives so only one contraction possible

$n_F = 4$: partial quenching argument

- Examples ($i \neq j$, $\alpha \neq \beta$, no sums):

$$\frac{\partial}{\partial s^{ij}(x)}\frac{\partial}{\partial s^{ji}(y)}\,(4, \text{“1”};\, s,\sigma=0)_{LQCD}\Big|_{s=0} \;=\; \frac{1}{4}\,\langle \mathrm{tr}\Big(\mathrm{G_j(x,y)G_i(y,x)}\Big)\rangle$$

$$= \frac{1}{4}\,\frac{\partial}{\partial \sigma^{\alpha\beta}(x)}\frac{\partial}{\partial \sigma^{\beta\alpha}(y)}\,(4, \text{“1”};\, s=0,\sigma)_{LQCD}\Big|_{\sigma=0}$$

$$\frac{\partial}{\partial s^{ii}(x)}\frac{\partial}{\partial s^{ii}(x)}(4, \text{“1”};\, s,\sigma=0)_{LQCD}\Big|_{s=0} =$$

$$= \frac{1}{4}\langle \mathrm{tr}\Big(\mathrm{G_i(x,y)G_i(y,x)}\Big)\rangle + \left(\frac{1}{4}\right)^2 \langle \mathrm{tr}\Big(\mathrm{G_i(x,x)}\Big)\mathrm{tr}\Big(\mathrm{G_i(y,y)}\Big)\rangle$$

$$= \left[\frac{1}{4}\frac{\partial}{\partial \sigma^{\alpha\beta}(x)}\frac{\partial}{\partial \sigma^{\beta\alpha}(y)} + \left(\frac{1}{4}\right)^2 \frac{\partial}{\partial \sigma^{\alpha\alpha}(x)}\frac{\partial}{\partial \sigma^{\beta\beta}(y)}\right](4, \text{“1”};\, s=0,\sigma)_{LQCD}\Big|_{\sigma=0}$$

- For $(4, 4, n_R)$ theory, just replace $1/4 \rightarrow n_R$

$n_F = 4$: partial quenching argument

- Can therefore write:

$$\prod_n \frac{\partial}{\partial s^{i_n j_n}(x_n)} (4, \text{``1''};\ s, \sigma = 0)_{LQCD}\Big|_{s=0} =$$

$$= \sum_C \left(\tfrac{1}{4}\right)^{L_C} \prod_n \frac{\partial}{\partial \sigma^{\alpha_n^C \beta_n^C}(x_n)} (4, \text{``1''}; s = 0, \sigma)_{LQCD}\Big|_{\sigma=0}$$

$$\prod_n \frac{\partial}{\partial s^{i_n j_n}(x_n)} (4, 4, n_R;\ s, \sigma = 0)_{LQCD}\Big|_{s=0} =$$

$$= \sum_C (n_R)^{L_C} \prod_n \frac{\partial}{\partial \sigma^{\alpha_n^C \beta_n^C}(x_n)} (4, 4, n_R;\ s = 0, \sigma)_{LQCD}\Big|_{\sigma=0}$$

- C labels a contraction with L_C valence quark loops
- Valence indices α_n^C, β_n^C adjusted so only one contraction
- Same arrangements of valence flavor indices & powers L_C work in both cases

$n_F = 4$: partial quenching argument

- Pass to corresponding chiral theories:

$$\prod_n \frac{\partial}{\partial s^{i_n j_n}(x_n)} (4,\, \text{“}1\text{”};\; s, \sigma = 0)_\chi \Big|_{s=0} =$$

$$= \sum_C \left(\tfrac{1}{4}\right)^{L_C} \prod_n \frac{\partial}{\partial \sigma^{\alpha_n^C \beta_n^C}(x_n)} (4,\, \text{“}1\text{”};\, s = 0, \sigma)_\chi \Big|_{\sigma=0}$$

$$\prod_n \frac{\partial}{\partial s^{i_n j_n}(x_n)} (4, 4, n_R;\; s, \sigma = 0)_\chi \Big|_{s=0} =$$

$$= \sum_C (n_R)^{L_C} \prod_n \frac{\partial}{\partial \sigma^{\alpha_n^C \beta_n^C}(x_n)} (4, 4, n_R;\; s = 0, \sigma)_\chi \Big|_{\sigma=0}$$

