Heavy-Light Decay Constants and Form Factors in Staggered Chiral Perturbation Theory

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Outline

- Lattice quarks
- Staggered Chiral Perturbation Theory (SχPT)
- Heavy Quark Effective Theory (HQET)
- Heavy quarks on the lattice
- Including taste violations with heavy-light mesons
- Calculation of $f_B$ to one loop
- Form factors for $B \rightarrow \pi\ell\nu$ and $D \rightarrow K\ell\nu$
Why?

- Accurate calculations of heavy-light quantities can help determine CKM matrix elements.

- Need \( \chi \)PT for lattice simulations for two reasons:
  - Physical light quark masses are hard to simulate; we need a systematic way to extrapolate from the large masses used in simulations to the physical values of these masses
  - All simulations are done at finite lattice spacing; need to extrapolate to the continuum limit.
Lattice Quarks

Discretize spacetime with lattice spacing $a$.

The derivative becomes a discrete difference:

$$\partial \psi(x) \rightarrow \sum_{\mu} \gamma_\mu \frac{\psi_{x+a\hat{\mu}} - \psi_{x-a\hat{\mu}}}{2a}$$

PROBLEM: In the continuum limit ($a \rightarrow 0$), this “naively” discretized theory gives 16 fermion species.

The fermion propagator is not a function of $p_\mu$, but $\sin(ap_\mu)/a$.

16 low energy excitations when components of $p$ are near $p = 0$ or $\pi/a$ as $a \rightarrow 0$. 
Lattice Quarks

Two solutions:

**Wilson quarks**: Give 15 doublers mass of $O(1/a)$—they decouple in the continuum limit.
- Break chiral symmetry
- Computationally expensive
- A variant of these is good for heavy quarks (“Fermilab” quarks)

**Staggered quarks**: Diagonalize $\gamma$ matrices, keep only one spinor component on each site—This reduces doubling by a factor of 4.
- Chiral symmetry at finite $a$
- Computationally cheap
- Still have four tastes $\Rightarrow$ Take fourth root of the quark determinant.
- Spinor components of these four tastes lie on different sites
Taste Violations

\[ \begin{align*}
    k &= 0 \\
    q &= \pi/a \\
    p &= \pi/a \\
    p' &= 0 \\
    k' &= \pi/a \\
\end{align*} \]
Staggered $\chi$PT for 3 light flavors

- Lee & Sharpe, PRD 60, 114503; CA & Bernard, PRD 68 034014 & 074011
- Light mesons: $\Sigma = \exp(i\Phi/f)$, with

$$
\Phi = \begin{pmatrix}
U & \pi^+ & K^+ \\
\pi^- & D & K^0 \\
K^- & \bar{K}^0 & S
\end{pmatrix}, \quad U = U_a T_a, \quad K^+ = K^+_a T_a, \ldots
$$

- $T_a \in \{\xi_5, i\xi_\mu \xi_5, i\xi_\mu \xi_\nu, \xi_\mu, I\}$, \quad $\{\xi_\mu, \xi_\nu\} = 2\delta_{\mu\nu}$
- Under chiral $SU(12)_L \times SU(12)_R$: $\Sigma \to L\Sigma R^\dagger$
- 16 tastes/flavor (in degenerate $SO(4)$ representations: $P, A, T, V, S$) with masses:

$$
m_t^2 = \mu(m_a + m_b) + a^2 \Delta_t, \quad (t = P, A, T, V, S)
$$

- Taste violations at finite lattice spacing $\Rightarrow \Delta_t \neq 0$
Staggered $\chi$PT

$L$ is an expansion in
- $m_\pi^2 \sim m_q$; $m_q$ is a light quark mass
- $a^2$, the lattice spacing

\[ L = \frac{f^2}{8} \text{Tr}[\partial_\mu \Sigma \partial^\mu \Sigma] + \frac{\mu f^2}{4} \text{Tr}[\mathcal{M}(\Sigma + \Sigma^\dagger)] + L_{\text{singlet}} - a^2 \mathcal{V}_\Sigma \]

