A Random Matrix Model for two-color QCD at finite chemical potential

[“From Lattice to Stars”, INT, Seattle]
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[Bertram Klein, Univ. Heidelberg]

BK, D. Toublan, and J.J.M. Verbaarschot,
Phys. Rev. D **68**, 014009 (2003). (3 colors)
BK, D. Toublan, and J.J.M. Verbaarschot, [arXiv:hep-ph/0405180]

Outline

- motivation
- random matrix theory
- random matrix model for two colors
- phase diagram for two colors
- conclusions

QCD phase diagram

- real systems at finite baryon and isospin density $\Rightarrow$ away from planes $\mu_B = 0$ and $\mu_I = 0$
- lattice gauge theory: $\mu_B \neq 0$ sign problem [C.R. Allton *et al.*, Phys. Rev. D **66** (2002) 074507, Z. Fodor, S. Katz, Phys. Lett. **B534** (2002) 87, JHEP 0203 (2002) 014]
- chiral perturbation theory [J.B. Kogut *et al.*, Nuc. Phys.**B582** (2000) 477; D.T. Son, M.A. Stephanov, Phys. Rev. Lett. **86** (2001) 592; K. Splittorff *et al.*, Phys. Rev. **D64** (2001) 016003]
- phenomenological models [J. Berges, K. Rajagopal, Nucl. Phys. **B538** (1999) 215]
- random matrix models: schematic models, respect global symmetries, topology of phase diagram [A.D. Jackson, J.J.M. Verbaarschot, Phys. Rev. D **53** (1996) 7223; M.A. Stephanov, Phys. Rev. Lett. **76** (1996) 4472; M.A. Halasz *et al.*, Phys. Rev. D **58** (1998) 096007]

Two applications of RMT to QCD

▷ eigenvalues of the QCD Dirac operator

$$\not{D}\psi_k = -i\lambda_k\psi_k$$

- exact theory of spectral correlations below λ_c
  finite volume $V_4 = L^4$: mesoscopic range

  $$\frac{1}{m_\pi} \gg L \gg \frac{1}{\Lambda}$$

  spectrum from random matrix theory
  up to a scale

  $$\boxed{\lambda_c \ll \frac{F^2}{\langle\bar{\psi}\psi\rangle L^2}}$$

  goal: spectral density of low-lying
  eigenvalues (→ Kim)

▷ *universal*

- schematic model for the phase diagram:
  chiral phase transition at finite T, μ
  goal: phase structure as determined
  by symmetries

▷ *non-universal*

Random matrix partition function

QCD partition function

$\rightarrow$ RMT partition function

- replace
  Dirac operator by random matrix
- replace
  gauge action by probability distribution

$$Z_{QCD} = \int \mathcal{D}A \, \exp[-S_{YM}] \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f & T \\ -T^\dagger & m_f \end{pmatrix}$$

$$\Downarrow$$

$$Z_{RMT} = \int \mathcal{D}W \, P(W) \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f & W \\ -W^\dagger & m_f \end{pmatrix}$$

W: $n \times n$, $P(W) = \exp[-nG^2 \mathrm{Tr} W^\dagger W]$

- respects symmetries
- no dynamics, static regime: $\frac{1}{m_\pi} \gg L$

What can RMT contribute to our knowledge of the QCD phase diagram?

- ingredients for the model:
  symmetries of QCD
- schematic model for the phase diagram:
  chiral phase transition at finite T, μ

To what extent is the phase diagram determined by the symmetries?

How can RMT describe phase transitions?

- order parameter for chiral symmetry breaking connected to spectral density of QCD Dirac operator:

$$|\langle\bar{\psi}\psi\rangle| = \Sigma = \lim_{m\to 0}\lim_{V_4\to\infty}\frac{\pi\rho(0)}{V_4}$$

[T. Banks, A. Casher, Nucl. Phys. B**169**, 103 (1980)]

$\rho(0) \to 0$: *non-universal*

QCD with two colors ($N_c = 2$)

- confinement
- baryonic states: color-neutral diquarks
- no sign problem for $\mu_B \neq 0$
- similar phase diagram as for $N_c = 3$
- enlarged chiral flavor symmetry (over $N_c = 3$) spontaneously broken:

$$SU(2N_f) \to Sp(2N_f)$$

- $\frac{1}{2}2N_f(2N_f - 1) - 1$ Goldstone bosons: baryonic diquark states (μ_B) and pion states (μ_I) of mass m_π
- pion condensation at finite μ_I
- diquark state condensation at finite μ_B

Symmetries of the QCD partition function for $N_c = 2$

$$Z_{QCD} = \left\langle \prod_{f=1}^{N_f} \det(D\!\!\!/ + m_f + \mu_f \gamma_0) \right\rangle$$

$$D\!\!\!/ = \gamma_\mu(\partial_\mu + iA_\mu)$$

- chiral symmetry

$$\{D\!\!\!/, \gamma_5\} = 0$$

$\longrightarrow \pm\lambda_k$ symmetry of the spectrum

- chiral flavor symmetry ($m_f = 0$)

$$SU(2N_f) \to Sp(2N_f)$$

- antiunitary symmetry
  $[iD\!\!\!/, C\tau_2 K] = 0$, $(C\tau_2 K)^2 = 1\!\!1$

- fermion determinant factorizes:

$$\det(D_\mu\sigma_\mu + \mu_1)\det(D_\mu\sigma_\mu - \mu_1) \\ \times \det(D_\mu\sigma_\mu + \mu_2)\det(D_\mu\sigma_\mu - \mu_2)$$

Random Matrix Model for $N_c = 2$

$$Z_{RMT}(m_1, m_2, \mu_1, \mu_2, T) =$$

$$\int \mathcal{D}W \exp(-\frac{n}{2} G^2 \mathrm{Tr} W^T W) \times$$

$$\times \prod_{f=1,2} \det \begin{pmatrix} m_f & W + \omega(T) + \mu_f \\ -W^T - \omega^\dagger(T) + \mu_f & m_f \end{pmatrix}$$

$$W \in \mathbb{R}^{n \times n}$$

- real temperature dependence

$$\omega(T) = \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}$$

has eigenvalues $\pm iT$

- pion condensate source term

$$\lambda \bar{\psi} \gamma_5 (i\tau_2) \psi$$

- diquark condensate source term

$$ij\psi^T (i\tau_2) \psi - \psi^\dagger (i\tau_2) \psi^*$$

Effective model for $N_c = 2$

exact mapping onto effective partition function:

$$Z^{\text{eff}} = \int \mathcal{D}A \, \exp(-\mathcal{L}(A, A^\dagger))$$

Lagrangian $\qquad A \in \mathbb{C}^{2N_f \times 2N_f}, \; A^T = -A$

$$\mathcal{L} = nG^2 \mathrm{Tr} A^\dagger A - \frac{n}{2} \mathrm{Tr} \log Q_+ Q_-$$

$$Q_\pm = \begin{pmatrix} A^\dagger + M & \pm iT + \mu_B B + \mu_I I_3 \\ \pm iT + \mu_B B + \mu_I I_3 & A + M^\dagger \end{pmatrix}$$

$B = \mathrm{diag}(1, 1, -1, -1), \; I_3 = \mathrm{diag}(1, -1, -1, 1)$

$$M = \begin{pmatrix} 0 & -ij & m_1 & \lambda \\ ij & 0 & -\lambda & m_2 \\ -m_1 & \lambda & 0 & -ij \\ -\lambda & -m_2 & ij & 0 \end{pmatrix}$$

