A variant of overlap fermions with staggered fermion kernel

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Abstract

A variant of overlap fermions with staggered fermion kernel is presented. It aims to combine the exact chiral symmetry of overlap fermions with the advantageous features of staggered fermions for numerical implementation.

- A remarkable feature of the construction is that the 4 flavours (tastes) of the staggered fermion get reduced to two flavours for the overlap fermion
- The construction breaks lattice rotation symmetry though, due to a need to choose lattice paths connecting opposite corners of lattice hypercubes. We discuss why the need to make such choices seems inevitable in any attempt to chirally improve the staggered fermion formulation.

Recall usual construction of overlap fermions

Insert massless Wilson-Dirac operator ($a=1$)

$$D_W = \gamma_\mu \nabla_\mu + \tfrac{1}{2}\Delta$$

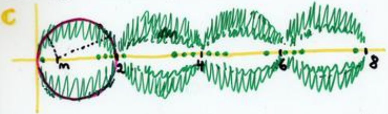
into overlap formula [Neuberger, PLB 1998]

$$D_{ov} = 1 + (D_W - m)\frac{1}{\sqrt{(D_W-m)^\dagger (D_W-m)}}$$

D_{ov} satisfies GW relation $\gamma_5 D_{ov} + D_{ov}\gamma_5 = a D_{ov}\gamma_5 D_{ov}$

$\Rightarrow$ exact lattice-deformed chiral symm.

The overlap formula "projects" the spectrum of D_W onto GW circle:



$\therefore$ choose $m \in]0,2[$ then no species doubling

low-lying real eigenmodes of D_W $\rightarrow$ exact zero-modes of D_{ov} with definite chirality

$\rightarrow$ index D_{ov} well-defined. Index theorem:

$$\text{index } D_{ov} = Q$$

has been checked numerically & analytically.

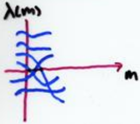
Another perspective on D_{ov} and its index

Introduce Hermitian Wilson-Dirac operator

$$H_W(m) = \gamma_5 (D_W - m)$$

$H_W(m)$ has zero-mode $\leftrightarrow$ m is real eigenvalue of D_W

$\rightarrow$ Can associate an integer "index" to the low-lying real eigenmodes of D_W from the spectral flow of $H_W(m)$ near $m \approx 0$:



"index D_W" $= n_- - n_+$

$n_\pm$ = # crossings near 0 with slope $\pm$

[Itoh et al., PRD 1987]

This coincides with usual index in continuum:

$$\left.\begin{array}{l} \not{D}\psi = 0 \\ \gamma_5\psi = \pm\psi \end{array}\right\} \Rightarrow H(m)\psi = \gamma_5(\not{D} - m)\psi = \mp m\psi$$

$$\rightarrow \lambda(m) = \mp m$$

Can express overlap operator as

$$D_{ov} = 1 + \gamma_5 \frac{H_W(m)}{\sqrt{H_W(m)^2}}$$

Can show: index D_{ov} = "index D_W" as defined above

Construction of overlap fermions with staggered kernel

i) The direct approach fails. To see this, recall

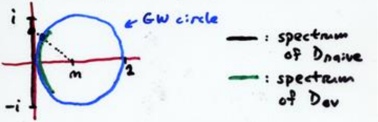
4 species staggered fermions $\simeq$ 1 species naive fermion

$\therefore$ Consider overlap construction with naive kernel

Naive Dirac operator:

$D_{naive} = \gamma_\mu \nabla_\mu$, purely imaginary spectrum

Spectrum of overlap Dirac op. with naive kernel:



$\Rightarrow$ D_{ov} has <u>same zero-modes</u> as D_{naive}

$\rightarrow$ D_{ov} has same fermion doubling as D_{naive}

More significantly,

$$\text{index } D_{ov} = \underbrace{\text{index } D_{naive}}_{\text{always vanishes on general grounds}} \equiv 0$$

$\therefore$ This construction fails. Shows crucial role of Wilson term in usual overlap construction

2) An indirect approach succeeds :