- $\mathcal{M}$: Light quark mass matrix
- $\mathcal{V}_\Sigma$: Taste-breaking potential arising from four-quark operators.
- $f$: tree-level pion decay constant
- $\mu$: tree-level ratio of $m_\pi^2$ to $m_q$
$\mathcal{N}_\Sigma$ and $\mathcal{L}_{\text{singlet}}$ give rise to terms of the form $+\frac{\lambda'_t}{2} (U_t + D_t + S_t)^2$, where $\lambda'_t$ depends on the channel we’re discussing:

$$\lambda'_V = a^2 \delta'_V, \text{ taste-vector}; \quad \lambda'_A = a^2 \delta'_A, \text{ taste-axial}; \quad \lambda'_I = \frac{4m_0^2}{3}, \text{ taste-singlet.}$$

We must rediagonalize the mass matrix for taste vector, axial and singlet flavor-neutral mesons.

Unlike $m_0$, we have no reason a priori to send $\delta'_{V(A)} \to \infty$, so the $\eta'_{V(A)}$ does not decouple like the $\eta'_I$. 
**HQET**

- Heavy quark $SU(2)$ spin symmetry

\[ H = \frac{1 + \frac{\phi}{2}}{2} \left[ \gamma^\mu B^*_\mu - \gamma_5 B \right] \]

\[ \overline{H} \equiv \gamma_0 H^\dagger \gamma_0 \]

- $v$ is the heavy-light velocity

- Use $\sigma$, where $\sigma^2 = \Sigma$, to make chiral invariants.

- Under chiral $SU(12)_L \times SU(12)_R$:

\[ H \rightarrow H U^\dagger \]

\[ \overline{H} \rightarrow U \overline{H} \]

\[ \sigma \rightarrow L \sigma U^\dagger = U \sigma R^\dagger \]

\[ \Sigma \rightarrow L \Sigma R^\dagger \]
The heavy-light portion of $\mathcal{L}$ is an expansion in $m_\pi$ (not $m_\pi^2$) and $a^2$, and also

- $k$ ($p_B = m_Q v + k$), the heavy-light residual momentum
- $1/m_Q$, $m_Q$ is the heavy quark mass

\[
\mathcal{L} = i \text{Tr}_D [\overline{H} v \cdot D H] + g_\pi \text{Tr}_D (\overline{H} H \gamma^\nu \gamma_5 A_\nu) + g_1 \text{Tr}_D (\overline{H} H M^+) + g_2 \text{Tr}_D (\overline{H} H) \text{Tr}(M^+) + \mathcal{L}_{3,m} + a^2 \mathcal{L}_{H,a}
\]

\[
D^\mu_{ab} = \delta_{ab} \partial^\mu - \frac{1}{2} \left[ \sigma^\dagger \partial^\mu \sigma + \sigma \partial^\mu \sigma^\dagger \right]_{ba}
\]

\[
A_\mu = i \frac{1}{2} \left[ \sigma^\dagger \partial_\mu \sigma - \sigma \partial_\mu \sigma^\dagger \right]
\]

\[
M^\pm = \sigma M \sigma \pm \sigma^\dagger M \sigma^\dagger
\]
HQET & SχPT

$\mathcal{L}_{3,m}$: 7 terms which include the light quark mass matrix and the heavy-lights and an additional single derivative, such as

$$\mathrm{Tr}_D \left[ \bar{H} i v \cdot D H \mathcal{M}^+ \right]$$

$$\mathrm{Tr}_D \left[ \bar{H} H \gamma_\mu \gamma_5 \{ A^\mu, \mathcal{M}^+ \} \right]$$

$$i \mathrm{Tr}_D \left[ \bar{H} H \gamma_\mu [ A^\mu, \mathcal{M}^- ] \right]$$

These terms at most contribute only to analytic terms.