- At any finite order in chiral perturbation theory both sides of last eqn are polynomial in n_R. Can take $n_R \to 1/4$

$n_F = 4$: partial quenching argument

- After $n_R \to 1/4$ in second eqn:

$$\prod_n \frac{\partial}{\partial s^{i_n j_n}(x_n)} (4, \text{“}1\text{”};\ s, \sigma = 0)_\chi \Big|_{s=0} =$$

$$= \sum_C \left(\tfrac{1}{4}\right)^{L_C} \prod_n \frac{\partial}{\partial \sigma^{\alpha_n^C \beta_n^C}(x_n)} (4, \text{“}1\text{”}; s = 0, \sigma)_\chi \Big|_{\sigma=0}$$

$$\prod_n \frac{\partial}{\partial s^{i_n j_n}(x_n)} (4, 4, \tfrac{1}{4};\ s, \sigma = 0)_\chi \Big|_{s=0} =$$

$$= \sum_C \left(\tfrac{1}{4}\right)^{L_C} \prod_n \frac{\partial}{\partial \sigma^{\alpha_n^C \beta_n^C}(x_n)} (4, 4, \tfrac{1}{4};\ s = 0, \sigma)_\chi \Big|_{\sigma=0}$$

- Right sides equal since $(4, \text{“}1\text{”};\ s = 0, \sigma)_\chi \doteq (4, 4, \frac{1}{4};\ s = 0, \sigma)_\chi$
- So left sides equal, which is what we wanted to show

$n_F = 4$: analyticity assumptions

- So all derivatives of $V[s]$ vanish at $s = 0$
- If $V[s]$ analytic in s — up to possible isolated singularities — it vanishes everywhere
- Strong assumption; is it obviously too strong?
  - “Don’t expect convergent expansions in QFT”
  - Factorial growth of large orders in perturbation theory: expansion at best asymptotic
  - But here every order is zero!

- How could analyticity go wrong?
  - Line of singularities, domain boundary
    - Ground state for $(4, \text{“}1\text{”})_\chi$ changes discontinuously from state assumed by $(4, 4, \frac{1}{4})_\chi$
    - Inside the range of m & a studied by MILC, such a singularity would have probably been detected
    - No evidence outside MILC range, though

$n_F = 4$: analyticity assumptions

- How could analyticity go wrong? (continued)
  - Essential singularity at $s = 0$
    - Term like $\exp(-1/V_1^2)$ is logically possible
    - Best I can say right now is there's no reason to expect it (no obvious IR problem; expanding around massive theory)
    - Speculations later
  - NB: Not assuming that $(4, \text{“}1\text{”})_\chi$ and $(4, 4, \frac{1}{4})_\chi$ are separately analytic, only that difference is
  - If $V[s]$ not analytic, then SχPT is wrong

$n_F = 3$: decoupling

- Try to get to $n_F = 3$ by taking one mass ("charm") large
- Take m_c large as possible w/o leaving region where SχPT applies
  - Nominally, say $m_c \sim 2m_s^{\rm phys}$
- Take other masses small for clean separation ($m_s \ll m_s^{\rm phys}$)
- Integrate out m_c from $(4, 4, \frac{1}{4})_\chi$
  - Should get $(3, 4, \frac{1}{4})_\chi$
  - Since perturbative, there is little doubt here
  - Explicit check is planned (CB & X. Du)
- So charm has decoupled from low energy physics when $m_c \sim 2m_s^{\rm phys}$
- Assume it remains decoupled from low energy physics as m_c increases to $\gg 1/a$

$n_F = 3$: decoupling

- When $m_c \gg 1/a$, it is much larger than all eigenvalues of D
  - $\sqrt[4]{\mathrm{Det}(D + m_c)}$ independent of gauge field
  - charm decouples from $(4, \text{“}1\text{”})_{LQCD}$, leaving $(3, \text{“}1\text{”})_{LQCD}$

$$\Longrightarrow \qquad (3, \text{“}1\text{”})_\chi \doteq (3, 4, \tfrac{1}{4})_\chi$$

