RG Analysis and Magnetic Instability in Gapless Superfluid

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Based on hep-ph/0506097
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I. Introduction

- Asymptotic freedom of QCD (Gross-Politzer-Wilczek 73)

\[ \epsilon_{ab3} \Delta \]

- Matter at extreme density is quark matter. (Collins-Perry 75)

- The ground state is a color superconductor. (Barrois 77)
QCD predicts phase transitions.
• Various phases of CSC due to stress ($m_i$, electroweak interaction) $\delta \mu \neq 0$

• Charge neutrality stabilizes the Sarma Phase ($\Delta < \delta \mu$): Huang-Shovkovy 03. (Cf. Bedaque-Caldas-Rupak 03: Forbes-Gubankova-Liu-Wilczek 05)
II. Meissner Mass and Instability

- Magnetic instability (Huang+Shovkovy 04, Wu+Yip, Casalbouni et al.):

\[ m^2_M = \frac{1}{3} \delta^{ij} \frac{\partial^2 V(A)}{\partial A_i \partial A_j} \bigg|_{A=0} < 0. \]

- Effective potential

\[ V(A) = V_0 - \frac{1}{2} m^2_D A_0^2 + \frac{1}{2} m^2_M \vec{A}^2 + \cdots. \]
- Two-point correlation functions (HDET):

1. Free fermions, $V^\mu = (1, \vec{v}_F)$:

$$\Pi_{\mu\nu}(p) = \frac{-g^2 p_F^2}{\pi} \sum \frac{V_\mu V_\nu}{v_F} \left[ 1 - \frac{p_0 + \vec{v}_F \cdot \vec{p}}{p_0 - \vec{v}_F \cdot \vec{p}} \right] + \delta \Pi_{\mu\nu}$$ (3)

$$\Pi_{00}(0) = m_D^2, \quad \Pi_{ij}(0) = 0$$ (4)

2. BCS superconductor

$$\Pi_{ij}(0) = \frac{1}{3} \delta_{ij} \int \frac{d^4 l}{(2\pi)^4} \frac{4 \Delta^2}{l_4^2 + (\vec{v}_F \cdot \vec{l})^2 + \Delta^2} = \frac{1}{3} \delta_{ij} m_D^2$$ (5)

3. Gapless superfluid (Goldstone field, $\varphi$)

$$\Pi_{\mu\nu}(p) = \tilde{\Pi}_{\mu\nu}(p_0 + i \partial_t \varphi, \vec{p} - i \vec{\nabla} \varphi)$$ (6)
• Energy dispersion relation:
\[ \omega(\vec{p}) = \pm \left( \delta \mu \pm \sqrt{\epsilon^2(\vec{p}) + \Delta^2} \right) \]

• Gapless modes near the Fermi surface:
\[
\omega(\vec{p}) \approx \begin{cases} 
\eta \vec{v}_i \cdot \vec{l}, & \text{if } \left| \vec{v}_i \cdot \vec{l} \right| < \delta p \\
\frac{(\vec{v}_i \cdot \vec{l})^2}{2\delta \mu}, & \text{if } \delta p < \left| \vec{v}_i \cdot \vec{l} \right| < \delta \mu \\
\vec{v}_i \cdot \vec{l}, & \text{if } \bar{\mu} \gg \left| \vec{v}_i \cdot \vec{l} \right| > \delta \mu 
\end{cases}
\]

\[ \delta p = \sqrt{\delta \mu^2 - \Delta^2}, \quad \eta = \delta \mu / \delta p \]

• When \( \delta \mu \approx \Delta \) or \( \delta p \approx 0 \), the gapless modes are quadratic all the way down to the Fermi surface.
The instability is due to the IR divergences by gapless modes.

Longitudinal mode suffers IR divergences = Magnetic Instability

One-loop Coleman-Weinberg effective potential

\[
V(A) = \frac{1}{2} N_g \text{Tr} \ln \left[ p_4^2 + \left( \delta \bar{\mu} - \sqrt{\bar{\epsilon}_1 \bar{\epsilon}_2 + \Delta^2} \right)^2 \right] + [- \to +] \tag{7}
\]

where \( \delta \bar{\mu} = \delta \mu - \delta q A_0 \) and \( \bar{\epsilon}_i = E(\vec{p} + q_i \vec{A}) - \bar{\mu} - \bar{q} A_0 \).

\[
V(A) = \begin{array}{c}
\includegraphics[width=1.5in]{V_A_diagram}
\end{array} + \cdots
\]
• When $\delta \mu \approx \Delta$, the gapless modes have quadratic dispersion relation, which is problematic:

1. The effective potential is unbounded from below, $V_* = (1, \vec{v}_*)$:

$$V = -\frac{16\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{4}\right)}{3\sqrt{\pi}2\delta \mu} \int \frac{\nu_*|e|^3}{|\vec{v}_*|} |q_1V_* \cdot A - \delta qA_0|^\frac{3}{2} |q_2V_* \cdot A + \delta qA_0|^\frac{3}{2} + \cdots .$$