Terms $\propto \mathcal{M}^-$ don’t contribute to $f_B$ or form factors.
Heavy quarks on the Lattice

We place two restrictions on the heavy quark mass:

- \( m_Q \gg \Lambda_{\text{QCD}} \)
  - This allows us to keep only terms which are \( O[1/m_Q]\) in HQET.
- \( am_Q \gg 1 \)
  - To decouple doublers, need \( \Delta M \gg \Lambda_{\text{QCD}} \)
  - \( \Delta M \approx \frac{1}{a^2m_Q} \) for large \( am_Q \)

For the “coarse” MILC lattices, and Wilson-type heavy quarks:

\[
\begin{align*}
  a &\approx 0.12\text{fm} \quad \Rightarrow \quad a^{-1} \approx 1.59\text{GeV} \\
  \kappa_{\text{crit}} &\approx 0.1378 \\
  \kappa_b &\approx 0.086 \\
  \Delta M &\approx 775\text{MeV}
\end{align*}
\]
Taste-breaking Four-quark operators

There are three types of four-quark operators that can possibly contribute to $\mathcal{L}_{H,a}$:

1. $Q(S \otimes I)Q(S' \otimes I)Q$
   - Taste singlet: No taste violations from these operators

2. $Q(S \otimes I)Qq_j(S' \otimes \xi I)q_j$
   - Correct the heavy-light masses at $O(a^2)$, do not involve the pions at this order (thus are not taste-violating)

3. $q_i(S \otimes \xi T)q_iq_j(S' \otimes \xi T')q_j$
   - Same operators which contribute to $\mathcal{V}_\Sigma$; give rise to the terms in $\mathcal{L}_{H,a}$
There are 8 operators $O_k^A$ which arise from the four-quark operators, such as

\begin{align*}
O_1^A &= \sigma \xi_5^{(3)} \Sigma^\dagger \xi_5^{(3)} \sigma \\
O_7^A &= (\sigma \xi_\nu^{(3)} \sigma) \text{Tr}(\xi^{(3)}\nu \Sigma^\dagger)
\end{align*}

\begin{align*}
\xi_\mu^{(3)} &= \begin{pmatrix}
\xi_\mu & 0 & 0 \\
0 & \xi_\mu & 0 \\
0 & 0 & \xi_\mu 
\end{pmatrix}
\end{align*}

These are related to the 8 operators which make up the $S\chi PT$ potential, $V_\Sigma$

These 8 operators can be written as Hermitian or anti-Hermitian, giving us (potentially) 16 operators, $O_k^{A,\pm} \equiv O_k^A \pm O_k^{A\dagger}$
Taste-breaking Lagrangian with Heavy-lights

$\mathcal{O}_k^{A,\pm}$ transform like $\mathcal{M}^{\pm}$

Take the 9 operators from before and send $\mathcal{M}^{\pm} \rightarrow \mathcal{O}_k^{A,\pm} \Rightarrow 72$ terms!

$$\mathrm{Tr}_D \left[ \overline{H} H \mathcal{O}_k^{A,+} \right] , \quad \mathrm{Tr}_D \left[ \overline{H} v \cdot D H \right] \mathrm{Tr}[\mathcal{O}_k^{A,+}]$$

$$\mathrm{Tr}_D \left[ \overline{H} H \gamma_\mu \gamma_5 A^\mu \right] \mathrm{Tr}[\mathcal{O}_k^{A,+}] , \quad i\mathrm{Tr}_D \left[ \overline{H} H \gamma_\mu A^\mu \right] \mathrm{Tr}[\mathcal{O}_k^{A,-}]$$

$$\mathrm{Tr}[\mathcal{O}_k^{A,-}] = 0 \text{ for } k = 1, 5, 6, 8$$

$\Rightarrow 68$ terms
Lorentz Violation

- We can also have Lorentz-violating terms in the Lagrangian at this order.
- Not in SχPT with only pions—these terms must have derivatives $\rightarrow O(\alpha^2 p^2)$.
- Here we have $\gamma_\mu, v_\mu, D_\mu, A_\mu$.
- Using the method of Sharpe & Van de Water, hep-lat/0409018, we find six operators, $O_{\mu,k}^{B,\pm}$ ($k = 1, 2, 3$) arising from four-quark operators that break Lorentz invariance.

