Muon $g-2$:
Reclaiming the theoretical calculation of the leading QCD contribution

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Stupid Question: Why?

Currently:

- Experiment: Very precise — 0.5 ppm (BNL)

“Theory”:

- Low compared with experiment
- Relates $g-2$ to
  1. $e^+ e^- \rightarrow$ hadrons cross section and
  2. $\tau$ decay cross section
- Discrepancy with experiment: $0.7\sigma \rightarrow 2.7\sigma$

Lattice: method to extract hadronic contributions without experimental input
Theory vs. Experiment

Experiment

\[ a_\mu^{\text{exp}} = \left( g - \frac{2}{2} \right)^{\text{exp}} = 11\,659\,208(6) \times 10^{-10} \]
Outline

• Muon $g-2$ and current theoretical predictions

• Calculating $g-2$ on the lattice, with Lattice Gauge Theory and Chiral Perturbation Theory

• $O(\alpha^2)$ Contribution: Vacuum Polarization

• Lattice results for vacuum polarization

• Fits and preliminary results for $g-2$
Muon $g - 2$

Full muon-photon vertex:

\[ \Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\mu}{2m_\mu} F_2(q^2) \]

\[ a_\mu = \frac{g - 2}{2} = F_2(0) \]

$O(\alpha)$: \hspace{1cm} $O(\alpha^2)$:

\[ \vdots \]

\[ \vdots \]
Hadronic Contributions

$O(\alpha^2)$, Hadronic contribution to the photon vacuum polarization:

$O(\alpha^3)$, Light-by-light scattering:

Hadronic contributions are $7 \times 10^{-5}$ times smaller than leading corrections
The $O(\alpha^2)$ hadronic contribution, $a_{\mu}^{HLO}$, cannot be calculated in perturbation theory.

Using the Optical Theorem, one can evaluate it using the cross section of $e^+e^- \rightarrow \text{hadrons}$:

$$a_{\mu}^{HLO} = \frac{\alpha^2}{3\pi^2} \int_{4m_{\pi}^2}^{\infty} \frac{ds}{s} K(s) R(s)$$

The kernel, $K(s)$ is known (dominated by small $s$), and $R(s)$ can be measured experimentally.

Not a theoretical problem since 1961!
The precision of the Standard Model prediction is limited by the experimental measurement of $R(s)$. 

(Davier et al, hep-ph/0208177)
Using $\tau$ decay

- In isospin limit, relate $\tau$ spectral data to isovector part of $\sigma(e^+e^-)$ using Conserved Vector Current (CVC) relations
- Result for $g-2$ is higher than "standard method"
- Contraversial: Studies have conflicting results on validity of CVC relations
- Either way, still is an experimental calculation, and we want a theoretical one

\[
\begin{align*}
B(\tau^- \rightarrow \nu_\tau \pi^0) &\quad \text{(in %)} \\
\text{CLEO} &\quad 25.42 \pm 0.42 \\
\text{OPAL} &\quad 25.44 \pm 0.34 \\
\text{ALEPH preliminary} &\quad 25.47 \pm 0.13 \\
\text{CVC} &\quad 23.98 \pm 0.30 \\
\text{Average} &\quad 25.46 \pm 0.12
\end{align*}
\]

(Davier et al, hep-ph/0208177)
Field Theory

Path Integral: $Z[J] = \int_{\phi(x_a)}^{\phi(x_b)} D\phi \exp \left\{ i \int d^4 x \left[ \mathcal{L}[\phi(x)] + J(x)\phi(x) \right] \right\}$

($D = $Sum over all paths)

There are an infinite number of paths!

Use PT if coupling constant is small (high-$E$ QCD, QED)
To calculate $Z$ (and physical quantities) on the lattice:

- Continue to Euclidean space: $t \rightarrow -it_E$
- Discretize space and time (with a lattice spacing $a$) and put system in a finite volume $V$
- Now a finite dimensional path integral
Field theory on a lattice 2

- This is still non-trivial: Finite, but large dimensional integral

- Use Monte Carlo techniques to evaluate \( Z \) and whatever matrix element you want (within reason)

- In the end, take \( a \to 0 \) and \( V \to \infty \) (the “continuum limit”) and continue back to Minkowski space

A few comments:
- We can vary external (valence) and internal (sea) quark masses separately
- Often \( m_{\text{sea}} \to \infty \) (Quenched approx) due to limited computational power
- Finite volume \( \Rightarrow \) discrete momenta.
- \( p_{\text{min}} = 2\pi/T \), where \( T \) is the size of the largest direction
- Quarks on the lattice are a problem...
Simulating Quarks