2. $\star$ Under a scaling $E \mapsto sE$, $\vec{A} \mapsto s^{1/2} \vec{A}$:

$$V(A) \sim |\vec{A}|^3 .$$

(cf. Liu-Wilczek-Zoler 04)

2. ABS construction of K theory on Fermi surface shows the gapless modes must be linear. (Horava 05)
III. RG Analysis and Secondary Gap

- Effective Lagrangian for quadratic gapless modes (cf. HDET):

\[
\mathcal{L}_{\text{eff}} = \sum_{\vec{v}_*} \psi^\dagger(\vec{v}_*, x) \left[ i \partial_t + \left( \frac{\vec{v}_* \cdot \vec{\nabla}}{2 \delta \mu} \right)^2 \right] \psi(\vec{v}_*, x) + \frac{\kappa}{2} (\psi^\dagger \psi)^2 + \cdots \tag{10}
\]

- Under the scaling \( E \mapsto s E \), the action \( S = \int dt d^3l \mathcal{L}_{\text{eff}} \) is invariant:

\[
\vec{l}_\parallel \mapsto s^{1/2} \vec{l}_\parallel, \quad \psi_{\vec{v}_*}(t, \vec{l}) \mapsto s^{-1/4} \psi_{\vec{v}_*}(t, \vec{l}) \tag{11}
\]

- The four Fermi-interaction is relevant for incoming fermions with opposite momenta:

\[
\kappa \mapsto s^{-1/2} \kappa \tag{12}
\]

(Cf. If not opposite, the interaction is marginal.)
• At tree level the interaction between the gapless modes and the gapped modes ($\psi_H$) are repulsive. However the repulsive interaction induces attraction among the gapless modes upon integrating out the gapped modes (cf. Kohn-Luttinger theorem):

\[
\kappa = \frac{\pi}{4} G_s^2 \frac{v_*}{v_*}
\]

(13)

• The Cooper-pair gap eq. for the secondary gap is given as

\[
\Delta_s = \kappa \int \frac{d^4l}{(2\pi)^4} \frac{\Delta_s}{l_4^2 + \left[\frac{(l \cdot \tilde{v}_*)}{2\delta \mu}\right]^2 + \Delta_s^2}.
\]

(14)
• The secondary gap is power-suppressed in $\kappa$ or $g \equiv [\ln (2\bar{\mu}/\Delta)]^{-1}$,

$$\Delta_s \simeq 6.85 \kappa^2 \left(\frac{\nu_*}{\nu^{**}}\right)^2 \delta \mu = 4.2 \left(\frac{G_s}{G}\right)^2 g^4 \delta \mu,$$

(15)

(cf. Alford-Wang 05)

• For $g2\text{CSC}$, it is spin-1 and color-antitriplet pairing while for $g\text{CFL}$ it has to be color-sextet pairing like $\langle \Psi_L C \gamma_0 \gamma_5 \Psi_R \rangle$. (cf. Alford et al. 02)
• Meissner mass due to the secondary gap (Hong-Sohn, to appear):

\[
V(A)|_{\text{quadratic}} = \frac{N_g}{2} \text{Tr} \ln \left\{ l_4^2 + \frac{1}{(2\delta\mu)^2} \left[ \vec{v}_* \cdot (\vec{l} + \vec{A}) \right]^4 + \Delta_s^2 \right\}
\] (16)

\[
\rightarrow \quad m_M^2|_{\text{quadratic}} \simeq \left( \frac{\pi}{2} - \frac{2}{3} \right) m_M^2|_{\text{BCS}}
\]

(N.B. The Meissner mass is independent of \(\delta q\).)
IV. New Stable Gapless Superfluids

- For a large mismatch the secondary gap does not open, since the RG running is not long enough for

\[ \Lambda_{\text{quadratic}}^{\text{IR}} > \Delta_s \quad \text{or} \quad \frac{1}{2\delta\mu} \left( \delta\mu^2 - \Delta^2 \right) > \Delta_s. \]

Instead, the condensate of Goldstone currents develops.

- The system is then dominated by linearly gapless modes at low energy, \( \omega < (\delta\mu^2 - \Delta^2)/(2\delta\mu) \).