$$O_{\mu,1}^{B,\pm} = \sum_\nu (\sigma^\dagger \xi_\mu \xi_\nu \Sigma^\nu \xi^\mu \sigma^\dagger) \pm h.c.$$ 

- Ultimately 27 more terms in $L_{H,a}$, such as

$$(v^\mu)^2 \text{Tr}_D [\bar{H}H O_{\mu,i}^{B,+}]$$

$$\text{Tr}_D [\bar{H}v^\mu D^\mu H] \text{Tr} [O_{\mu,i}^{B,+}]$$

$$\text{Tr}_D [\bar{H}v^\mu [A^\mu, O_{\mu,i}^{B,-}]]$$
Disaster?

- We now have a Lagrangian with 95 additional terms than found in the continuum!

- Not a major problem
  - Many don’t contribute to $f_B$ or form factors to this order
  - Those that do—only analytic terms
  - All the taste-violating terms contribute in exactly the same way

- Still is predictive
  - Coefficients of analytic terms are independent of mass—$f_D$ and $f_B$ have the same analytic terms
Discrete Symmetry

- Staggered quarks have a discrete symmetry, given by
  \[ q \rightarrow (1 \otimes \xi_{\mu})q \]

- At the meson level, this takes the form
  \[
  \Sigma \rightarrow \xi_{\mu}^{(3)} \Sigma \xi_{\mu}^{(3)}, \quad \sigma \rightarrow \xi_{\mu}^{(3)} \sigma \xi_{\mu}^{(3)} \\
  H \rightarrow H \xi_{\mu}^{(3)}, \quad \overline{H} \rightarrow \xi_{\mu}^{(3)} \overline{H}
  \]

- \( \mathcal{L} \) is invariant under this symmetry, which implies that matrix elements bilinear in the heavy-light fields can be written as an average over tastes
NLO calculation of $f_B$

- The $B$ decay constant can be extracted from the matrix element

$$\langle 0 | L_\alpha^\mu | B_\alpha(v) \rangle = -i f_{Bx} m_{Bx} v^\mu,$$

where $L_\alpha^\mu$ is the axial current which destroys a $B_\alpha$ meson

$$L_\alpha^\mu = \frac{i\kappa}{2} \text{Tr}_D \left[ \gamma^\mu (1 - \gamma_5) H \sigma^\dagger \Lambda_\alpha \right] + \cdots$$

- $\alpha = \{x, a\}$ is a flavor/taste index.
- $(\Lambda_\alpha)_\beta = \delta_{\alpha\beta}$
- The $\cdots$ denote terms which will arise at this order but only contribute to analytic terms.
- We will write the decay constant as:

$$f_{Bx} = \frac{\kappa}{\sqrt{m_{Bx}}} \left( 1 + \frac{1}{16\pi^2 f^2} \delta f_{Bx} \right),$$
NLO calculation of $f_B$

The only non-zero diagrams are:

- Crosses are one or more hairpin insertions from $\mathcal{L}_{S\chi PT}$:

  $-ia^2\delta'_V$ (taste-vector); $-ia^2\delta'_A$ (taste-axial); $-i4m_0^2/3$, (taste-singlet)

Moving from 4 → 1 tastes per flavor is no different than in the calculations for light meson quantities.
4 \rightarrow 1 \text{ tastes per flavor}

Understand the diagrams at the quark level, for every quark loop, we multiply the corresponding term by $1/4$ to account for taking the $\sqrt[4]{\text{Det}}$ in simulations.