Quarks are anti-commuting fields → Must integrate over them first in the path integral:

$$Z = \int_{A_\mu, \psi, \bar{\psi}} e^{-S_{QCD}} = \int_{A_\mu} \det K[A] e^{-S_{gluons}}$$

- $K[A]$ is the Dirac operator for a given set of gauge fields
- $\det K$ is slow to simulate (very non-local), quenched approximation sets this to 1
- For example, pion propagator:

$$\langle \pi^+ \pi^- \rangle = \frac{1}{Z} \int_{A_\mu, \psi, \bar{\psi}} (\bar{u} \gamma_5 d)(\bar{d} \gamma_5 u) e^{-S_{QCD}}$$

Wick contract the quarks to give us quark propagators, which we can evaluate on a given gauge background
Lattice Quarks

Discrete version of the theory has the 15 “doubling symmetries”

\[ \psi_x \rightarrow e^{i\pi x \cdot p} \Gamma_p \psi_x \quad \bar{\psi}_x \rightarrow e^{i\pi x \cdot p} \bar{\psi}_x \Gamma_p^\dagger \]

\[ a_p \in \{(1, 0, 0, 0), (0, 1, 0, 0), \ldots, (1, 1, 0, 0), \ldots, (1, 1, 1, 1)\} \]

\[ \Gamma_p = \prod_\mu (i\gamma_5 \gamma_\mu)^{a_p\mu} \]

⇒ 16 species (“tastes”) when \( a \rightarrow 0 \)

If \( \psi_0^x \) satisfies the lattice Dirac equation, we have 15 other solutions, \( \psi_p^x \), which are degenerate in mass in the continuum limit
Lattice Quarks

Many solutions to the doubling problem:

- **Wilson quarks**: Slow, breaks chiral symmetry at finite $a$, difficult to renormalize, but gets rid of all doublers
- **Staggered quarks**: Fast, has a remnant chiral symmetry at finite $a$, still has four species as $a \to 0$
- **Domain-Wall quarks**: Slow, has controlled and small chiral symmetry breaking at finite $a$, no doubling remnants
- **Overlap quarks**: VERY slow, but perfect chiral symmetry

For now we'll choose staggered:

- Dynamical simulations with Full QCD with very light quark masses
- Lightest quark masses $\Rightarrow$ easier to take chiral limit
- Largest volumes
- These lattices already exist (MILC Collaboration)
Staggered Quarks

- On the lattice, the usual continuum $SO(4)$ rotation symmetry is broken to allow only hypercubic rotations.

- A unitary transformation on $\psi$ can diagonalize the $\gamma$ matrices.

- This decouples the four spinor components of the fermion $\Rightarrow$ we can keep only one component per species.

- We have 16 one-component fields, *staggered* on separate sites of a hypercube $\Rightarrow$ 4 four-component Dirac *tastes*, degenerate as $a \rightarrow 0$. 
Aside: $4 \rightarrow 1$ tastes

- Evaluate the staggered quark path integral $\Rightarrow \det K$
- $\det K$ describes four degenerate tastes in the continuum limit
- $\Rightarrow (\det K)^{1/4}$ describes 1 taste

Can we do this before taking the continuum limit?

- At finite $a$, we have violations of the taste symmetry (i.e., the four quark species are not degenerate in mass for $a \neq 0$)

Won’t worry about this now:

- There is evidence that this isn’t a problem
- *Lots* of people trying to figure out if it is/isn’t a problem (e.g., earlier part of this workshop)
- “Fourth-root” can be taken into account in chiral perturbation theory with staggered quarks.
Apply Feynman rules, take external $q^2 \to 0$, go to Euclidean space, and performing angular rotations, we get

$$a^{(1)}_{\mu} = \frac{\alpha}{\pi} \int_0^\infty dK^2 f(K^2)$$

$f(K^2)$ is a known function of $K^2$ and $m^2_{\mu}$

Integral is finite and gives precisely

$$a^{(1)}_{\mu} = \frac{\alpha}{2\pi}$$

Lot of work for something we already know...
We want to insert the quark loop into the vacuum polarization:

We can apply this procedure to the $O(\alpha^2)$ hadronic contribution to $a_\mu$ to get (Blum, 2003)

\[
a_{\mu}^{(2)\text{had}, \text{LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2 f(K^2) \hat{\Pi}(K^2)
\]

\[
\hat{\Pi}(K^2) = 4\pi^2 \sum_i Q_i^2 [\Pi_i(K^2) - \Pi_i(0)]
\]
Leading Hadronic Contribution