(ansatz for A from M)

isospin and baryon chemical potential

$$\mu_I = \frac{1}{2}(\mu_1 - \mu_2), \;\; \mu_B = \frac{1}{2}(\mu_1 + \mu_2)$$

Chiral Lagrangian for $N_c = 2$

▷ expansion around saddle point solution at $\mu_B = \mu_I = m = 0$

$$A = \sigma(T)\,\Sigma, \quad \Sigma^\dagger\Sigma = \mathbb{1}, \quad \Sigma^T = -\Sigma$$

- power counting

$$\mu_B, \mu_I \sim \epsilon, \quad m \sim m_\pi^2 \sim \epsilon^2$$

- expansion to $\mathcal{O}(\epsilon^2)$, $c_k = c_k(T)$, $k = 1, 2$

$$\mathcal{L}(\Sigma) = -nG^2\Big[c_2 \mathrm{Tr}(\Sigma^\dagger(\mu_I I_3 + \mu_B B)\Sigma(\mu_I I_3 + \mu_B B)) + c_1 \frac{m}{G}\mathrm{Tr}(\Sigma^\dagger \widehat{M}^\dagger + \Sigma\widehat{M})\Big]$$

- chiral Lagrangian

$$\mathcal{L}_{\mathrm{stat}}(\Sigma) = -\frac{F^2}{2}\Big[\mathrm{Tr}(\Sigma^\dagger(\mu_I I_3 + \mu_B B)\Sigma(\mu_I I_3 + \mu_B B) + \frac{m_\pi^2}{2}\mathrm{Tr}(\Sigma^\dagger \widehat{M}^\dagger + \Sigma\widehat{M})\Big]$$

[K. Splittorff, D.T. Son, and M.A. Stephanov, Phys. Rev. D **64** (2001) 016003]

▷ equivalent to zero momentum part of chPT

[Chiral Lagrangian for $N_c = 2$]

- minimized by

$$\Sigma \ = \ \cos\alpha\,\Sigma_\sigma + \sin\alpha(\cos\eta\Sigma_\triangle + \sin\eta\Sigma_\rho)$$

in condensation phases ($\alpha \neq 0$):

$$\eta = 0 \quad \text{for} \quad \mu_B^2 > \mu_I^2$$

$$\eta = \tfrac{\pi}{2} \quad \text{for} \quad \mu_I^2 > \mu_B^2$$

$$\alpha = 0 \qquad \text{for} \quad \mu_I < \frac{m_\pi}{2} \quad \text{and} \quad \mu_B < \frac{m_\pi}{2}$$

$$\cos\alpha = \frac{m_\pi^2}{4\mu_I^2} \quad \text{for} \quad \mu_I > \frac{m_\pi}{2} \quad \text{and} \quad \mu_I^2 > \mu_B^2$$

$$\cos\alpha = \frac{m_\pi^2}{4\mu_B^2} \quad \text{for} \quad \mu_B > \frac{m_\pi}{2} \quad \text{and} \quad \mu_I^2 < \mu_B^2$$

Effective potential: Ansatz

- ▷ large-n-limit: evaluate effective partition function in saddle point approximation
- ▷ treat as a Landau-Ginzburg model!
- ▷ determine phase diagram (mean field result)
  - flavor dependent chemical potentials break flavor symmetry: $\mu_1 \neq \mu_2$
  - expect independent behavior of chiral condensates: $\langle \bar{u}u \rangle \neq \langle \bar{d}d \rangle$
  - expect pion condensation above critical isospin chemical potential
  - expect diquark condensation above critical baryon chemical potential
- ▷ ansatz

$$A = \begin{pmatrix} 0 & -i\triangle & -\sigma_1 & -\rho \\ i\triangle & 0 & \rho & -\sigma_2 \\ \sigma_1 & -\rho & 0 & -i\triangle \\ \rho & \sigma_2 & i\triangle & 0 \end{pmatrix}$$

Order parameters

- chiral condensates

$$\begin{aligned}\langle\bar{u}u\rangle &= \frac{1}{2n}\partial_{m_1}\log Z^{\mathrm{eff}}\\ &= G^2\sigma_1\end{aligned}$$

$$\begin{aligned}\langle\bar{d}d\rangle &= \frac{1}{2n}\partial_{m_2}\log Z^{\mathrm{eff}}\\ &= G^2\sigma_2\end{aligned}$$

- pion condensate

$$\begin{aligned}&\frac{1}{2}(\langle\bar{u}\gamma_5 d\rangle - \langle\bar{d}\gamma_5 u\rangle)\\ &\qquad= \frac{1}{4n}\partial_\lambda\log Z^{\mathrm{eff}}\Big|_{\lambda=0}\\ &\qquad= G^2\rho\end{aligned}$$

- diquark condensate

$$\begin{aligned}&\frac{1}{4}i(\langle d^\dagger u^*\rangle - \langle u^\dagger d^*\rangle + \langle d^T u\rangle - \langle u^T d\rangle)\\ &\qquad= \frac{1}{4n}\partial_j\log Z^{\mathrm{eff}}\Big|_{j=0}\\ &\qquad= G^2\Delta\end{aligned}$$

Order parameters 2:

▷Why can we use RMT for condensates *other* than $\langle\bar{\psi}\psi\rangle$?

chiral condensate

$$\langle\bar{\psi}\psi\rangle_{V_4} = \frac{1}{V_4}\left\langle\sum_k \frac{1}{m+i\lambda_k}\right\rangle, \quad \lambda_k \text{ ev. of}$$

$$\begin{pmatrix} m & W+\mu_B \\ -W^\dagger+\mu_B & m \end{pmatrix}$$

$$\langle\bar{\psi}\psi\rangle = \lim_{m\to 0}\lim_{V_4\to\infty}\frac{\pi\rho(0)}{V_4}$$

pion condensate

$$\langle\pi\rangle_{V_4} = \frac{1}{V_4}\left\langle\sum_k \frac{1}{\lambda+i\lambda_k}\right\rangle, \quad \lambda_k \text{ ev. of}$$

$$\begin{pmatrix} \lambda & 0 & -m & W-\mu_I \\ 0 & \lambda & -W^\dagger-\mu_I & -m \\ m & W+\mu_I & \lambda & 0 \\ -W^\dagger+\mu_I & m & 0 & \lambda \end{pmatrix}$$

[D. Toublan, J.J.M. Verbaarschot, arXiv:hep-th/0208021]

Chiral phase transition

- assume absence of pion ($\rho = 0$) and diquark ($\Delta = 0$)condensates
- effective potential: flavors separate

$$
\begin{aligned}
\frac{1}{n}\mathcal{L} &= \sum_{f=1,2} G^2 \sigma_f^2 \\
&\quad -\frac{1}{2}\log((\sigma_f + m)^2 - \mu_f^2 + T^2)^2 + 4\mu_f^2 T^2)
\end{aligned}
$$

▷ depends only on μ_f^2

▷ symmetry between μ_B and μ_I

$$
\left.\begin{array}{lll} \mu_1 = -\mu_2 & \to \mu_I \neq 0, & \mu_B = 0 \\ \mu_1 = \mu_2 & \to \mu_I = 0, & \mu_B \neq 0 \end{array}\right\} \text{equivalent}
$$

