Aspects of High Density Effective Theory in QCD

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`QCD and Dense Matter:`

From Lattices to Stars (INT-04-1)

June 17, 2004

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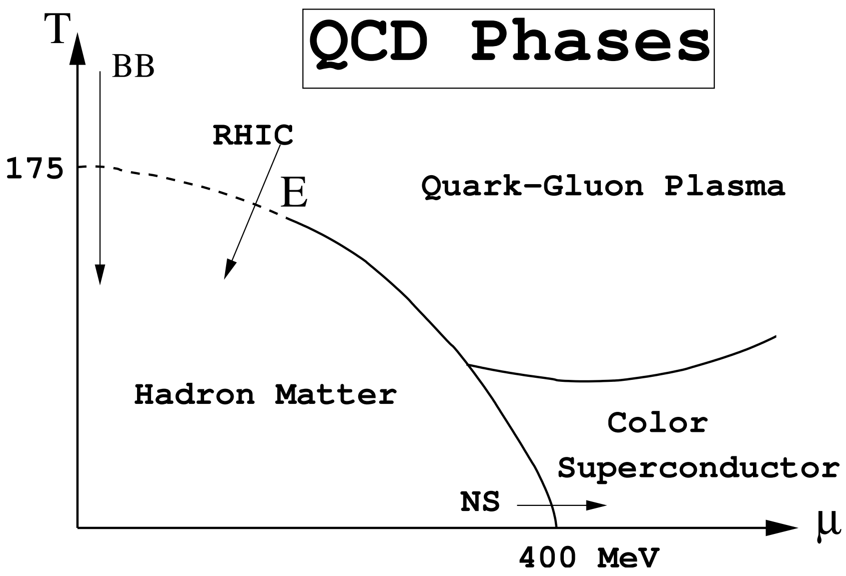
1. Introduction

- QCD describes the strong interaction:
  1. The prediction on high energy hadron interaction is confirmed.
  2. The low energy hadron dynamics is in good agreement with χSB of QCD.
- QCD should tell us how matter behaves at extreme environments:

  heavy ion collision, early universe, compact stars
- QCD predicts phase transitions; $T_C,\ \mu_C \sim \Lambda_{\rm QCD}$



Confirmed partially by Lattice QCD at zero density:

