A variant of overlap fermions with staggered fermion kernel

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Abstract
A variant of overlap fermions with staggered fermion kernel is presented. It aims to combine the exact chiral symmetry of overlap fermions with the advantageous features of staggered fermions for numerical implementation.

A remarkable feature of the construction is that the 4 flavours (tastes) of the staggered fermion get reduced to two flavours for the overlap fermion.

The construction breaks lattice rotation symmetry though, due to a need to choose lattice paths connecting opposite corners of lattice hypercubes. We discuss why the need to make such choices seems inevitable in any attempt to sharply improve the staggered fermion formulation.
Recall usual construction of overlap fermions

Insert massive Wilson-Dirac operator \((\alpha \alpha)\)

\[
D_\alpha = \bar{\psi}_\alpha D_\alpha + \frac{\alpha}{2} \Delta
\]

into overlap formula [Neuburger, PLB 1998]

\[
D_\alpha = 1 + (D_\alpha - m)^{-1}/(D_\alpha - m)^2
\]

\(D_\alpha\) satisfies GW relation \(\bar{\psi}_\alpha D_\alpha + D_\alpha \psi = \alpha D_\alpha \bar{\psi}_\alpha \psi\)

an exact lattice-deformed chiral symmetry.

The overlap formula "projects" the spectrum of \(D_\alpha\) onto GW circle:

- choose \(m \geq 2\) then no species doubling
- low-lying real eigenmodes of \(D_\alpha\) → exact zero modes of \(D_\alpha\) with definite chirality
- index \(D_\alpha\) well-defined. Index theorem:
  \[
  \text{index } D_\alpha = \chi
  \]

has been checked numerically & analytically.
Another perspective on $D_W$ and its index

Introduce Hermitian Wilson-Dyson operator

$$H_W(m) = \delta_S (D_W - m)$$

$H_W(m)$ has zero-mode $\rightarrow m$ is real eigenvalue of $D_W$

$\rightarrow$ Can associate an integer index to the low-lying real eigenvalues of $D_W$ from the spectral flow of $H_W(m)$ near $m = 0$:

\[
\text{index } D_W = n_+ - n_-
\]

$n_+ = \# \text{ crossings near } 0 \text{ with slope } \pm 1$

[Itak et al., PRB 1989]

This coincides with usual index in continuum:

\[
\begin{align*}
\psi = 0 & \Rightarrow H(m) \psi = \chi(m) (\beta - m) \psi = 0 \Rightarrow \\
\chi(m) = \pm m
\end{align*}
\]

Can express overlap operator as

$$D_{ov} = 1 + \frac{\chi(m)}{\sqrt{H_W(m)^2}}$$

Can show: index $D_{ov} = \text{"index } D_W\text{" as defined above}
Construction of overlap fermions with staggered kernel.

1. The direct approach fails. To see this, recall 4 species staggered fermions vs. 2 species naive fermions.

2. Consider overlap construction with naive kernel.

Naive Dirac operator:

$D_{naive} = \gamma^\mu \not{D}_\mu$, purely imaginary spectrum.

Spectrum of overlap Dirac op. with naive kernel:

$\Rightarrow D_{ov}$ has same boson modes as $D_{naive}$.

$\Rightarrow D_{ov}$ has same fermion doubling as $D_{naive}$.

More significantly,

index $D_{ov} = index D_{naive} \equiv 0$ always vanishes on general grounds.

$\Rightarrow$ This construction fails. Shows crucial role of Wilson term in usual overlap construction.
2) An indirect approach succeeds:

Usual staggered fermion action is \((a_{01}, a_{03})\)

\[
S_{st} = \sum_x \bar{\psi}(x) \gamma_\mu \psi(x) \nabla_\mu \%
\]

\(\gamma_\mu(x) = \delta(1)^{x_1 \cdots x_4}\), \(\nabla_\mu\) : Dirac covariant derivative operator

\(\Rightarrow\) Staggered Dirac operator is

\[
D_\mu = \gamma_\mu \nabla_\mu \nabla_\mu + D_\mu^+ = -D_\mu \Rightarrow\) purely imaginary spectrum

If we try to construct overlap fermion by replacing

\[
D_\mu \rightarrow D_\mu^+ \Rightarrow \exists ~ \Phi \in \Psi_{a_{01}, a_{03}}
\]

in overlap construction, it will fail just like in case of naive fermion kernel.