▷ chiral transition in μ_B- T–plane and μ_I-T–plane identical

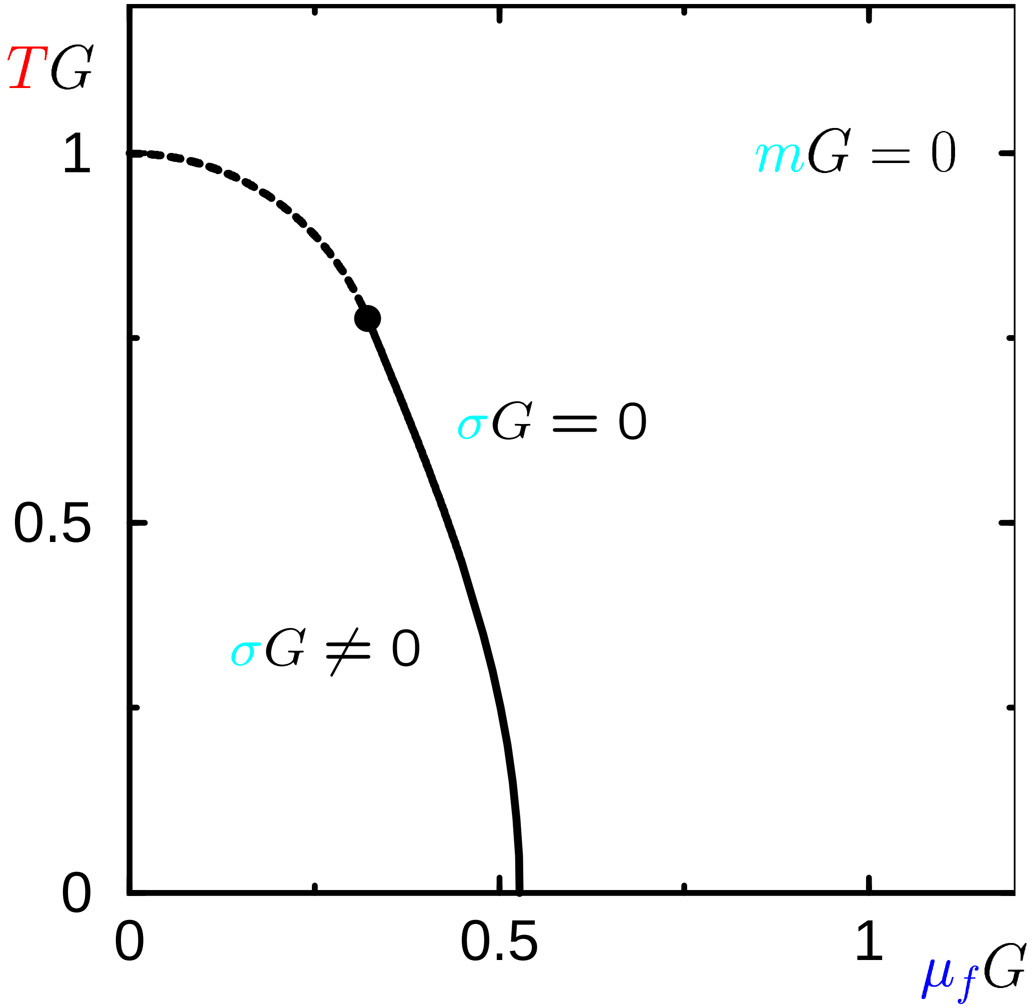
- tricritical point in the chiral limit $mG = 0$

$$
\begin{aligned}
\mu_{f3} G &= \tfrac{1}{2}\sqrt{\sqrt{2} - 1} \approx 0.3218 \\
T_3 G &= \tfrac{1}{2}\sqrt{\sqrt{2} + 1} \approx 0.7769
\end{aligned}
$$

Chiral Phase transition

Phase diagram in the μ_f-T-plane
for $\rho = 0$ and $\triangle = 0$

- symmetry $\mu_I \leftrightarrow \mu_B$
- identical to μ_I-T−plane and μ_B-T−plane
- $mG = 0$: tricritical point

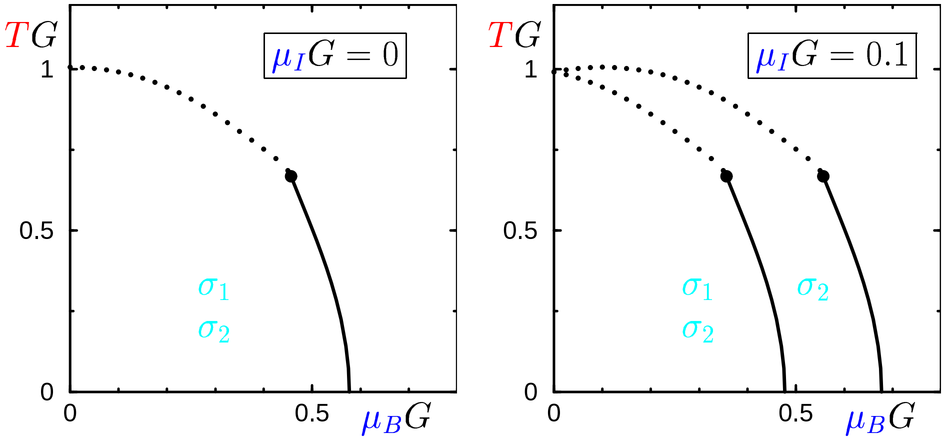


▷ what about the μ_B-T-**plane**
at **nonzero** μ_I and **nonzero** m ?

- $\boxed{\text{two}}$ transition lines for $\rho = 0$

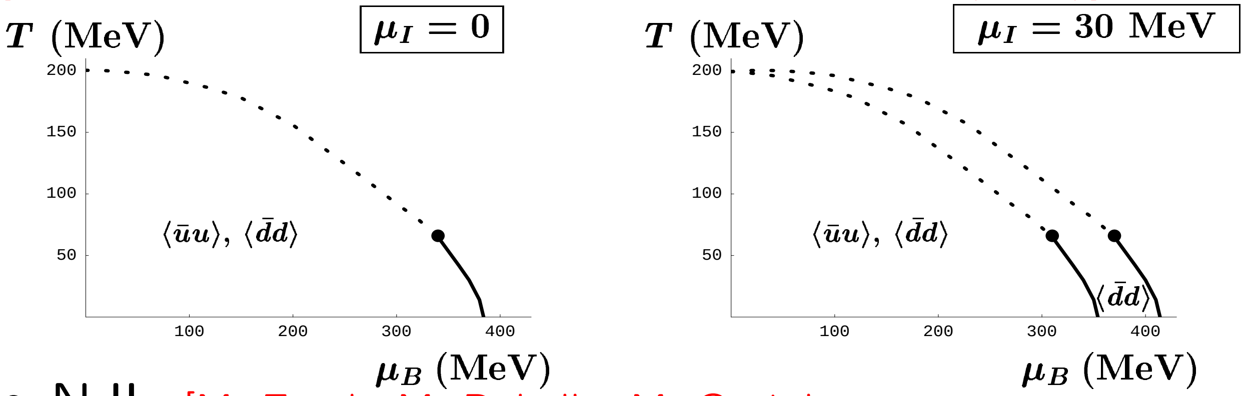
$$\begin{aligned} {\mu_{Bc}}^{(1)}(T) &= \mu_c(T) - \mu_I \\ {\mu_{Bc}}^{(2)}(T) &= \mu_c(T) + \mu_I \end{aligned}$$