Usual staggered fermion action is ($a=1$, $m=0$)

$$S_{st} = \sum_x \bar{\chi}(x)\, \eta_\mu(x) \nabla_\mu \chi(x)$$

$\eta_\mu(x) = (-1)^{x_1+\ldots+x_{\mu-1}}$, ∇_μ : symm. cov. finite diff operator

→ Staggered Dirac operator is

$$D_{st} = \eta_\mu \nabla_\mu$$

$D_{st}^\dagger = -D_{st}$ ⇒ purely imaginary spectrum

If we try to construct overlap fermion by replacing

$$D_W \rightarrow D_{st}$$
$$\gamma_5 \rightarrow \eta_5 = \eta_1 \eta_2 \eta_3 \eta_4$$

in overlap construction, it will fail just like in case of naive fermion kernel.

∴ Another approach is needed. To make progress, a better understanding of "chirality" for staggered fermions is needed. Can get this by considering the flavor (taste) field representation of staggered fermions. The insights obtained then need to be carried back to the usual Staggered formulation

Flavor (taste) field rep. of staggered fermions

x : site on original lattice
y : site on blocked lattice

$$x = 2y + \rho \quad , \quad \rho_\mu \in \{0,1\}$$

$$\chi(x) = \chi(2y+\rho) \equiv \chi_\rho(y)$$

Choose Dirac gamma matrices $\{\gamma_\mu\}$

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu} \; , \; \gamma_\mu^\dagger = \gamma_\mu \; , \; \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$$

Define

$$\Gamma_\rho = \gamma_1^{\rho_1} \cdots \gamma_4^{\rho_4}$$

and set

$$\psi_{\alpha f}(y) = \tfrac{1}{2} \sum_\rho (\Gamma_\rho)_{\alpha f} \, \chi_\rho(y)$$

$$\bar\psi_{\alpha f}(y) = \tfrac{1}{2} \sum_\rho \bar\chi_\rho(y) (\Gamma_\rho^*)_{\alpha f}$$

Then, in free field case,

$$S_{sf} = \sum_x \bar\chi(x) \eta_\mu(x) \nabla_\mu \chi(x)$$

$$= \frac{1}{2^4} \boxed{\sum_y \bar\psi(y) \left[(\gamma_\mu \otimes 1) \tilde\nabla_\mu + (\gamma_5 \otimes t_\nu t_5) \tfrac{1}{2} \hat\Delta_\nu \right] \psi(y)}$$

where

$$\left.\begin{aligned} \tilde\nabla_\mu &= \text{symm. finite diff. op.} \\ \sum_\nu \tilde\Delta_\nu &= \text{Laplace operator} \end{aligned}\right\} \text{ on blocked lattice}$$

and $t_\nu = \gamma_\nu^*$ rep of Dirac alg. in flavour space

The staggered Dirac operator in the taste field rep is

$$D_{st} = (\gamma_\mu \otimes 1)\tilde{\nabla}_\mu + (\gamma_5 \otimes t_\nu t_5)\tfrac{1}{2}\tilde{\Delta}_\nu$$

Can be re-expressed as

$$D_{st} = (\gamma_\mu \otimes 1)\nabla_\mu + i(\gamma_5 \otimes \tilde{t}_\nu)\tfrac{1}{2}\tilde{\Delta}_\nu \quad (*)$$

where $\tilde{t}_\nu = -i t_\nu t_5$ is another rep of Dirac alg:

$$\tilde{t}_\mu \tilde{t}_\nu + \tilde{t}_\nu \tilde{t}_\mu = 2\delta_{\mu\nu}\,, \quad \tilde{t}_\nu^{\dagger} = \tilde{t}_\nu$$

Clearly the appropriate chirality matrix in this rep is $\gamma_5 \otimes 1$

- Note that η_5 in original staggered fermion rep corresponds to $\gamma_5 \otimes \tilde{t}_5$ rather than $\gamma_5 \otimes 1$

$\therefore$ Should not use η_5 as chirality matrix in original rep; instead should use operator corresponding to $\gamma_5 \otimes 1$

- Note that D_{st} in $(*)$ has form of Wilson-Dirac operator, except that the Wilson term $i(\gamma_5 \otimes \tilde{t}_\nu)\frac{1}{2}\tilde{\Delta}_\nu$ is purely imaginary.