\(\Rightarrow\) Another approach is needed. To make progress, a better understanding of chirality for staggered fermions is needed. Can get this by considering the flavor (taste) field representation of staggered fermions. The insights obtained then need to be carried back to the usual staggered formulation.
Flavor (octet) field rep. of staggered fermions

\[ x: \text{site on original lattice} \]
\[ y: \text{site on blocked lattice} \]
\[ x = 2y + s, \quad s, y \in \{0, 1\} \]
\[ X(x) = X(2y + s) = \hat{x}_y(y) \]

Choose Dirac gamma matrices $\{\hat{\gamma}_\mu\}$
\[ \hat{\gamma}_\mu \hat{\gamma}_\nu + \hat{\gamma}_\nu \hat{\gamma}_\mu = 2\delta_{\mu\nu}, \quad \hat{\gamma}_0 = \hat{x}_0, \quad \hat{x}_5 = 2\hat{x}_0 \hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_4 \]

Define $\hat{\Gamma}_5 = \hat{x}_0, \ldots \hat{x}_4$

and set
\[ \Phi^{\mu \gamma}(y) = \frac{1}{2} \frac{\delta}{\delta x^\mu} \langle x \gamma \rangle \]
\[ \Phi^{\mu \gamma}(y) = \frac{1}{2} \frac{\delta}{\delta \hat{x}^\mu} \langle \hat{x} \gamma \rangle \]

Then, in free field case,
\[ S_{\text{SF}} = \frac{\lambda}{2} \langle x \rangle \hat{\gamma}_\mu(x) \hat{\gamma}_\nu(x) \nabla \hat{\gamma}_\mu \hat{\gamma}_\nu \]

when
\[ \hat{\gamma}_\mu = \text{symm. Ante dirp. op.} \] on blocked lattice
\[ \nabla = \text{Laplace operator} \]
and $\hat{x}^\mu$ rep of Dirac alg. in flavour space
The staggered Dirac operator in the three field rep is
\[ D_{\text{st}} = (\gamma_5 \otimes 1) \tilde{D}_\mu + (\gamma_5 \otimes i \tau_3 \gamma_2) \tilde{D}_5 \]
Can be re-expressed as
\[ D_{\text{st}} = (\gamma_5 \otimes 1) \tilde{D}_\mu + i (\gamma_5 \otimes \tau_3 \gamma_2) \tilde{D}_5 \]
where \( \tilde{D}_\mu \) is another rep of Dirac alg:
\[ \tilde{D}_\mu \tilde{D}_\nu + \tilde{D}_\nu \tilde{D}_\mu = 2 \delta_{\mu\nu}, \quad \tilde{D}_\mu^\dagger = \tilde{D}_\mu \]
Clearly the appropriate chirality matrix in this rep is \( \gamma_5 \otimes 1 \)
- Note that \( \gamma_5 \) in original staggered fermion rep corresponds to \( \gamma_5 \otimes 1 \) rather than \( \gamma_5 \otimes \gamma_5 \)
- Should not use \( \gamma_5 \) as chirality matrix in original rep; instead should use operator corresponding to \( \gamma_5 \otimes 1 \)
- Note that \( D_{\text{st}} \) in (a) has form of Wilson-Dirac operator, except that the Wilson term \( i (\gamma_5 \otimes \tau_3 \gamma_2) \tilde{D}_5 \) is purely imaginary.
- Need to rewrite this stuff in such a way that a new (i.e. Hermitian) "Wilson term" arises. Will do this in following.
Consider \( D_{\mathbf{s}} \) with "twisted" mass term \( \Sigma \)
\[
\Sigma = -i \left( \chi \gamma \right) m
\]

Then
\[
-i(\partial_{\mu} + M) = -i \left( \chi \gamma_{\mu} \right) \chi + i \left( \chi \gamma \right) \gamma_{\mu} \gamma_{\nu} - i \left( \chi \right) m
\]
\[
= \chi \left[ (-i \gamma_{\mu} \gamma_{\nu} \gamma_{\mu} \gamma_{\nu} - (i \gamma_{\mu}) m \right]
\]

Define \( \chi_{\mu} = -i \chi \gamma_{\mu} \)

This is a new rep of Dirac algebra with \( \chi_{\nu} = \chi \)

Define the Hermitian operator
\[
\Delta = \chi \gamma_{\mu} \Delta_{\mu}
\]

Then
\[
-i(\partial_{\mu} + M) = \chi \left[ \chi_{\mu} \gamma_{\mu} + \frac{i}{2} \Delta - m \right]
\]

This is a Hermitian operator analogous to the Hermitian Wilson-Diak operator,
\[
H_{\text{Diak}}(m) = \chi \left[ \chi_{\mu} \gamma_{\mu} + \frac{i}{2} \Delta - m \right]
\]