- Integrating out modes with \( \omega \gtrsim \delta\mu \), we may write the effective potential Eq. (7) as

\[
V(A) = -\frac{N_g}{2} \int_{\Lambda} \frac{d^4 p_E}{(2\pi)^4} \ln \left[ p_4^2 + \left( \sqrt{\bar{\epsilon}_1 \bar{\epsilon}_2 + \Delta^2 - \delta\bar{\mu}} \right)^2 \right] + \\
+ C_0(\Lambda) + \frac{C_2(\Lambda)}{2} A^2 - \frac{B_2(\Lambda)}{2} A_0^2 + \cdots
\]

(18)
Integrating further out the gapless modes till they become linearly gapless, we get with $V_1^\mu = (\eta \bar{q} + \delta q, \eta \bar{q} \bar{v}_1)$ and $V_2^\mu = (\eta \bar{q} - \delta q, \eta \bar{q} \bar{v}_2)$.

$$V(A) = -\nu_1 \eta^{-1} \int \frac{d\Omega \bar{v}_1}{4\pi} (eV_1 \cdot A)^2 \left[ \ln \frac{\Lambda^2}{(eV_1 \cdot A)^2} + 1 \right] + (1 \to 2)$$

$$+ C_0(\Lambda) + \frac{C_2(\Lambda)}{2} \bar{A}^2 - \frac{B_2(\Lambda)}{2} A_0^2 + \ldots$$

Counter-terms are fixed by the renormalization conditions,

$$\left. \frac{\partial^2 V}{\partial A^2} \right|_{A_0=M} = -m_D^2, \quad \frac{1}{3} \delta_{ij} \left. \frac{\partial^2 V}{\partial A_i \partial A_j} \right|_{\bar{A}^2=M^2} = m_M^2,$$
• The effective potential becomes \((\vec{A} \mapsto \vec{A} - \vec{\nabla} \varphi)\)

\[
V(A) = \frac{1}{2} m^2_M \vec{A}^2 - \frac{\eta}{3} (\nu_1 v_1^2 + \nu_2 v_2^2) e^2 q^2 \vec{A}^2 \left[ \ln \left( \frac{M^2}{\vec{A}^2} \right) + 3 \right] \\
- \frac{1}{2} m^2_D A_0^2 - \frac{1}{\eta} (\nu_1 + \nu_2) (\eta q + \delta q)^2 e^2 A_0^2 \left[ \ln \left( \frac{M^2}{A_0^2} \right) + 3 \right]
\]

\((21)\)

\[\star\] Since the linear modes are scale-invariant, it is logarithmically divergent.

• The minimum occurs at

\[
\left\langle (\vec{A} - \vec{\nabla} \varphi)^2 \right\rangle \simeq \delta \mu^2 \exp \left[ 2 - \frac{4 \nu^*_1 v^*_2}{\eta (\nu_1 v_1^2 + \nu_2 v_2^2)} \right].
\]

\((22)\)

(Cf. Huang, 05)
This slide is added to answer some questions raised during the talk.

- The renormalization condition Eq. (20) fixes the curvature of the potential, Eq. (21), at $A = M$. Since the potential is independent of $M$, the Meissner mass at different scale $M'$ is related to the one at $M$ as

$$m^2_M(M') = m^2_M(M) - c^2_M \ln \left( \frac{M^2}{M'^2} \right)$$

(23)

where $c^2_M = \frac{2n}{3} \left( \nu_1 v_1^2 + \nu_2 v_2^2 \right) e^2 \bar{q}^2$.

- As $M'$ approaches zero, the log becomes large and one has to sum all the log’s to get

$$m^2_M(M') = m^2_M(0) + \frac{m^2_M(M) - m^2_M(0)}{1 + b \ln \left( \frac{M^2}{M'^2} \right)}$$

(24)

where $b = c^2_M / [m^2_M(M) - m^2_M(0)]$. 

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- The ground state breaks the rotational symmetry. (Cf. FFLO state)

- The ground state, however, does not carry any net current, which agrees with the London theorem:

\[
\langle \vec{J} \rangle \equiv - \frac{\partial V (A)}{\partial \vec{A}} \bigg|_{\vec{A}=\vec{A}_c} = 0
\] (25)

- Under an external field \( \vec{A} \), the current is given as

\[
J_i (x) = - \frac{\partial^2 V (A)}{\partial A_i \partial A_j} \bigg|_{\vec{A}=\vec{A}_c} \left( A_j - \partial_j \varphi \right).
\] (26)

Taking the curl of the current, we get

\[
\nabla^2 \vec{B} = c_M^2 \left( \vec{B} - \vec{B} \cdot \vec{A}_c \vec{A}_c \right), \quad c_M^2 = \frac{2\eta}{3} \left( \nu_1 v_1^2 + \nu_2 v_2^2 \right) e^2 q^2
\] (27)
• The new state is stable and has a non-negative, but directional, Meissner mass.
V. Conclusion

- The magnetic instability of gapless superfluids is studied in terms of effective field theories.

- The instability arises due to the IR divergences associated with gapless modes.

- The unstable gapless superfluids make phase transition to
  
  1. a fully gapped phase, opening a secondary gap, if \( \delta \mu \approx \Delta \) or \( \delta \mu - \Delta < \Delta_s \).

  2. Gapless superfluids with condensed supercurrents, if \( \delta \mu \gtrsim \Delta \) or \( \delta \mu - \Delta > \Delta_s \).