random matrix model, $mG = 0.1$



Nambu–Jona-Lasinio model, $m = 10$MeV

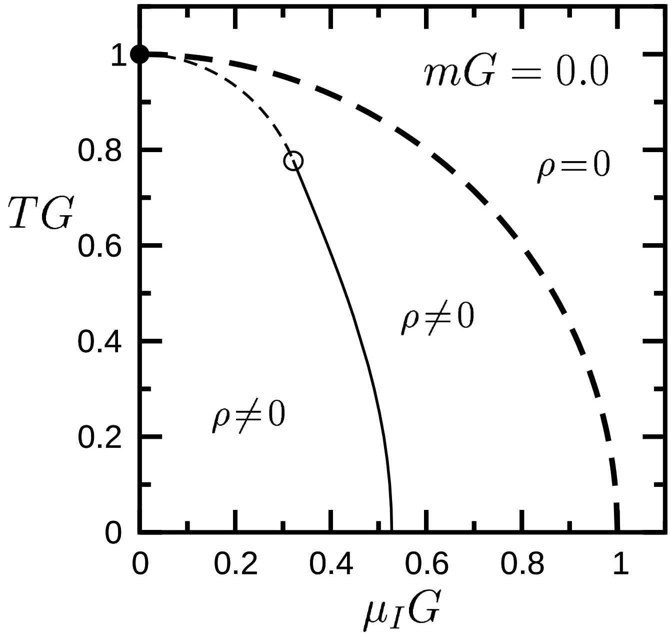
[J. B. Kogut, D. Toublan, Phys. Lett. **B564**, 212 (2003)]

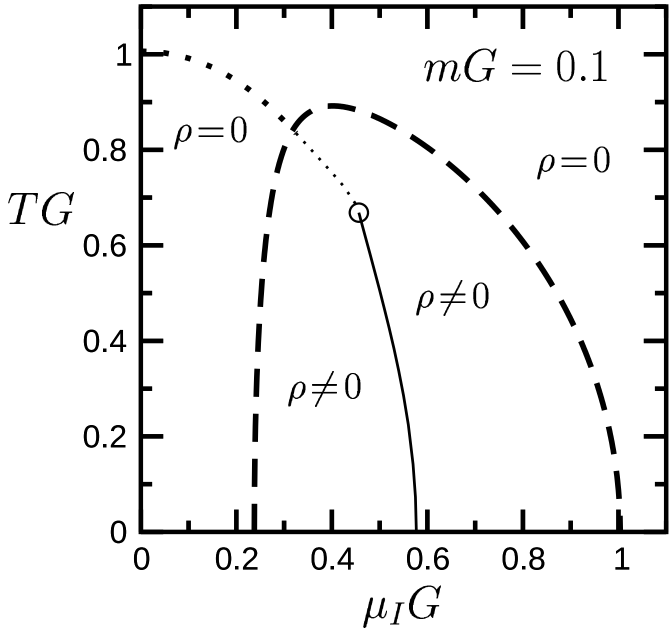


- NJL [M. Frank, M. Buballa, M. Oertel, Phys. Lett. **B562**, 221 (2003)]
- ladder-QCD [A. Barducci, R. Casalbuoni, G. Pettini, L. Ravagli, Phys. Lett. **B564**, 217 (2003); arXiv:hep-ph/0402104]

Phase diagram in the μ_I-T-plane
for different values of mG

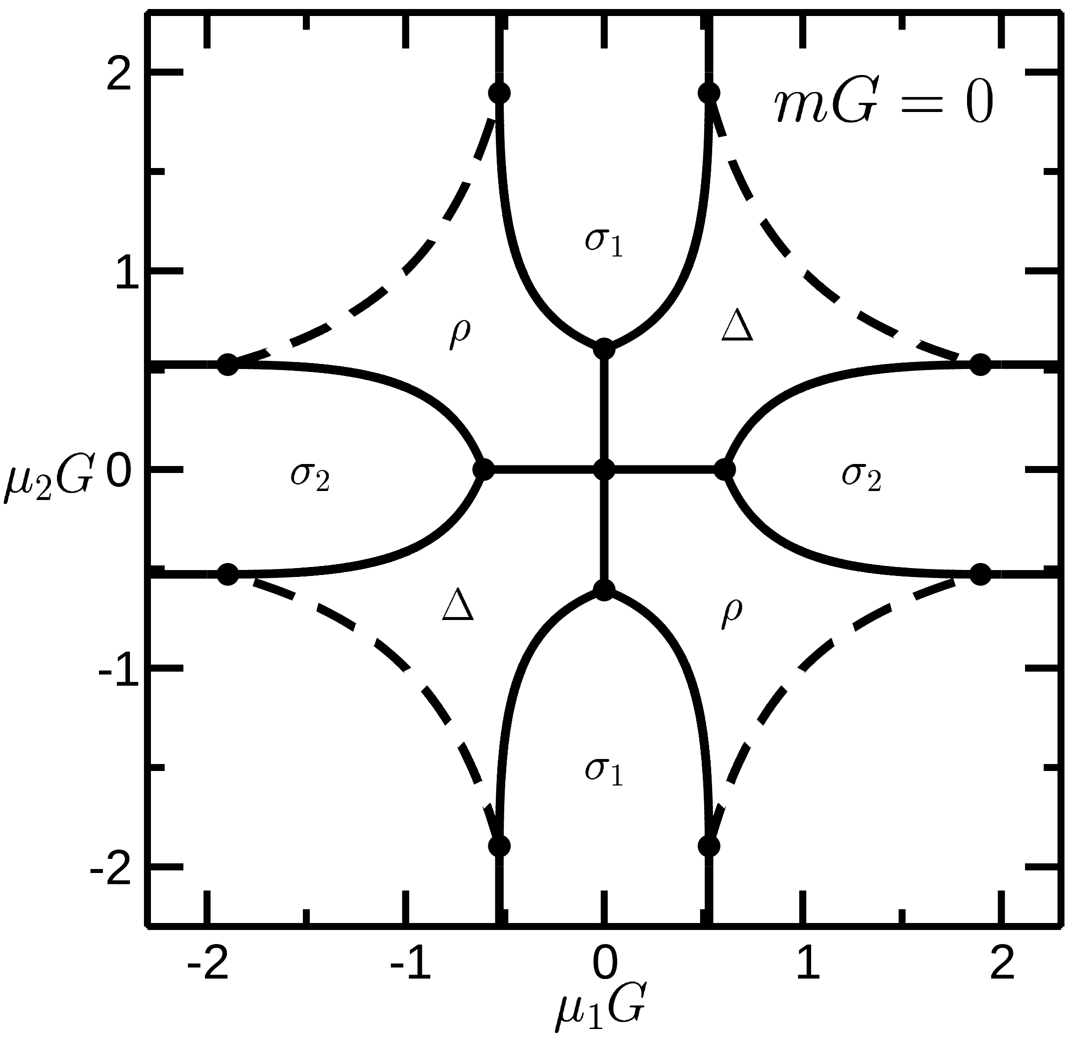
▷ **Problem:** Can we see the chiral transition in the presence of pion/diquark condensates?