As the Hermitian staggered Dirac op we use
\[
H_{\text{Stag}}(m) = -i \left( D_{\mathbf{s}} + i M \right) = -i D_{\mathbf{s}} - i \left( \gamma \right) m
\]

Note that \( H_{\text{Stag}}(m) \) is related to \( D_{\mathbf{s}} \) in a quite different way than Wilson case when
\[
H_{\text{Wilson}}(m) = \chi \left( D_{\mathbf{s}} - m \right)
\]
We have hereby determined appropriate staggered fermion analogues of $Y_5$ and $H_{56}(m)$ in the 
free taste field representation:

\[ Y_5 \to Y_5 \otimes 1 \]

\[ H_{56}(m) = -i \, D_{56} \, - (X_{56})_{m} \]

One can construct free field overlap operator with staggered fermion kernel by making 
these replacements in the overlap formula

\[ D_{ov} = 1 + Y_5 \, \frac{H_{56}(m)}{D_{56}(m)} \]

We denote the resulting operator by $D_{ov}$. Following issues now need to be addressed:

(i) Translate $Y_5 \otimes 1$, $H_{56}(m)$, and thereby $D_{ov}$ into the usual staggered fermion rep., i.e., as operators on $\mathbb{R}^Z$ on original lattice, and couple these operators to the 
link variables in a suitable way.

(This is required to get an overlap Dirac op., of interest for numerical implementation.)

(ii) Determine suitable range for $m$ in $H_{56}(m)$. Check that index $D_{ov}$ satisfies Index Theorem.
Recalling
\[ v(y) = \frac{1}{3} \sum_{k} (\Gamma_{5})_{ab} \chi_{k}(y), \quad \Gamma_{5} = \chi_{0} \ldots \chi_{N} \]
and
\[ \Phi(y) = \frac{1}{2} \sum_{k} \chi_{k}(y)(\Gamma_{5})_{ab} \]
one finds
\[ \Phi(\Gamma_{5} g) = e^{ig} \frac{1}{2} \text{tr} \left( \Gamma_{5} \chi_{5} \Gamma_{5} \right) \chi_{5} \]
\[ \rightarrow \Gamma_{5} \] on \( \chi(x) = \chi_{g}(y), \quad x = 2y + g \)
given by \( (\Gamma_{5})_{gg'} = \frac{1}{2} \text{tr} \left( \Gamma_{5} \Gamma_{5} \right) \)
Explicit calculation leads to
\[ (\Gamma_{5})_{gg'} = e^{ig} g_{g} g' \]
where \( g \) denotes opposite corner of hypercube to \( g. \)

E.g.
\[ g = (0,1,1,0) \rightarrow \bar{g} = (1,1,0,1) \]
\[ g = (0,1,0,0) \rightarrow \bar{g} = (1,0,0,1) \]
\( \phi \) couples opposite corners of hypercube:

\[
\phi_2 \cdot \phi_3 = (-1)^{\phi_2 + \phi_3} \phi_2 \phi_3
\]

- \( \phi \) couples different sites, it needs to be gauged (i.e., coupled with link variables) to maintain gauge covariance.
- Need to choose lattice paths connecting opposite corners of hypercube.

E.g.,

\( \phi \) is then gauged in obvious way by taking products of link variables along paths. This breaks lattice rotation symmetry.

- Raises renormalizability issue.

E.g., no guarantee that the divergences in

\( \phi \) can be canceled by tuning parameters of bare action (needs to be checked explicitly).
Thus in the full interacting case we now have an appropriate version \( \Phi \) of \( \Phi \equiv \Phi \) and Hermitian staggered Dirac operator

\[ H_{\Phi}(\eta) = -i \partial_\eta v - m \Phi \]

acting on the original staggered fermion field \( \Phi \).

Constant corresponding overlap Dirac operator,

\[ D_{\text{over}}(\eta) = 1 + i \Phi \frac{H_{\Phi}(\eta)}{\sqrt{H_{\Phi}(\eta)}} \]

\[ = 1 + (i \Phi \partial_\eta v - m) \frac{1}{\sqrt{(-i \partial_\eta v - m \Phi)}} \]

General remark: the necessity of having to choose lattice paths inside the hypercubes (used in the construction of \( \Phi \) in the interacting case) appears inevitable for any attempt to "chirally improve" the staggered fermion formulation. Such an attempt must involve the appropriate chirality matrix, which, as we have seen, is \( \Phi \) (corresponding to \( \Phi \equiv \Phi \) in the taste field rep.).
Remains to resolve issue (ii) regarding appropriate range of $m$ and whether index Bozor set should Index Theorem.