- If true for small u, d, s masses, then analyticity assumption implies still true for physical ones
- Can repeat to argue $(2, \text{“}1\text{”})_\chi \doteq (2, 4, \frac{1}{4})_\chi$ and $(1, \text{“}1\text{”})_\chi \doteq (1, 4, \frac{1}{4})_\chi$
- Decoupling assumption not only sufficient but also necessary for $n_F = 3$ SχPT:
  - Any new physical effects entering for $2m_s^{\mathrm{phys}} \lesssim m_c \lesssim 1/a$ automatically violate chiral theory

One-flavor paradox

- Theory with 1 flavor should have only heavy pseudoscalar, η', no light pseudo-Goldstone bosons
- SχPT for 1 rooted-staggered flavor has 16 pseudoscalars ("pions"); only the taste-singlet is heavy
- Different weightings (factors of $1/4$) in rooted case compared to unrooted case, but otherwise similar — all pions contribute at $a \neq 0$
- For consistency, light pions must decouple from pure-glue correlation functions when $a \to 0$
- Work by CB, DeTar, Fu, Prelovsek; more details in DeTar's talk tomorrow

One-flavor paradox

- Mock up the kind of pure-glue correlation function that can persist in continuum limit: add taste-singlet scalar source to rooted one-flavor theory:

$$\mathcal{L}_{\rm source} = s(z)\bar\Psi(z)\Psi(z)$$

- To show factors resulting from rooting, take the R^{th} power of the determinant; set $R = 1/4$ at end

$$(1,\text{``}1\text{''})_{LQCD} = \frac{\int \mathcal{D}A\ \exp\{-S_G(A) + R\,{\rm tr}\,(\ln\,({\rm D} + {\rm m} + {\rm s}))\}}{\int \mathcal{D}A\ \exp\{-S_G(A) + R\,{\rm tr}\,(\ln\,({\rm D} + {\rm m}))\}}$$

- Look at connected part of

$$G(x-y) = \left(\frac{\partial}{\partial s(x)}\frac{\partial}{\partial s(y)}\ (1,\text{``}1\text{''})_{LQCD}\right)_{s=0}$$

[“connected” $\Rightarrow$ subtract $\langle\bar\Psi\Psi\rangle^2$]

One-flavor paradox

- Calculate G(x-y) for large $|x-y|$ in LO SχPT
- First rewrite in terms of valence Green's functions

$$G(x-y) = R\left(\frac{\partial}{\partial\sigma^{\alpha\beta}(x)}\frac{\partial}{\partial\sigma^{\beta\alpha}(y)}(1,\text{“}1\text{”})_{LQCD}\right)_{\sigma=0} + R^2\left(\frac{\partial}{\partial\sigma^{\alpha\alpha}(x)}\frac{\partial}{\partial\sigma^{\beta\beta}(y)}(1,\text{“}1\text{”})_{LQCD}\right)_{\sigma=0}$$

One-flavor paradox

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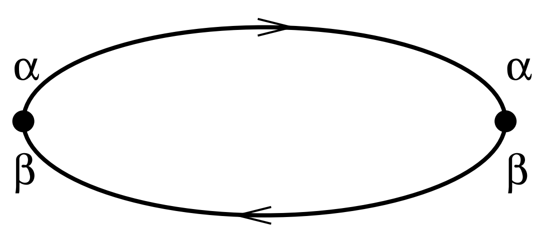
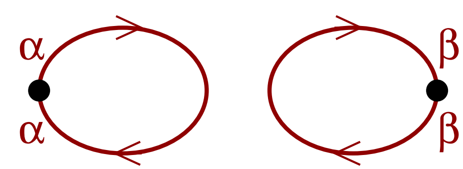
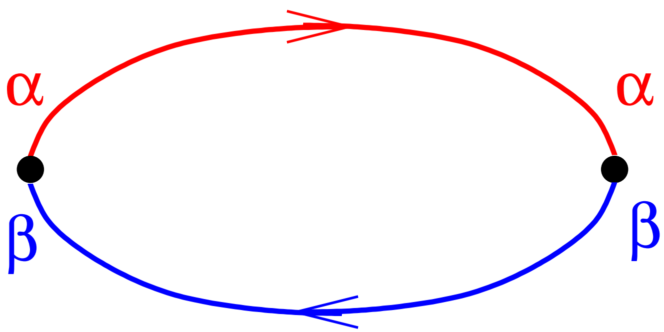