So now we just need to evaluate $\Pi(q^2)$ on the lattice, and plug it into our expression for $a_\mu$

First some comments about $f(K^2)$:

- $f(K^2) \sim 1/(2m_\mu \sqrt{K^2})$ for small $K^2$
- diverges as $K^2 \to 0 \implies$ dominated by low momentum region
- Need large lattices to reach these low momenta
Lattice Calculation of $\Pi^{\mu\nu}$

Calculate the vacuum polarization using the conserved current

$$\Pi^{\mu\nu}(q) = \int d^4 x e^{i q \cdot (x-y)} \langle J^\mu(x) J^\nu(y) \rangle = (q^2 g^{\mu\nu} - q^\mu q^\nu) \Pi(q^2)$$

Continuum $J^\mu$ satisfies $\partial_\mu J^\mu = 0$:

$$J^\mu = \bar{\psi} \gamma^\mu \psi$$

On the lattice this is a point-split current:

$$J_\mu(x) = \frac{1}{2} \left[ \bar{\psi}(x + a\hat{\mu}) U_\mu(x)(1 + \gamma^\mu) \psi(x) - \bar{\psi}(x) U_\mu(x)(1 - \gamma^\mu) \psi(x + a\hat{\mu}) \right]$$

Satisfies

$$\sum_\mu \frac{J_\mu(x) - J_\mu(x - a\mu)}{a} = 0$$
Discrete version satisfies a discrete Ward Identity, so

\[ \Pi^{\mu\nu}(q) = (\hat{q}^2 \delta^{\mu\nu} - \hat{q}^\mu \hat{q}^\nu) \Pi(\hat{q}^2) \]

with

\[ \hat{q}^\mu = \frac{2}{a} \sin \left( \frac{a q^\mu}{2} \right) \]

and

\[ q^\mu = \frac{2\pi n^\mu}{a L_\mu} \]

WI provides strong check on the calculation!
Lattice Calculation of $\Pi^{\mu\nu}$

To perform lattice calculation: Wick contract the quark fields in $\langle J^\mu(x) J^\nu(y) \rangle$, giving two types of contractions:

- Fourier transform to get $\Pi^{\mu\nu}$
- We neglect second contraction (probably suppressed, also very noisy)
- Hard to fit low-$q^2$ region — Also most important part
- For more details on the lattice calculation, see
  - T. Blum, PRL 91 052001, 2003—Quenched Domain-Wall Quarks
  - T. Blum, Confinement 2003 (hep-lat/0310064)—Includes staggered calculations
Simulation parameters

On the lattice, “Full QCD” = 2+1 flavors (c, b, t integrated out):

- 1 “heavy” flavor, the strange quark at physical $m_s$
- 2 light flavors: $m_u = m_d \equiv m_l \gtrsim m_s / 10$ (can’t yet simulate at “real” $m_u$ or $m_d$)

These are “Improved staggered” configurations (so we have smaller lattice spacing errors)

MILC 2+1-flavor Configurations

<table>
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<th>$a$ (fm)</th>
<th>Volume</th>
<th>$a m_l$</th>
<th>$a m_s$</th>
<th>$a m_{val}$</th>
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<td>0.086(2)</td>
<td>$28^3 \times 96$</td>
<td>0.0124</td>
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<td>0.031</td>
</tr>
</tbody>
</table>
Simulation Results \((2 + 1\) Staggered\)

\[
\Pi(q^2) = \sum a_m \delta(q^2 - m^2)
\]

- \(a_m = 0.0031\)
- \(a_m = 0.0062\)
- \(a_m = 0.0124\)

Exploration of Hadron Structure and Spectroscopy using Lattice QCD, INT, Seattle – p.25
Fitting $\Pi(q^2)$

- **High-$q^2$** easy: Use continuum PT

- **Low-$q^2$** is tough:
  - Simple polynomials? These undershoot the data for lowest $q^2$
  - Physics-based models, like Chiral Perturbation Theory ($\chi$PT)?
    - $\chi$PT is an expansion in mass/energy of pions
    - Since it’s good for low-energy processes, could work here, for the low-$q^2$ region
As $m_q \to 0$ ($q = u, d, s$), QCD has an $SU(3)_L \times SU(3)_R$ chiral symmetry.

$$q_L \to Lq_L, \quad q_R \to Rq_R$$

$SU(3)_L \times SU(3)_R \to SU(3)_V$ by a nonvanishing quark condensate

$$\langle \bar{q}_R q_L \rangle \neq 0$$

$\Rightarrow$ 8 massless bosons: $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$

Put the pions in the field $\Sigma$ ($\Sigma \to L\Sigma R^\dagger$ under the chiral symmetry)

To leading order in the pion momentum

$$\mathcal{L}_{\text{kin}} \propto \text{Tr}[\partial_\mu \Sigma \partial^\mu \Sigma^\dagger]$$
Mass in $\chi$PT

We know the pions are not massless, and neither are the light quarks.