For this, return to previous expression for $H_{sp}(m)$ on free field fields:

$$H_{sp}(m) = \frac{1}{2} \left[ (\frac{1}{2} \sigma \cdot \partial) \bar{\phi} + \frac{1}{2} \delta - m \right]$$

$$\equiv \frac{1}{2} \left[ \delta - m \right]$$

where

$$\bar{\delta} = (\bar{\phi}, \bar{\sigma} \cdot \bar{\partial} + \frac{i}{2} \bar{\delta})$$  "Wilson-type" Dirac. up.

$$\bar{\delta} = \bar{\phi} \bar{\sigma} \bar{\partial}$$  Hermitian but not positive.

$$\bar{\Delta} \psi = \lambda \psi \Rightarrow \bar{\Delta} (t_3 \psi) = -\lambda (t_3 \psi)$$

$$\bar{\Delta}^2 = \bar{\sigma} \bar{\delta} \bar{\sigma}$$

The real eigenvalues of $\bar{\delta}$ correspond to momentum eigenstates with momenta at the corners of Brillouin zone:

$$\pm \sqrt{\frac{1}{2}} \vec{p}$$

$$\vec{p} = \{ 0, \pi \}, \vec{p} = \{ 2 \pi, \pi \}$$

$$\Rightarrow$$ The real eigenvalues of $\bar{\delta}$ are

$$\pm \sqrt{\frac{1}{2} \frac{\pi}{\sigma}}$$

(In usual Wilson case, real eigenvalues are $\pm \frac{\pi}{\sigma}$)
Recall that the value of $m$ in $H_\nu(m) = \psi(\nu - m)$ determines which real eigenmodes are preserved in the overlap construction. The appropriate range for $m$ in this case is $-4 < m < 2\sqrt{5}$.

Then the overlap projection of the spectrum of $B$ is as indicated:

In particular, the corresponding overlap operator $D_{\nu,\psi}$ has $2\nu$ zero modes, i.e., the index is reduced from 4 to 2.
The eigenvalues \( \lambda_n \) of \( H_\phi(m) \) cross the origin at precisely the values of \( m \) which are real eigenvalues of \( B \). In the interesting case, we expect that the eigenvalues of \( H_\phi(m) \) will continue to occur at values of \( m \) close to the free field ones when the background gauge field is reasonably smooth. (This is the experience from the Wilson case.) By general argument get

\[
\text{index } D_\phi(m) = \text{index of the eigenvalue crossings of } H_\phi(m) \\
\text{for } m^2 < m_0.
\]

Since \( D_\phi(m) \) has two states, index theorem is

\[
\text{index } D_\phi(m) = 2 \alpha \quad (\alpha = \text{top. charge of gauge field})
\]

This should be checked in two ways:

1. Numerically study spectral flow of \( H_\phi(m) \) in smooth gauge backgrounds with top. charge \( \alpha \) to confirm index theorem.

2. Analytically evaluate the axial anomaly (top charge density) in \( a \to 0 \) limit in presence of smooth gauge field. Should reduce to \( 2 \times \) usual anomaly. (These checks are currently underway... )
Summary

Can construct viable version of overlap Dirac operator with staggered fermion kernel by making following replacements in usual overlap construction:

\[ \gamma_5 \rightarrow \Gamma_5 \]

\[ H_{sp}(m) = \frac{1}{2} D_{2\mu} m \Gamma_5, \quad m \in \{3, -3\} \]

where \( D_{2\mu} = \gamma_{2\mu} \gamma_5 \) is usual massless staggered Dirac operator and \( \Gamma_5 \) given by (for \( x = 2y + z \))

\[ (\Gamma_5 \gamma_5)_L (x) = (-1)^{x^2} U(1; 1, 1, 1) \gamma_5 (x) \]

i.e. \( \Gamma_5 \) couples opposite sites of hypercube:

\[ \begin{array}{cccc}
  & & x & \\
  & x & & \\
 x & & & \\
 & & & x
\end{array} \]

and \( U(1; 1, 1, 1) \) is product of link variables along path from \( x \) to \( 3y + z \) to \( x = 2y + z \) specified by choice of path from \( x \) to \( 3y + z \) in standard hypercube.

Remarkable property: \( \gamma_5 \) factors reduced from 4 to 2 for resulting overlap operator \( D_{2\mu} m \Gamma_5 \). However, choice of paths = breaking of lattice symm. \( \Rightarrow \) renormalizability issue (needs to be clarified).