R term



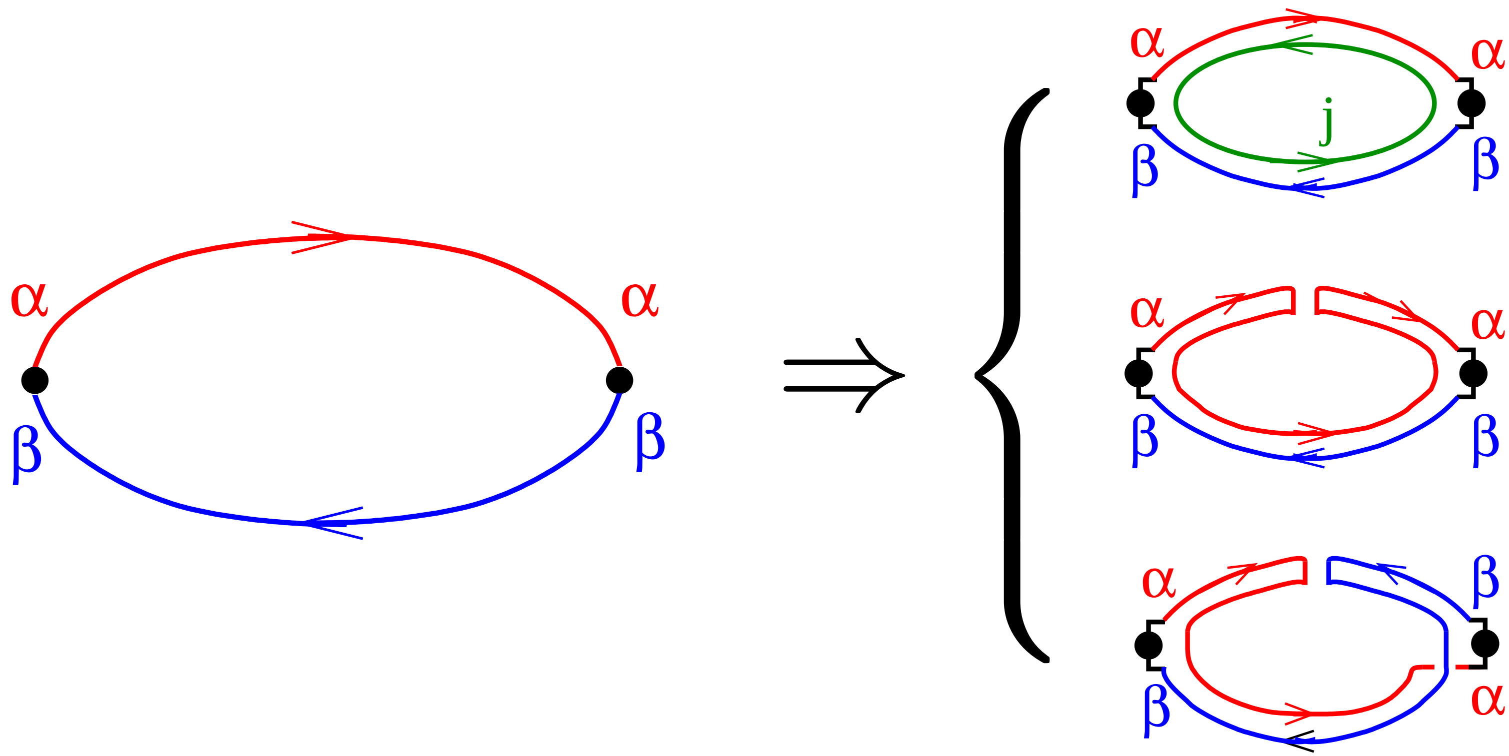
R^2 term

One-flavor paradox: diagrams



QCD valence contraction
(term proportional to R)

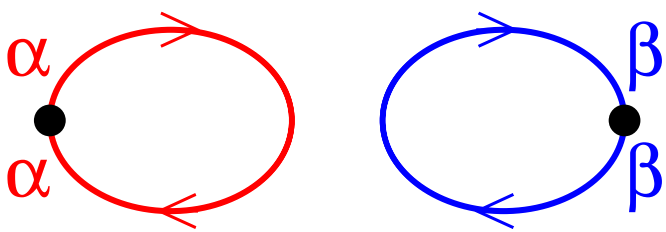
One-flavor paradox: diagrams



QCD valence contraction
(term proportional to R)

chiral quark flow (note hairpins)