Mass term in QCD looks like

$$\mathcal{L}_{\text{QCD},m} = \bar{q}_L M q_R + \bar{q}_R M q_L$$

where $M$ is the $3 \times 3$ light quark mass matrix.

Mass term in $\chi$PT should transform like the QCD mass term, so we have

$$\mathcal{L}_{\text{mass}} \propto \text{Tr}[M \Sigma + \Sigma^\dagger M]$$
Staggered $\chi$PT for 3 light flavors

Lee & Sharpe, PRD 60, 114503; CA & Bernard, PRD 68 034014 & 074011

- Light mesons: $\Sigma = \exp(i\Phi/f)$, with

$$\Phi = \begin{pmatrix}
U & \pi^+ & K^+ \\
\pi^- & D & K^0 \\
K^- & \bar{K}^0 & S
\end{pmatrix}$$

- Components above are $4 \times 4$ matrices.
- Under chiral $SU(12)_L \times SU(12)_R$: $\Sigma \to L\Sigma R^\dagger$
- $\mathcal{L}$ is an expansion in
  - $m^2_\pi \sim m_q$; $m_q$ is a light quark mass
  - $a^2$, the lattice spacing
Staggered $\chi PT$

$$\mathcal{L} = \frac{f^2}{8} \text{Tr}[\partial_\mu \Sigma \partial^\mu \Sigma] + \frac{\mu f^2}{4} \text{Tr}[\mathcal{M}(\Sigma + \Sigma^\dagger)] - a^2 \mathcal{V}_\Sigma$$

- $\mathcal{M}$: Light quark mass matrix
- $\mathcal{V}_\Sigma$: Taste-breaking potential arising from four-quark operators.
- $f$: tree-level pion decay constant
- For each pion: 16 tastes [in degenerate $SO(4)$ representations: $P, A, T, V, S$] with masses:
  $$m^2_t = \mu (m_a + m_b) + a^2 \Delta_t, \quad (t = P, A, T, V, S)$$
- Taste violations at finite lattice spacing $\Rightarrow \Delta_t \neq 0$
- Remnant chiral symmetry $\Rightarrow \Delta_P = 0$
- To include photons:
  $$\partial_\mu \Sigma \rightarrow \partial_\mu \Sigma + ie A_\mu [Q, \Sigma]$$
One-loop pion contribution:

\[ \Pi_M(q^2) = \frac{\alpha}{4\pi} \left\{ \frac{1}{3} (1 + x_M)^{3/2} \ln \left( \frac{\sqrt{1 + x_M} + 1}{\sqrt{1 + x_M} - 1} \right) - \frac{2x_M}{3} - \frac{8}{9} + \frac{1}{3} \ln \left( \frac{m_M^2}{\Lambda^2} \right) \right\} \]

\[ \Pi(q^2) = \frac{1}{16} \sum_t \left[ \Pi_{\pi_t}(q^2) + \Pi_{K_t}(q^2) \right] + \text{c. t.} \]

\[ x = 4m^2/q^2 \]

- Sum over \( t \) is a sum over the 16 tastes
- Nice: No free parameters (besides counterterm—this is just a constant)
- Bad: Two orders of magnitude too small!
SXPT with vectors

Without sea quarks (quenched), \( \Pi(q^2) \) is dominated by effects of the \( \rho \) (QCDSF), perhaps they play a role here...