$\therefore$ Need to rewrite this stuff in such a way that a real (i.e. Hermitian) "Wilson term" arises. Will do this in following.

Consider D_{st} with "twisted" mass term 8

$$\mathcal{M} = -i(\gamma_5 \otimes 1)m$$

Then

$$-i(D_{st} + \mathcal{M}) = -i\left[(\gamma_\mu \otimes 1)\tilde{\nabla}_\mu + i(\gamma_5 \otimes \tilde{t}_\nu)\tfrac{1}{2}\hat{\Delta}_\nu - i(\gamma_5 \otimes 1)m\right]$$

$$= \gamma_5\left[(-i\gamma_5\gamma_\mu \otimes 1)\tilde{\nabla}_\mu + (1 \otimes \tilde{t}_\nu)\tfrac{1}{2}\hat{\Delta}_\nu - (1 \otimes 1)m\right]$$

Define

$$\tilde{\gamma}_\mu = -i\gamma_5\gamma_\mu$$

This is a new rep of Dirac algebra with $\tilde{\gamma}_5 = \gamma_5$

Define the Hermitian operator

$$\tilde{\Delta} = 1 \otimes \tilde{t}_\nu \Delta_\nu$$

Then

$$-i(D_{st} + \mathcal{M}) = \tilde{\gamma}_5\left[(\tilde{\gamma}_\mu \otimes 1)\nabla_\mu + \tfrac{1}{2}\tilde{\Delta} - m\right]$$

This is a Hermitian operator analogous to the Hermitian Wilson-Dirac operator,

$$H_W(m) = \gamma_5\left[\gamma_\mu\nabla_\mu + \tfrac{1}{2}\Delta - m\right]$$

$\therefore$ As the Hermitian staggered Dirac op we use

$$\boxed{H_{st}(m) \equiv -i(D_{st} + \mathcal{M}) = -iD_{st} - (\gamma_5 \otimes 1)m}$$

Note that $H_{st}(m)$ is related to D_{st} in a quite different way than Wilson case where $H_W(m) = \gamma_5(D_W - m)$

We have hereby determined appropriate staggered fermion analogues of γ_5 and $H_W(m)$ in the free taste field representation:

$$\gamma_5 \rightarrow \gamma_5 \otimes 1$$

$$H_W(m) \rightarrow H_{st}(m) = -i D_{st} - (\gamma_5 \otimes 1) m$$

$\therefore$ Can construct free field overlap operator with staggered fermion kernel by making these replacements in the overlap formula

$$D_{ov} = 1 + \gamma_5 \frac{H(m)}{\sqrt{H(m)^2}}$$

Will denote the resulting operator by $D_{ov(st)}$

Following issues now need to be addressed:

(i) Translate $\gamma_5 \otimes 1$, $H_{st}(m)$, and thereby $D_{ov(st)}$ into the usual staggered fermion rep., i.e. as operators on $\chi(x)$ on original lattice, and couple those operators to the link variables in a suitable way.
(This is required to get an overlap Dirac op. of interest for numerical implementation.)

(ii) Determine suitable range for m in $H_{st}(m)$. Check that index $D_{ov(st)}$ satisfies Index Theorem.

Re (i): It suffices to translate $\gamma_5 \otimes 1$ into operator on $\chi(x)$ on original lattice, since

$$H_{st}(m) = -i D_{st} - (\gamma_5 \otimes 1) m$$

and D_{st} on $\chi(x)$ is just given by usual expression:

$$D_{st} = \eta_\mu \nabla_\mu$$

Recalling

$$\psi_{\alpha f}(y) = \tfrac{1}{2} \sum_{\xi} (\Gamma_\xi)_{\alpha f}\, \chi_\xi(y) \,, \quad \Gamma_\xi = \gamma_1^{\xi_1} \cdots \gamma_4^{\xi_4}$$

$$\bar{\psi}_{\alpha f}(y) = \tfrac{1}{2} \sum_{\xi} \bar{\chi}_\xi(y) (\Gamma_\xi^*)_{\alpha f}$$

one finds

$$\bar{\psi} (\gamma_5 \otimes 1) \psi = \bar{\chi}_\xi \, \tfrac{1}{4} \mathrm{tr}(\Gamma_\xi^\dagger \gamma_5 \Gamma_{\xi'}) \chi_{\xi'}$$