One-flavor paradox: diagrams



QCD valence contraction
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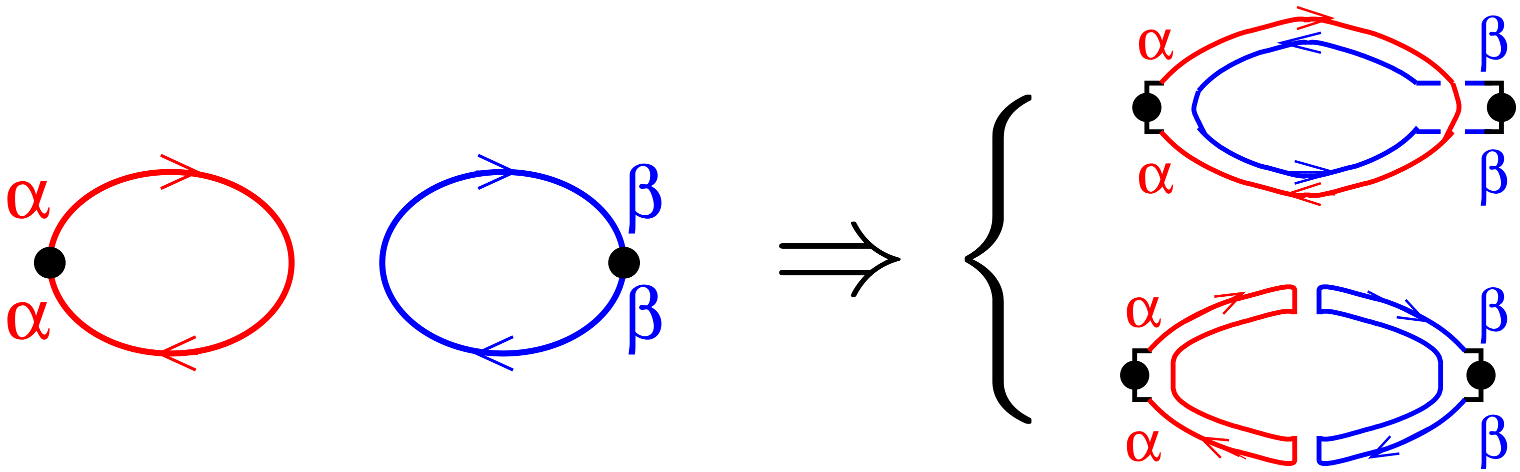

QCD valence contraction
(term proportional to R^2)

chiral quark flow (note hairpins)

One-flavor paradox: resolution

$$\tilde{G}(q) = \mu^2 \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{2R^2}{n_R^2} \frac{1}{\left(p^2 + M_{\eta'_I}^2\right)\left((p+q)^2 + M_{\eta'_I}^2\right)} + \right.$$

$$+ (Rn_R + R^2) \sum_{\Xi} \frac{1}{(p^2 + M_\Xi^2)\left((p+q)^2 + M_\Xi^2\right)} - \left(\frac{4R}{n_R} - \frac{2R^2}{n_R^2}\right) \frac{1}{(p^2 + M_I^2)\left((p+q)^2 + M_I^2\right)}$$

$$\left. + \left(\frac{2R}{n_R} - \frac{2R^2}{n_R^2}\right) \left(\frac{1}{(p^2 + M_I^2)\left((p+q)^2 + M_{\eta'_I}^2\right)} + \frac{1}{\left(p^2 + M_{\eta'_I}^2\right)\left((p+q)^2 + M_I^2\right)} \right) \right\}$$

- $M_{\eta'}$ heavy
- M_Ξ light (for all Ξ; including M_I)
- Setting $R = 1/4 = n_R$, red terms vanish
- When $a \to 0$, all 16 of M_Ξ degenerate $\Rightarrow$ blue terms vanish
- So only η' left in intermediate state in continuum $\surd$