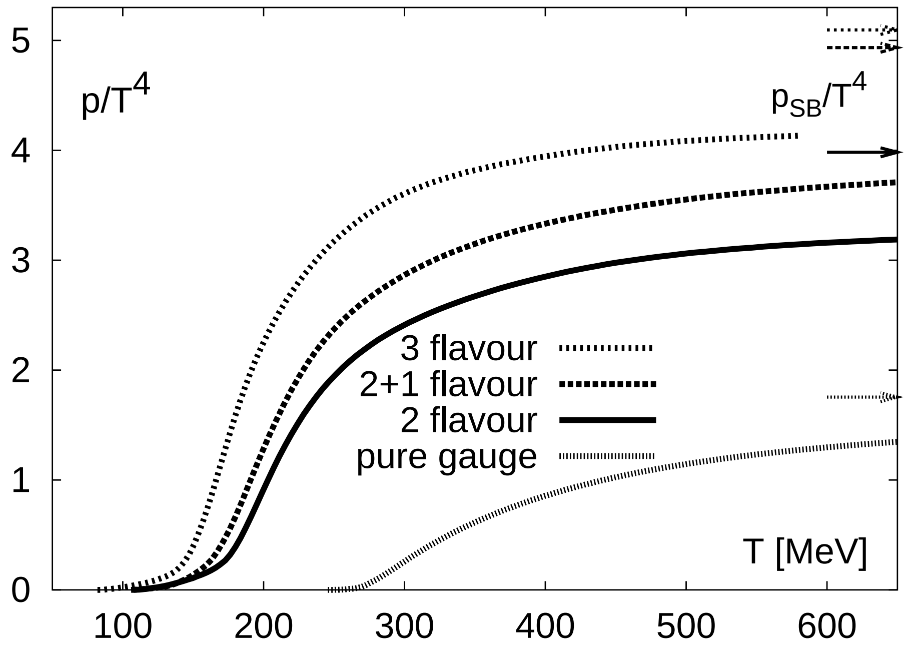


Figure 1: Karsch et. al

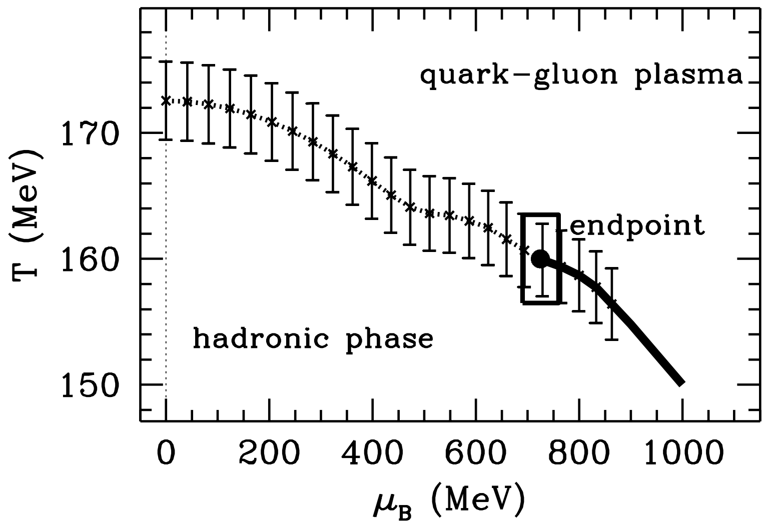
- Lattice QCD at finite density has a notorious sign problem

$$Z(\mu) = \int \mathrm{d}A \det(M)\, e^{-S(A)}$$

where $M = \gamma_E^\mu D_E^\mu + \mu\gamma_E^4 \neq P^{-1} M^\dagger P$.

- Recent progress by reweighting : Fodor and Katz, nucl-th/0201071.

$$Z(\alpha) = \int d\phi \ \det(M,\alpha)\, e^{-S(\phi,\alpha)} = \int d\phi \ \det(M,\alpha_0) e^{-S(\phi,\alpha_0)}\, W(\phi,\alpha)$$



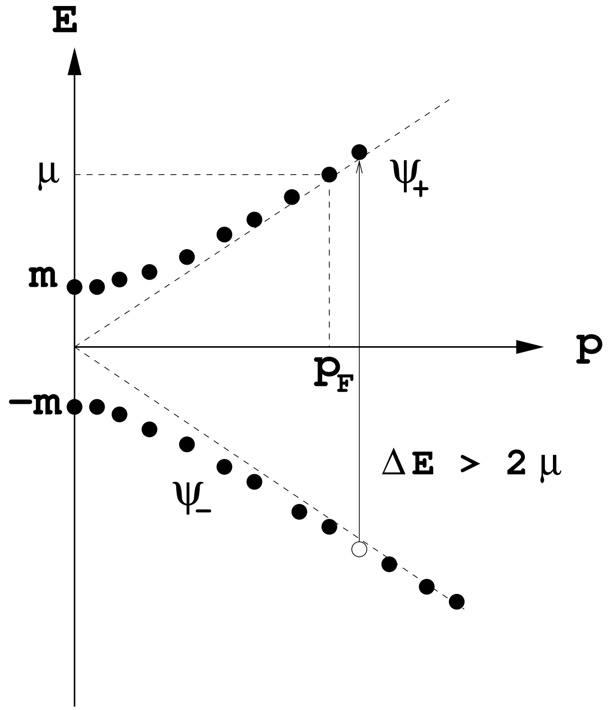
- However, the complexness is due to fast modes ($\omega \gtrsim \mu$): DKH+Hsu, PRD 02, 03.

- The sign problem is mild or absent for physics near the Fermi surface ($\omega \lesssim \mu$).