$\therefore \gamma_5 \otimes 1 \longrightarrow \Gamma_5$ on $\chi(x) = \chi_\xi(y)$, $x = 2y + \xi$

given by $(\Gamma_5)_{\xi \xi'} = \tfrac{1}{4} \mathrm{tr}(\Gamma_\xi^\dagger \gamma_5 \Gamma_\xi)$

Explicit calculation leads to

$$\boxed{(\Gamma_5)_{\xi\xi'} = (-1)^{\xi_1 + \xi_3} \delta_{\bar{\xi}\xi'}}$$

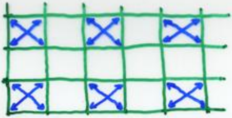
where $\bar{\xi}$ denotes opposite corner of hypercube to ξ.

E.g. $\xi = (0,0,0,0) \rightarrow \bar{\xi} = (1,1,1,1)$

$\xi = (0,1,1,0) \rightarrow \bar{\xi} = (1,0,0,1)$

$$\Rightarrow \quad (\Gamma_5 \chi)_\rho(y) = (-1)^{\rho_1+\rho_3}\, \chi_{\bar\rho}(y)$$

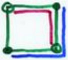
I.e. Γ_5 couples opposite corners of hypercubes:



Since Γ_5 couples different sites, it needs to be gauged (i.e. coupled to link variables) to maintain gauge covariance.

$\Rightarrow$ Need to choose lattice paths connecting opposite corners of hypercube

E.g.



Γ_5 is then gauged in obvious way by taking products of link variables along paths.
This breaks lattice rotation symmetry
$\rightarrow$ raises renormalisability issue:
E.g. no guarantee that the divergencies in

can be canceled by tuning parameters of bare action (needs to be checked explicitly)

Thus in the full interacting case we now have an appropriate version Γ_5 of $\gamma_5 \otimes 1$ and Hermitian staggered Dirac operator

$$H_{st}(m) = -i D_{st} - m \Gamma_5$$

acting on the original staggered fermion field $\chi(x)$.

$\Rightarrow$ Construct corresponding overlap Dirac operator

$$D_{ov(st)} = 1 + \Gamma_5 \frac{H_{st}(m)}{\sqrt{H_{st}(m)^2}}$$

$$= 1 + (-i\Gamma_5 D_{st} - m) \frac{1}{\sqrt{(-iD_{st} - m\Gamma_5)^2}}$$

General remark: the necessity of having to choose lattice paths inside the hypercube (used in the construction of Γ_5 in the interacting case) appears inevitable for any attempt to "chirally improve" the staggered fermion formulation. Such an attempt must involve the appropriate chirality matrix, which, as we have seen, is Γ_5 (corresponding to $\gamma_5 \otimes 1$ in the taste field rep.).

Remains to resolve issue (ii) regarding appropriate range of m and whether index D_{overl} satisfies Index Theorem.

For this, return to previous expression for $H_{st}(m)$ on free taste fields:

$$H_{st}(m) = \gamma_5 \left[(\gamma_\mu \otimes \mathbb{1}) \tilde{\nabla}_\mu + \tfrac{1}{2} \tilde{\Delta} - m \right]$$

$$\equiv \gamma_5 \left[\tilde{D} - m \right]$$

where

$\tilde{D} = (\gamma_\mu \otimes \mathbb{1}) \tilde{\nabla}_\mu + \frac{1}{2} \tilde{\Delta}$ "Wilson-type" Dirac op.