Three-flavor paradox

- Creutz: Continuum QCD with $n_F = 3$ (or any odd n_F) is not even under $m \to -m$, but rooted staggered determinant is even
  - staggered D is anti-hermitian, eigenvalues of $D + m$ come in pairs $m \pm i\lambda$, so $\mathrm{Det}(D + m)$ is function of m^2
- In standard continuum χPT, mass term (take degenerate for simplicity) is

$$-m\mathrm{Tr}(\Sigma + \Sigma^\dagger)$$

  - For $n_F = 3$, $m \to -m$ cannot be rotated away by non-anomalous chiral transformation
  - for $m < 0$ ground state is $\Sigma = \exp(\pm 2\pi i/3)$ instead of $\Sigma = 1$
  - theory with $m < 0$ is physically different from $m > 0$
  - $m < 0$ violates parity
- In finite volume, expansion of QCD level theory around $m = 0$ must have odd powers of m as well as even

Three-flavor paradox: resolution

- In SχPT for $n_F = 3$, there are an even number of flavors $\times$ tastes for any integer n_R
  - Can rotate $-m \to m$ for each n_R
  - $(3, 4, \frac{1}{4})_\chi$ SχPT is a function of $|m|$ only
  - But, in continuum limit, $(3, 4, \frac{1}{4})_\chi$ reproduces continuum χPT correctly, as long as $m > 0$
- At LQCD level, $\sqrt[4]{\mathrm{Det}(D+m)}$ means that theory does not have to be analytic function of m around $m = 0$, even in finite volume
  - Can be function of $\sqrt[4]{m^4} = |m|$
  - Can be even under $m \to -m$, and yet not just depend on even powers of m
  - Perfectly possible that gives correct odd powers of m for $m > 0$ (as SχPT says it does) without getting the $m < 0$ case right $\surd$

Consequences: health of rooted theory

- If SχPT is correct, what are the implications for validity of rooted theory itself?
- When $a \to 0$, $(n_F, 4, n_R)_\chi$ becomes ordinary χPT for $4n_F \cdot n_R$ "flavors"
  - For given flavor combo, all 16 taste pions become degenerate in continuum (before including anomaly effects)
  - Anomaly affects only taste singlet, flavor singlet meson, as always
- Taking $n_R \to 1/4$ order by order produces standard, continuum χPT for n_F flavors
  - NB: assumes vacuum of $(n_F, 4, n_R)_\chi$ ($\Sigma = 1$) is same as vacuum of continuum χPT— why $m < 0$ doesn't work
- Existing SχPT calculations all show this behavior explicitly

Consequences: health of rooted theory

- Since SχPT $\rightarrow$ χPT in continuum, low energy sector of n_F-flavor lattice QCD with rooted staggered quarks becomes indistinguishable in structure from ordinary n_F-flavor QCD
  - No violations of unitarity
  - No unphysical nonlocal scales
- Says nothing about sectors not described by χPT, but
  - can probably extend to heavy-light physics using SχPT for heavy-lights (Aubin & CB)
  - in $n_F = 4$ case, can probably extend to baryons with heavy-baryon SχPT (Bailey & CB)
    - baryon mass scale might give difficulties in decoupling to get to $n_F < 4$, though

Consequences: health of rooted theory

- Note: saying SχPT is valid doesn't necessarily $\Rightarrow$ LECs are correct
  - For $n_F = 4$, LECs are correct in degenerate case (locality $\Rightarrow$ universality)
  - LECs mass independent, so also correct for four nondegenerate flavors (if SχPT right)
  - For $n_F < 4$, decoupling assumptions not strong enough to guarantee correct LECs
    - Would need universality at the lattice QCD level (hope Shamir succeeds)
    - Agreement of simulations with experiment is nice; agreement between different lattice fermions would be better!

Consequences: mixed theory?

- “Mixed” theories have different lattice actions for sea and valence quarks
  - Sea and valence mass renormalizations different $\Rightarrow$ no simple way to enforce $m_S = m_V$
  - Continuum symmetries that rotate valence and sea quark into each other are violated by discretization effects
  - If quark masses adjusted to make meson masses $M_{SS} = M_{VV}$, then M_{SV} still differs by terms $\mathcal{O}(a^n)$
  - Such terms show up as new operators in mixed theory χPT (Bär, Rupak, Shoresh, . . .)

Consequences: mixed theory?