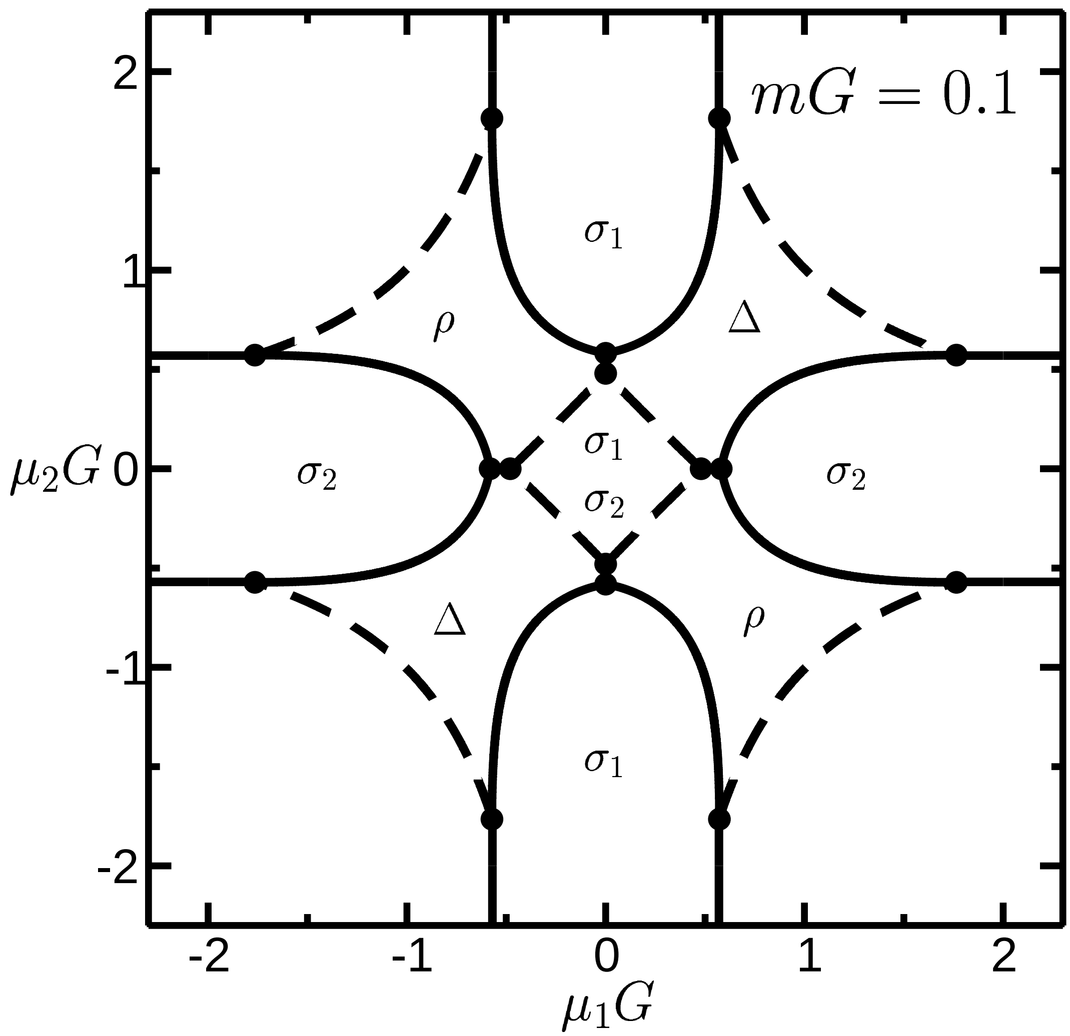


In RM model: Not for $\mu_I = 0$ or $\mu_B = 0$!

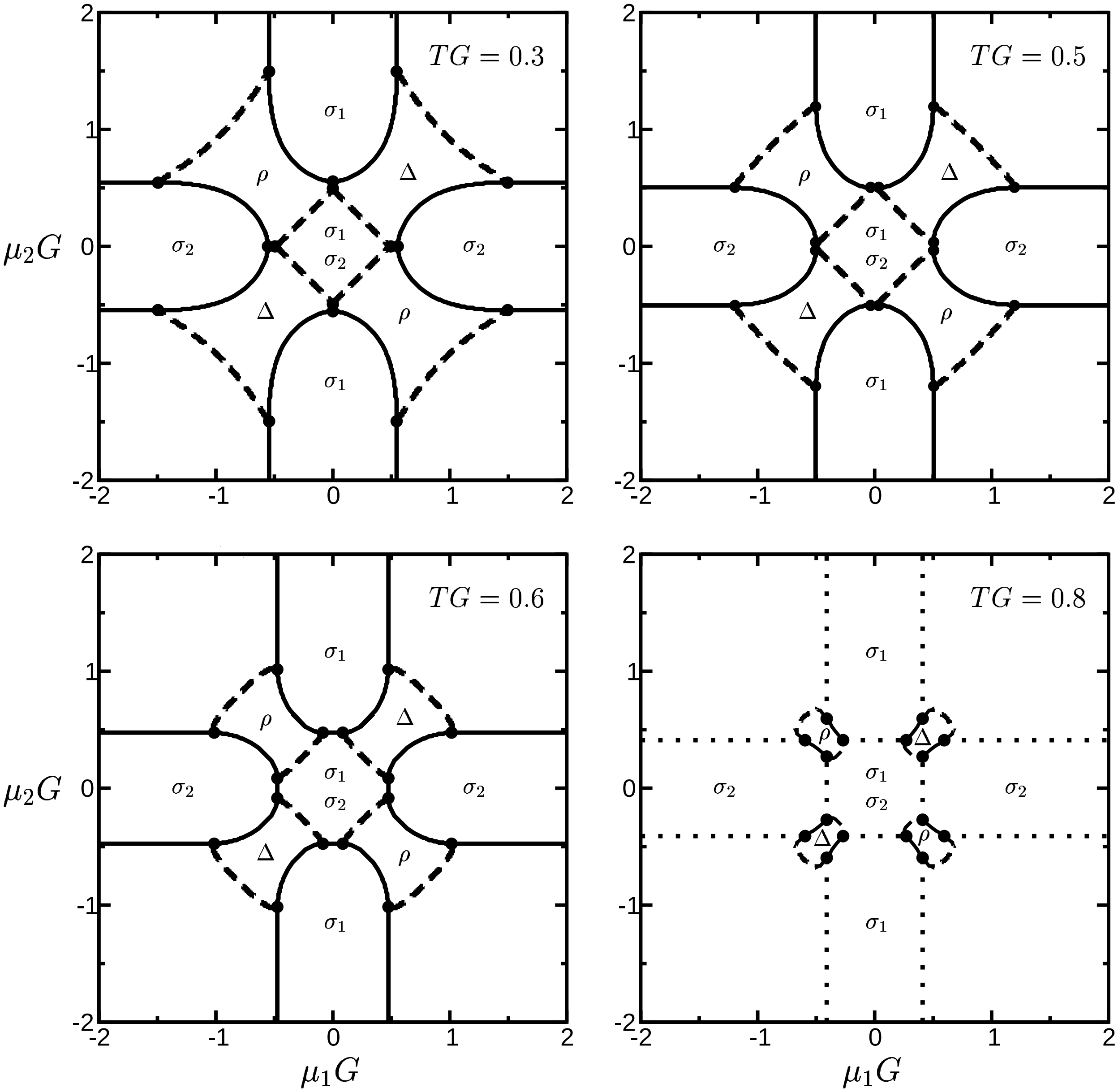
Phase diagram for $N_c = 2$ in the μ_B-μ_I-plane in the chiral limit $mG = 0$