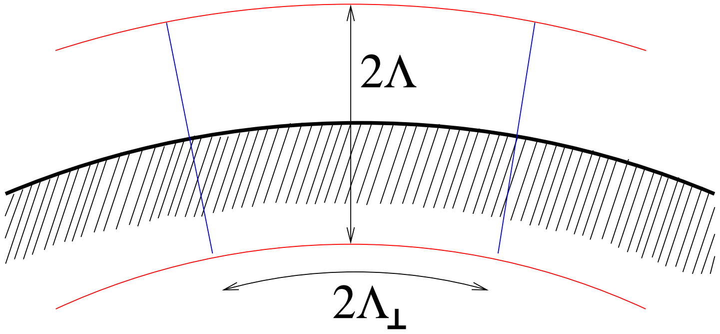
2. High Density Effective Theory

DKH PLB 473 (2000) 118; NPB 582 (2000) 451

- Fermi surface phenomena are determined by modes near F.S.
- **At** $\mu \gg \Lambda_{\rm QCD}$**,** $(\vec{\alpha}\cdot\vec{p}-\mu)\,\psi_{\pm} = E_{\pm}\psi_{\pm}$**.**
- At energy $E \ll 2\mu$, the states near F.S. ($|\vec{p}| \sim p_F$) are easily excited.



- Introduce patches that cover FS only once.



- Pick a quark near F. S. and decompose the quark momentum as

$$p_\mu = \mu v_\mu + l_\mu, \quad |l_\mu| < \Lambda\,,\ \Lambda_\perp\ (\ll \mu).$$

- In the leading order in $1/\mu$ expansion, the energy is independent of $\vec{l}_\perp$;

$$\sum_{\text{patches}} \int_{\Lambda_\perp} \mathrm{d}^2 l_\perp = 4\pi p_F^2.$$

- Some degrees of freedom in QCD are irrelevant in cold QM;

$$\Psi(x) = \sum_{\vec{v}_F} e^{-i\mu\vec{x}\cdot\vec{v}_F} \left[\psi_+(\vec{v}_F, x) + \psi_-(\vec{v}_F, x)\right], \quad \frac{1 \pm \vec{\alpha}\cdot\vec{v}_F}{2}\psi = \psi_\pm \tag{1}$$

- The relevant modes are soft gluons and quasi-quarks near the F.S.

$$\psi_+(\vec{v}_F, x) = \frac{1 + \vec{\alpha}\cdot\vec{v}_F}{2} e^{-i\mu\vec{v}_F\cdot\vec{x}}\psi(x)$$

since at low energy $E < \mu$ the Fermi velocity does not change.

- The quark Lagrangian becomes, $P_\pm = (1 \pm \vec{\alpha}\cdot\vec{v}_F)/2$,

$$\begin{aligned} \mathcal{L} \;&\ni\; \bar{\psi}\left(i\not{D} + \mu\gamma^0\right)\Psi = \sum_{\vec{v}_F} \bar{\psi}(\vec{v}_F, x)\left(P_+ + P_-\right)\left(\mu\not{V} + \not{l}\right)\left(P_+ + P_-\right)\psi(\vec{v}_F, x) \\ &= \bar{\psi}_+ i\not{D}_\parallel \psi_+ + \bar{\psi}_-(2\mu\gamma^0 + i\not{D}_\parallel)\psi_- + \left[\bar{\psi}_- i\not{D}_\perp \psi_+ + \text{h.c.}\right] \end{aligned}$$

- Propagators:

$$S_F^+ = P_+ \frac{i}{\not{l}_\parallel} \quad S_F^- = P_- \frac{i\gamma^0}{2\mu}\left[1 - \frac{i\gamma^0 \not{l}_\parallel}{2\mu} + \cdots\right] \tag{2}$$

- By integrating out ψ_- and hard gluons, we obtain the high density effective theory.
- Tree level matching: We eliminate the irrelevant modes by EOM:

$$\psi_-(\vec{v}_F,x) = -\frac{i\gamma^0}{2\mu + i\not{D}_\parallel}\not{D}_\perp\psi_+(\vec{v}_F,x) = -\frac{i\gamma^0}{2\mu}\sum_{n=0}^{\infty}\left(-\frac{i\not{D}_\parallel}{2\mu}\right)^n\not{D}_\perp\psi_+(\vec{v}_F,x)$$

$$\bar{\psi}_+ i\not{D}_\perp\psi_-(\vec{v}_F,x)\bar{\psi}_- i\not{D}_\perp\psi_+(\vec{v}_F,y) =$$