- Some (e.g. Kennedy) have suggested that rooted staggered sea + staggered valence ("rooted staggered") is a mixed theory
- But not hard to show that perturbative renormalization of sea and valence masses are the same
- Also does not look like a mixed theory non-perturbatively, at least in context of SχPT
  - $(n_F, 4, \frac{1}{4})_\chi$ obtained order by order from $(n_F, 4, n_R)_\chi$
  - $(n_F, 4, n_R)_\chi$ have symmetries interchanging valence and sea quarks
  - full symmetry group: $SU(4n_Rn_F + 4n_V|4n_V)_L \times SU(4n_Rn_F + 4n_V|4n_V)_R$.
  - Taste symmetries broken on lattice at $\mathcal{O}(a^2)$
  - But flavor subgroup ("residual chiral group") $U(n_Rn_F + n_V|n_V)_\ell \times U(n_Rn_F + n_V|n_V)_r$ is exact (up to mass terms)

Consequences: mixed theory?

- Chiral ops that split M_{SV} from M_{VV} & M_{SV} (when $m_V = m_S$) are forbidden by flavor subgroup in $(n_F, 4, n_R)_\chi$
- Corresponding sea-sea, valence-valence, and valence-sea mesons degenerate (when quark masses degenerate) in $(n_F, 4, n_R)_\chi$, and therefore in $(n_F, 4, \frac{1}{4})_\chi$
- Within SχPT, rooted staggered behaves like partially quenched theory, not like mixed theory
- NB: valence sector "richer" than sea sector
  - Valence sector includes particles in continuum limit whose sea-sector analogues have decoupled from physical theory
  - In normal partially quenched theory, can take more valence quarks than sea quarks & create valence states with no sea-quark analogues
  - Here, there's no choice: physical sea-quark states are always a proper subspace of valence states

Conclusions, Remarks, Speculations

- Most important assumptions:
  1) Taste symmetry restored in continuum limit of unrooted staggered theory
  2) Difference $V[s]$ between SχPT theory $(4, 4, \frac{1}{4})_\chi$ and true chiral theory $(4, \text{“1”})_\chi$ is analytic in s (for space-time independent s), up to possible isolated singularities
  3) As “charm” mass increases from $2m_s^{\text{phys}}$, when it has decoupled from chiral theory, to $\gg 1/a$, it remains decoupled from low energy physics
- Assumption 1) unproven but “non-controversial”
- Assumption 2) could be violated by essential singularities at $s = 0$ or by phase boundaries away from $s = 0$
  - Some numerical evidence against phase boundaries in regions of mass (and a) investigated by MILC
  - So essential singularity issue seems more pressing

Conclusions, Remarks, Speculations

- Assumption 3) will be tested by MILC in near future (I hope)
  - Simulate $n_F = 4$ theory in region $2m_s^{\text{phys}} \lesssim m_c \lesssim 1/a$
  - See if describable by $(3, 4, \frac{1}{4})_\chi$ at low energy
- Assumptions $\Longrightarrow$ SχPT
- But assumptions $\Longleftarrow$ SχPT, so testing assumptions tests SχPT
- One-flavor and three-flavor theories do not provide counter-examples to validity of SχPT or the fourth-root trick itself
  - But phase with odd number of negative masses not amenable to this approach (luckily QCD is not in that phase)

Conclusions, Remarks, Speculations

- If SχPT valid, then
  - Rooted theory ok at low energy (pseudoscalar meson sector)
  - Rooted theory not "mixed" (at least as far as χPT can tell)

- Looks like almost all of my arguments would go through for third root of theory with $n_F \le 3$!
  - $\Rightarrow (n_F, \text{“}4/3\text{”})_\chi \doteq (n_F, 4, \frac{1}{3})_\chi$
  - But that SχPT has no sensible continuum limit

Conclusions, Remarks, Speculations

- Can the essential singularity be eliminated as a possibility?
- Try to show that all complex derivatives of $V[s]$ vanish at $s = 0$, not just the real derivatives
  - Essential singularity doesn't have well defined complex derivatives: think of $\exp(-1/z^2)$ when $z = iy$
  - Formally, all arguments from before go through if s is complex
  - But big issue is now that Det is complex — can we choose phase of $\sqrt[4]{\text{Det}}$ consistently and continuously?
    - See Golterman, Shamir, & Svetitsky, hep-lat/0602026; Golterman's talk

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2) "There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact."

—Mark Twain