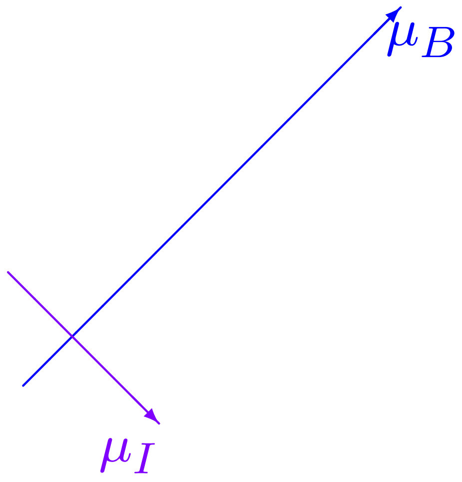


Phase diagram for $N_c = 2$ in the μ_B-μ_I-plane for $mG = 0.1$

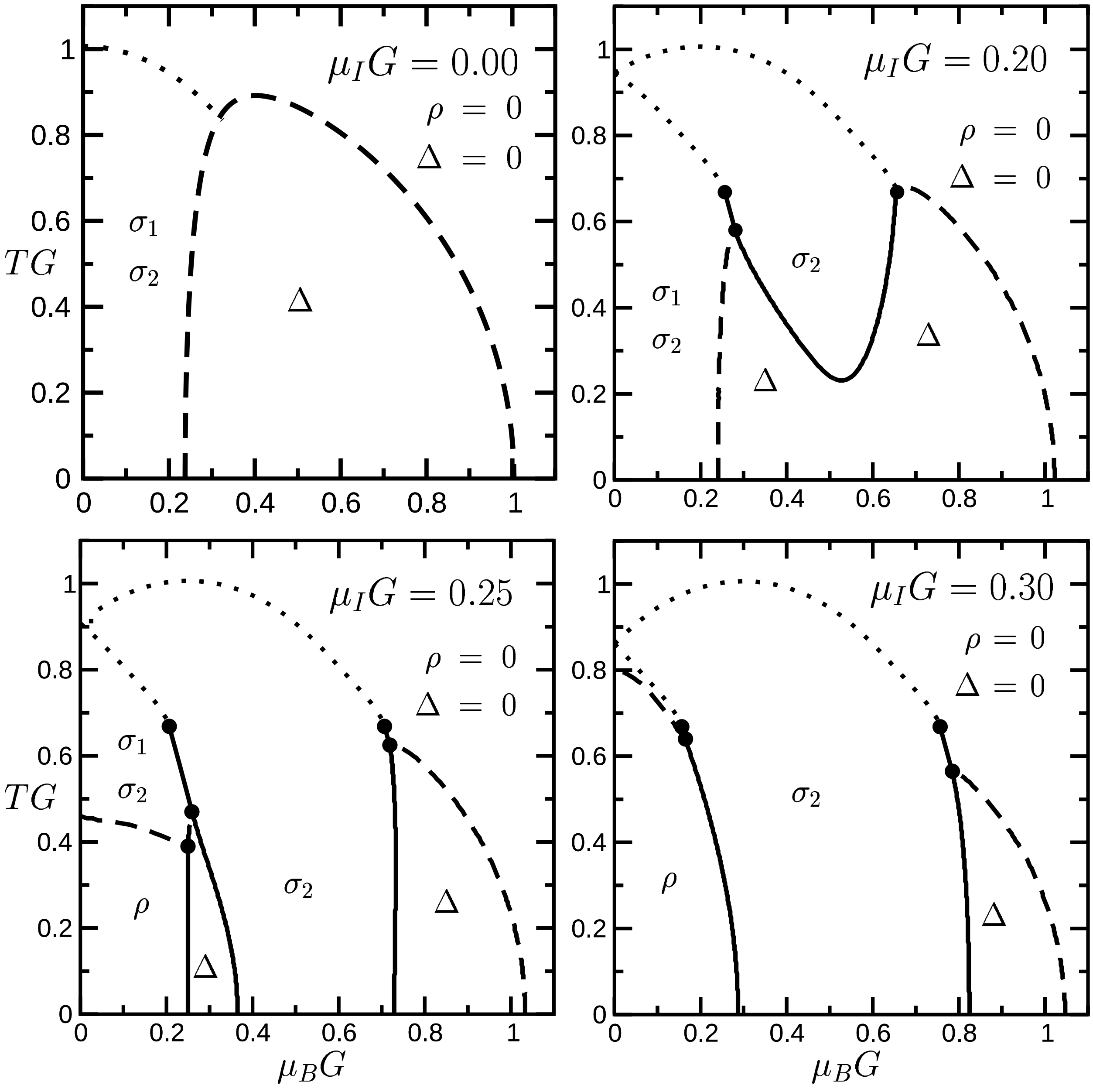


Phase diagram for $N_c = 2$ in the μ_1-μ_2-plane at $mG = 0.1$, TG nonzero