Use resonance formalism of Ecker, Gasser, and Pich [NPB 321 311 (1989)]

Incorporate vectors into field \( V_{\mu\nu} \) so that under chiral \( SU(12)_L \times SU(12)_R \):

\[
V_{\mu\nu} \rightarrow UV_{\mu\nu}U^\dagger
\]

where \( U \in SU(12) \) is defined as

\[
\sigma \rightarrow L\sigma U^\dagger = U\sigma R^\dagger
\]

with \( \sigma^2 = \Sigma \)
SxPT with vectors

So we have the interaction Lagrangian

\[ \mathcal{L}_{\text{vec}} = \frac{f_V}{2\sqrt{2}} \text{Tr} \left[ V_{\mu\nu}(\sigma F^{\mu\nu} \sigma^\dagger + \sigma^\dagger F^{\mu\nu} \sigma) \right] + \ldots \]

\[ F^{\mu\nu} = eQ(\partial^\mu A^\nu - \partial^\nu A^\mu) \]

- \( V_{\mu\nu} \) is a 12 \times 12 matrix with the 8 lightest vector mesons (each with 16 tastes)
- Empirically taste violations among vectors are small–Will ignore them here
- Leading contribution to the photon vacuum polarization is at tree level:
SχPT with $\rho$

Tree-level result:

$$\Pi_V(q^2) = -\frac{\alpha}{4\pi} \left( \frac{(4\pi)^2 f_V^2}{f^2} \right) \left[ \frac{3}{q^2 + m_{\rho}^2} + \frac{1}{q^2 + m_{\omega}^2} \right]$$

Although the masses are heavy, the numerator has enhancement of $(4\pi)^2 f_V^2$.

There are no free parameters: The masses and $f_V$ can be measured directly in the simulations ($f_V$ not measured yet).

One-loop calculation: only tadpole corrections to $\rho$–photon vertex

$$\Pi_V^{1-loop}(q^2) = \frac{\alpha}{4\pi} \left( \frac{4 f_V^2}{f^2} \right) \sum_t \left[ 2 \frac{m_{\pi_t}^2 \ln m_{\pi_t}^2}{q^2 + m_{\rho}^2} + \frac{m_{K_t}^2 \ln m_{K_t}^2}{q^2 + m_{\rho}^2} + \frac{m_{K_t}^2 \ln m_{K_t}^2}{q^2 + m_{\omega}^2} \right]$$
Fit to $\chi$PT+$\rho$ result

- $\Pi(q^2)$ vs $q^2$ (GeV$^2$)

- $\Pi(q^2)$ values for different $am_i$:
  - $am_i = 0.0124$
  - $am_i = 0.0062$
  - $am_i = 0.0031$

Exploration of Hadron Structure and Spectroscopy using Lattice QCD, INT, Seattle – p.35
Preliminary Results

\[
\begin{align*}
    a_{\mu}^{\text{had,VP}}(\infty) &= 367(12) \times 10^{-10} \\
    a_{\mu}^{\text{had,VP}}(0.0124) &= 431(7) \times 10^{-10} \\
    a_{\mu}^{\text{had,VP}}(0.0062) &= 509(14) \times 10^{-10} \\
    a_{\mu}^{\text{had,VP}}(0.0031) &= 636(8) \times 10^{-10} \\
    a_{\mu}^{\text{had,VP,\text{pert}}(\text{phys})} &\lesssim 10 \times 10^{-10} \\
    a_{\mu}^{\text{had,disp}(\text{phys})} &= 693.4(5.3)(3.5) \times 10^{-10}
\end{align*}
\]

- Statistical errors only
- Possibly large uncertainties:
  - Low-\(q^2\): Still undershoots at small mass, although not as much as a simple polynomial fit
  - Last line is from \(e^+e^-\) data and dispersion relation [A. Hocker, ICHEP 2004]
Preliminary Results

How to extrapolate?

- We have $f_V$, $m_V$, and pion/kaon masses all as functions of the light quark mass $m_l \Rightarrow$ Could extrapolate these to physical point $a m_l \approx 0.0001$...

- Extrapolation to physical point: Must go through the $2\pi$ threshold (and $m_V$ is not a linear function of $m_l$ for light quark masses)

- The three values for $a_{\mu}^{\text{had},\text{VP}}$ show significant curvature as a function of $m_l$:
  Quadratic fit?

  Quadratic fit of $a_{\mu}^{\text{had},\text{VP}}$ vs. $m_l$ gives:

  $$a_{\mu}^{\text{had},\text{VP}}(\text{phys}) \approx 755 \times 10^{-10}$$

  (Errors are not shown on purpose!)
Summary

- Haven’t included “disconnected diagrams” in lattice calculation (noisy)

- Functional form from $\chi$PT+$\rho$ fits well to lattice data with few unknown parameters, but not ideal

- Need to understand why fit undershoots data: Bad fitting form or are we missing something?

Issues/Future needs:

- Study possible finite volume problems
- More calculations on different lattice spacings (coarse MILC lattices, extra-fine coming soon)
- Twisted BCs to get more low-$q^2$ points?
- Spline fits instead of phenomenological fit?

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Thanks to MILC for configurations