$\tilde{\Delta} = \sum_\nu t_\nu \tilde{\Delta}_\nu$ Hermitian but not positive:

$$\tilde{\Delta} \psi = \lambda \psi \Leftrightarrow \tilde{\Delta} (t_5 \psi) = -\lambda (t_5 \psi)$$

$$\tilde{\Delta}^2 = \sum_\nu \Delta_\nu^2$$

The real eigenvalues of $\tilde{D}$ correspond to momentum eigenstates with momenta at the corners of Brillouin zone:

$$\tfrac{1}{2} \tilde{\Delta}_\nu(p) = \begin{cases} 0 & p_\nu = 0 \\ 2 & p_\nu = \pi \end{cases}$$

$\Rightarrow$ The real eigenvalues of $\tilde{D}$ are

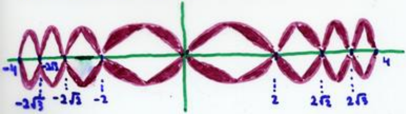
$$\pm \sqrt{\sum_\nu \epsilon_\nu^2} \quad , \quad \epsilon_\nu \in \{0, 2\}$$

(In usual Wilson case, real eigenvalues are $\sum_\nu \epsilon_\nu$)

real eigenvalue of $\tilde{D}$	0	± 2	$\pm 2\sqrt{2}$	$\pm 2\sqrt{3}$	± 4
# eigenmodes	4	8	12	8	2

Total # eigenmodes is $64 = 4 \times 16$

Spectrum of $\tilde{D}$:



Recall that the value of m in $H_{SP}(m) = \gamma_5(\tilde{D}-m)$ determines which real eigenmodes are projected to zero-modes in the overlap construction

$\Rightarrow$ The appropriate range for m in this case is

$$-4 < m < -2\sqrt{3}$$

Then the overlap projection of the spectrum of $\tilde{D}$ is as indicated:

In particular, the corresponding overlap operator $D_{ov(st)}$ has <u>two</u> zeromodes, ie. # tastes is reduced from 4 to 2.

The eigenvalues $\lambda(m)$ of $H_{st}(m) = \gamma_5(\tilde{D} - m)$ cross the origin at precisely the values of m which are real eigenvalues of $\tilde{D}$. In the full interacting case, we expect that the eigenvalue crossings for $H_{st}(m)$ will continue to occur at values of m close to the free field ones when the background gauge field is reasonably smooth. (This is the experience from the Wilson case.) By general argument get

$$\text{index } D_{ov(st)}(m_0) = \text{index of the eigenvalue crossings of } H_{st}(m) \text{ for } m < m_0$$

Since $D_{ov(st)}$ has two tastes, index theorem is

$$\text{index } D_{ov(st)} = 2Q \quad \left(\begin{array}{l} Q = \text{top. charge of} \\ \text{gauge field} \end{array}\right)$$

This should be checked in two ways:

(1) Numerically study spectral flow of $H_{st}(m)$ in smooth gauge backgrounds with top. charge Q to confirm index theorem.

(2) Analytically evaluate the axial anomaly ($\sim$ top. charge density) in $a \to 0$ limit in presence of smooth gauge field. Should reduce to $2\times$ usual anomaly

(These checks are currently underway...)

Summary

Can construct viable version of overlap Dirac operator with staggered fermion kernel by making following replacements in usual overlap construction:

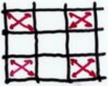
$$\gamma_5 \to \Gamma_5$$

$$H_W(m) \to H_{st}(m) = -iD_{st} - m\Gamma_5 \quad , m \in]-4, -2\sqrt{3}[$$

where $D_{st} = \eta_\mu \nabla_\mu$ is usual massless staggered Dirac operator and Γ_5 given by $\left(\begin{array}{l}\text{for } x = 2y+\xi \\ \chi(x) = \chi_\xi(y)\end{array}\right)$

$$(\Gamma_5 \chi)_\xi(y) = (-1)^{\xi_1+\xi_3} U(y;\xi,\bar{\xi})\, \chi_{\bar{\xi}}(y)$$

i.e. Γ_5 couples opposite sites of hypercube:



and $U(y,\xi,\bar{\xi})$ is product of link variables along path from $\bar{x} = 2y+\bar{\xi}$ to $x = 2y+\xi$ specified by choice of path from $\bar{\xi}$ to ξ in standard hypercube.

Remarkable property: # tastes reduced from 4 to 2 for resulting overlap operator $D_{ov(st)}$.
However, choice of paths → breaking of lattice rot. symm. → renormalisability issue (needs to be clarified)