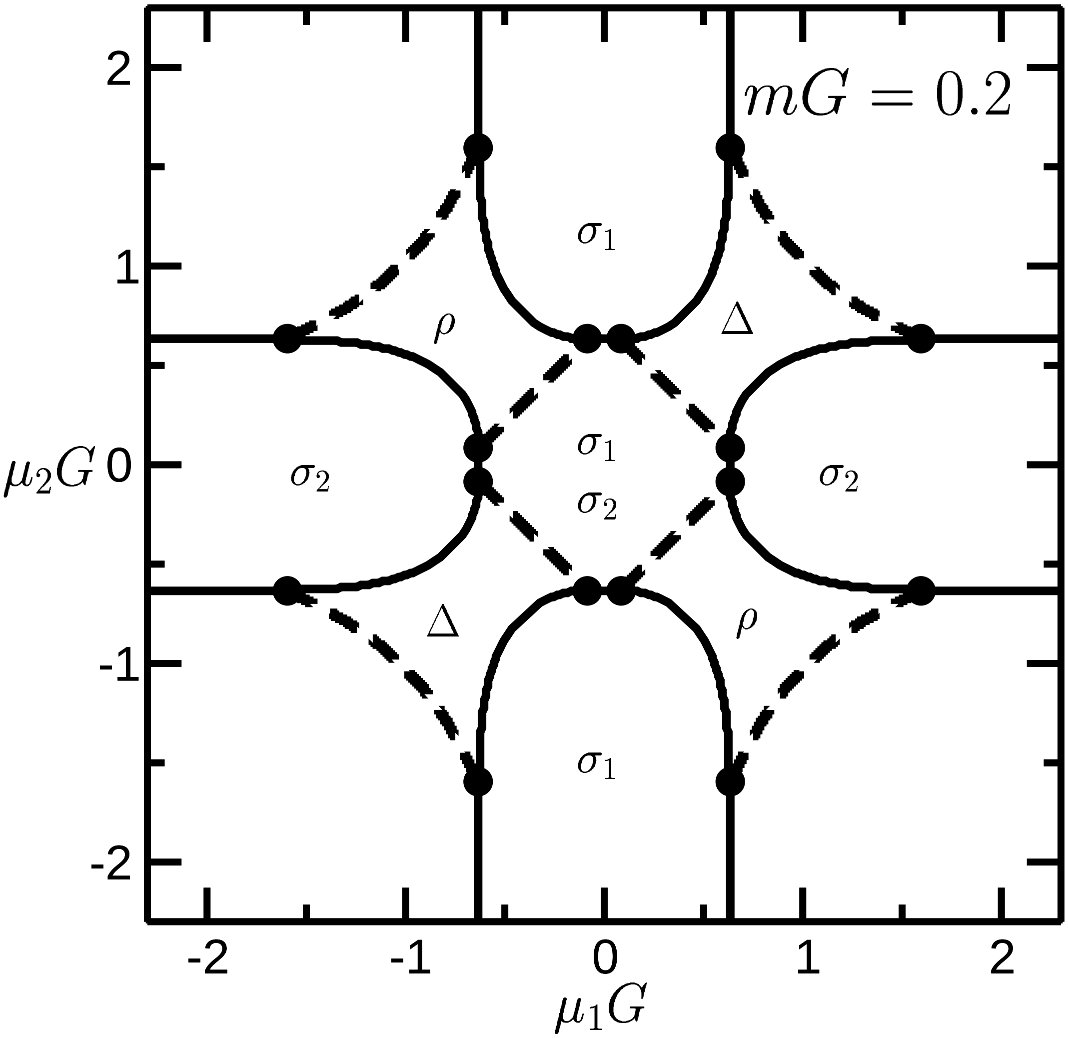

μ_B
μ_I

Phase diagram for $N_c = 2$ in the μ_B-T-plane for fixed $\mu_I G$, at $mG = 0.1$,

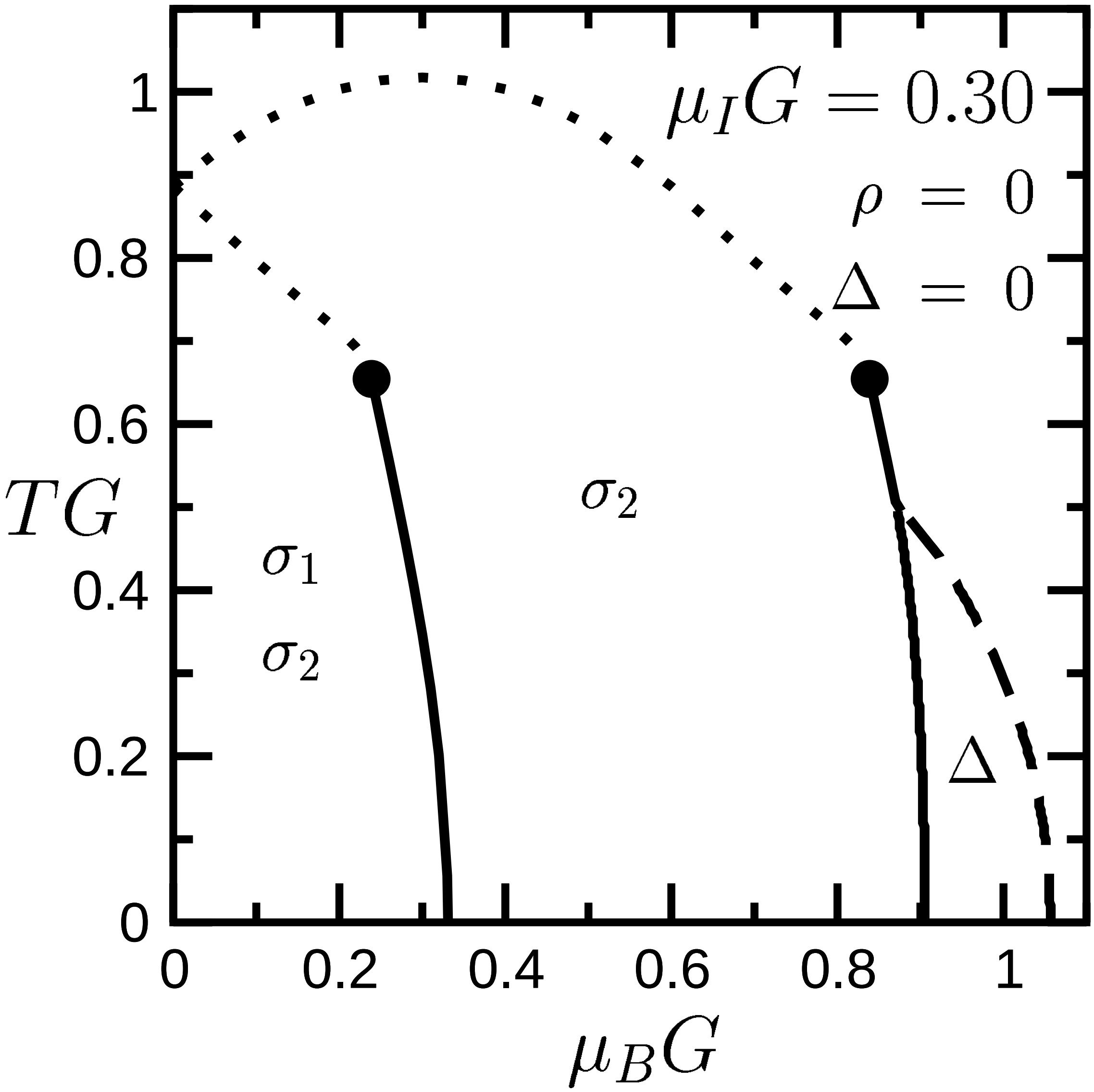


▷ make the quark mass larger (as you can do on the lattice)!