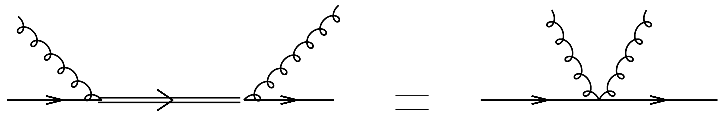


Figure 2: tree-level matching

- New marginal operators for Cooper pairs at one-loop matching :

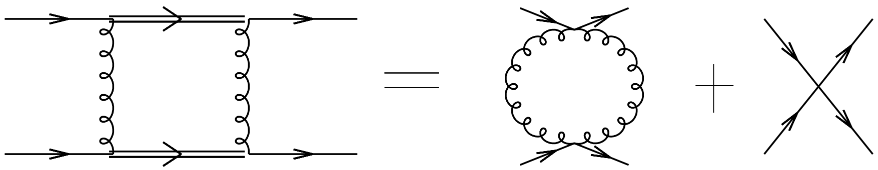


Figure 3: One-loop matching

- HDET has a systematic expansion in $1/\mu$ and α_s:

$$\mathcal{L}_{\rm HDET} = b_1 \bar{\psi}_+ i\gamma_{\shortparallel}^{\mu} D_\mu \psi_+ - \frac{c_1}{2\mu}\bar{\psi}_+\gamma^0 (\not{D}_\perp)^2 \psi_+ \; + \; \cdots, \tag{3}$$

$$b_1 = 1 + O(\alpha_s), \; c_1 = 1 + O(\alpha_s), \cdots$$

- Power counting in HDET:

$$\left(\frac{D_{\shortparallel}}{\mu}\right)^n \cdot \left(\frac{D_\perp}{\mu}\right)^m \cdot \psi_+^l \sim \left(\frac{\Lambda}{\mu}\right)^{n+m} \Lambda^{3l/2}. \tag{4}$$

- To be consistent with the power counting, we impose in loop integration

$$\int_{\Lambda_\perp} \mathrm{d}^2 l_\perp \, l_\perp^n = 0 \quad \text{for} \quad n > 0. \tag{5}$$

3. More on Matching

- Current conservation: In the effective theory, the currents are given in terms of particles and holes (but with no antiparticles) as

$$J^\mu = \sum_{\vec{v}_F} \bar{\psi}(\vec{v}_F, x)\gamma_{\|}^\mu \psi(\vec{v}_F, x) - \frac{1}{2\mu}\psi^\dagger(\vec{v}_F, x)\left[\gamma_\perp^\mu, \, i\not{D}_\perp\right]\psi(\vec{v}_F, x) + \cdots$$

- The HDET current is not conserved unless one adds a counter term,

$$\langle J^\mu(x)J^\nu(y)\rangle = \frac{\delta^2\Gamma_{\mathrm{eff}}}{\delta A_\mu(x)\delta A_\nu(y)} = \int_p e^{-ip\cdot(x-y)}\Pi^{\mu\nu}(p)$$

$$\Pi_{ab}^{\mu\nu}(p) = -\frac{iM^2}{2}\delta_{ab}\int \frac{\mathrm{d}\Omega_{\vec{v}_F}}{4\pi}\left(\frac{-2\vec{p}\cdot\vec{v}_F V^\mu V^\nu}{p\cdot V + i\epsilon\vec{p}\cdot\vec{v}_F}\right)$$

which is not transversal, $p_\mu \Pi_{ab}^{\mu\nu}(p) \neq 0$.