The vertices coming from our (a) Lagrangian and (b) current are:
4 → 1 tastes per flavor
\( f_B \text{ for } 2+1 \text{ flavors} \)

\[
\delta f_B = \left( \frac{1 + 3g_\pi^2}{2} \right) \left\{ -\frac{1}{16} \sum_t \left[ 2\ell(m_{\pi_t}^2) + \ell(m_{K_t}^2) \right] - \frac{1}{2} \ell(m_{\pi_0}^2) + \frac{1}{6} \ell(m_{\eta_I}^2) \\
- a^2 \delta' \left[ \frac{(m_{\pi_0}^2 - m_{\eta_V}^2)}{(m_{\pi_0}^2 - m_{\eta_{V'}}, m_{\pi_0}^2 - m_{\eta_{V'}})} \ell(m_{\pi_0}^2) + (\pi_0 \to \eta_V \to \eta_{V'}) \right] \right\} \\
+ [V \to A] + c_1(2m_l + m_s) + c_2m_l + c_a a^2 \right\}
\]

\[ \ell(m^2) = m^2 \ln m^2 + \cdots \]
$f_B$ \textbf{with} $am = 0.010$

- Set $N_{\text{sea}} = 2$

- Degenerate sea mass of $am = 0.010$

- Define the ratio $R = f_B / f_B^{\text{tree}}$

- Plot as a function of the valence quark mass in lattice units, $am_x$. 
$f_B$ with $am = 0.010$
$f_B$ with $am = 0.010$
Form Factors

- Comparing previous SχPT results with Continuum PQχPT results (with $N_{\text{sea}}$ degenerate sea quarks) allows us to generalize all results to SχPT.

- Terms $\propto N_{\text{sea}}$ come from quark-level diagrams like

- For the staggered case, this becomes an average over the 16 tastes.
Form Factors

- Terms $\propto \frac{1}{N_{\text{sea}}}$ are disconnected; in SχPT we have $S$, $V$, and $A$ pieces.

- Possible sign changes when commuting taste matrices:

$$\langle \pi_5 | \pi V \pi_5 \pi V | B \rangle$$
Form Factors example

- $s$: sea quark, $v$: valence light quark.
- Continuum, PQ$\chi$PT, $N$ degenerate sea quarks (Bećirević, et al, PRD68 074003):

\[
- \frac{1}{2(4\pi f)^2} \left\{ N\ell(m_{vs}^2) + \frac{1}{N} \frac{\partial}{\partial m_{vv}^2} \left[ (m_{ss}^2 - m_{vv}^2)\ell(m_{vv}^2) \right] \right\}
\]

- $S\chi$PT, $N$ degenerate sea quarks:

\[
- \frac{1}{2(4\pi f)^2} \left\{ \frac{N}{16} \sum_t \ell(m_{(vs)_t}^2) + \frac{1}{N} \frac{\partial}{\partial m_{(vv)_I}^2} \left[ (m_{(ss)_I}^2 - m_{(vv)_I}^2)\ell(m_{(vv)_I}^2) \right] 
+ a^2\delta'_V \frac{\partial}{\partial m_{(vv)_V}^2} \left[ \frac{m_{(ss)_V}^2 - m_{(vv)_V}^2}{m_{\eta'_V}^2 - m_{(vv)_V}^2} \ell(m_{(vv)_V}^2) \right] \right\} + [V \rightarrow A]
\]
Conclusions

- Calculations involving heavy-lights are a simple extension of HQET and $\chi$PT.

- We have a lot of taste-violating terms—Don’t contribute to loops.

- There is an easy generalization to $\chi$PT from $\chi$PT.

- Calculations for $f_B$ and form factors have been completed.

- Other calculations, such as $B$ parameters or other relevant quantities are straightforward.