Phase diagram for $N_c = 2$ in the μ_B-μ_I-plane for $mG = 0.2$



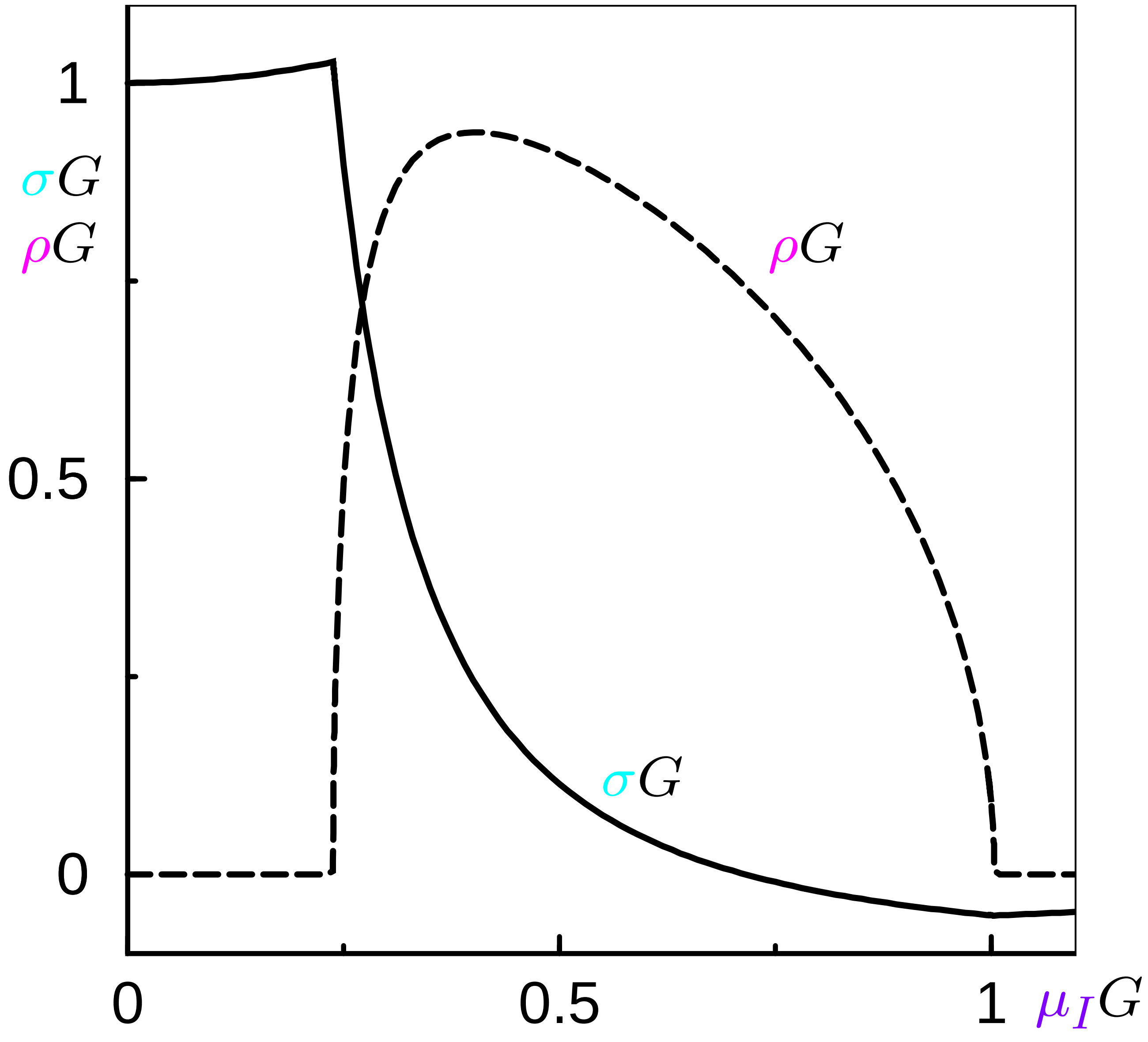
Phase diagram for $N_c = 2$ in the μ_B-T-plane for $\mu_I G = 0.30$, at $mG = 0.2$,



Conclusions

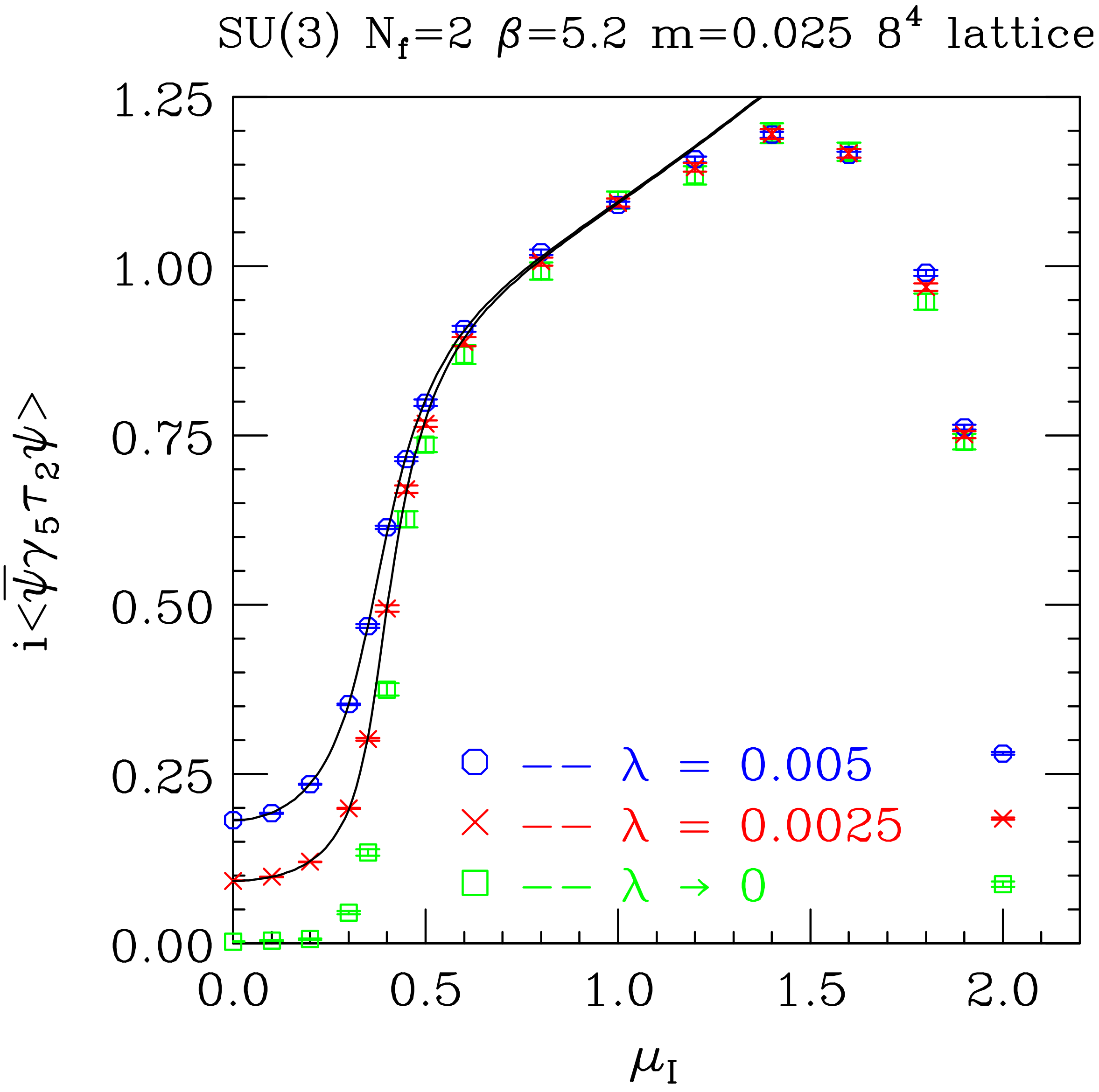
- agreement between RMT models for $N_c = 2$ and chiral Lagrangians at $T = 0$ for small μ_I and μ_B
- rich phase diagram beyond universal region
- symmetry $\mu_I \leftrightarrow \mu_B$ (for $\rho = 0$ and $\triangle = 0$)
  $\Rightarrow T_c(\mu)\Big|_{\mu_I=0} = T_c(\mu)\Big|_{\mu_B=0}$
- two flavor-dependent chiral condensates $\langle\bar{u}u\rangle \neq \langle\bar{d}d\rangle$ for $\mu_1 \neq \mu_2$
- two chiral first order transitions in μ_B-T-plane for appropriate $\mu_I \neq 0$
- two critical second order endpoints
- no tricritical point in the pion and diquark condensation transition
  → not explained by symmetries
- relevant for lattice simulations: regions exist for which critical point should be accessible!

RMT results: Pion and chiral condensates for finite μ_I at $\mu_B = 0$, $mG = 0.1$, $TG = 0$



Lattice results: Pion condensate

[J.B. Kogut, D.K. Sinclair, Phys. Rev. **D66** (2002) 034505]



Lattice results: Chiral condensate

[J.B. Kogut, D.K. Sinclair, Phys. Rev. **D66** (2002) 034505]