- For the current conservation, we need to add DeBye mass term due to ψ_-.

$$\Gamma^{\rm eff} \mapsto \tilde{\Gamma}^{\rm eff} = \Gamma^{\rm eff} - \int_x \frac{M^2}{2} \sum_{\vec{v}_F} A_\mu A_\nu g_\perp^{\mu\nu}. \tag{6}$$

$$\Pi^{\mu\nu}(p) \mapsto \tilde{\Pi}^{\mu\nu}(p) = \Pi^{\mu\nu} - \frac{i}{2} \sum_{\vec{v}_F} g_\perp^{\mu\nu} M^2, \quad p_\mu \tilde{\Pi}^{\mu\nu} = 0. \tag{7}$$

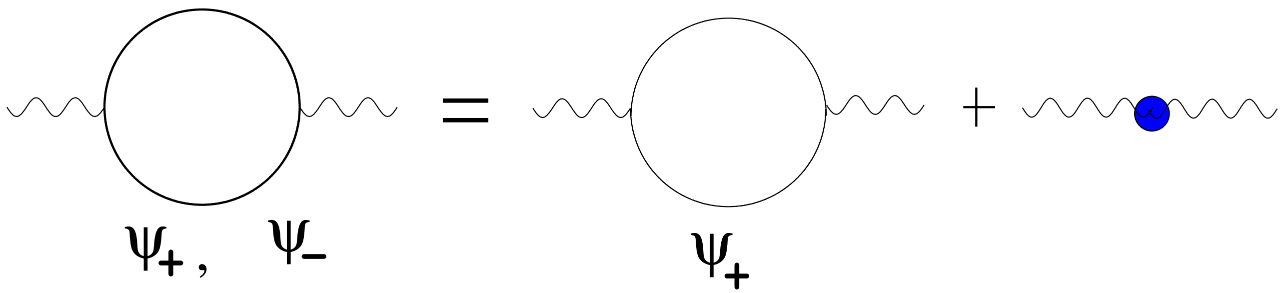


Figure 4: Matching two-point functions

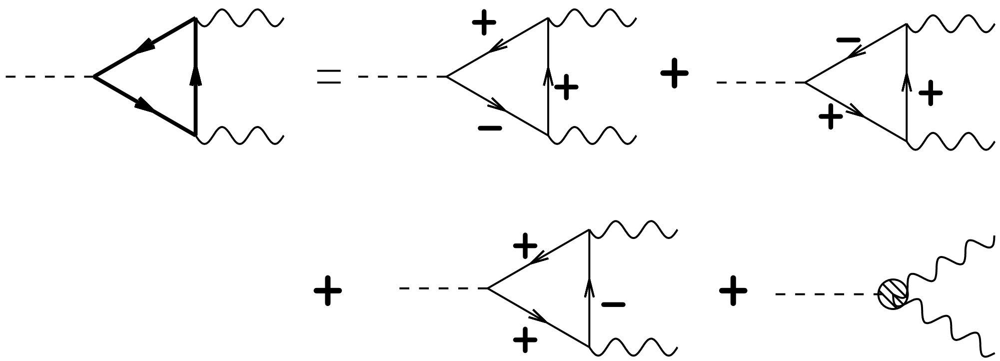
- Axial anomaly in dense QCD is independent of μ (DKH+Hur+Son+Park, to appear)

$$\left\langle \partial_\mu J_5^\mu \right\rangle = \frac{e^2}{8\pi^2}\tilde{F}_{\mu\alpha}F^{\mu\alpha} + \Delta^{\alpha\beta}(\mu)A_\alpha A_\beta, \quad \Delta^{\alpha\beta} = 0. \tag{8}$$

- Axial anomaly due to modes near F.S. is given as

$$\sum_{\vec{v}_F}\int_{x,y} e^{ik_1\cdot x+ik_2\cdot y}\left\langle \partial_\mu J_5^\mu(\vec{v}_F,0)J^\alpha(\vec{v}_F,x)J^\beta(\vec{v}_F,y)\right\rangle = \Delta_{\rm eff}^{\alpha\beta},$$

$$\Delta_{\rm eff}^{0i}(k_1,k_2) = -\frac{e^2}{2\pi^2}\cdot\frac{1}{3}\left(\vec{k}_1\times\vec{k}_2\right)^i, \; \Delta_{\rm eff}^{ij} = \frac{e^2}{2\pi^2}\frac{2}{3}\epsilon^{ijl}\left(k_{10}k_{2l} - k_{1l}k_{20}\right)$$



4. Color superconductivity in dense QCD

- At $\mu > \Lambda_{\rm QCD}$, matter becomes quark matter due to asymptotic freedom.
- Cooper instability of Fermi surface

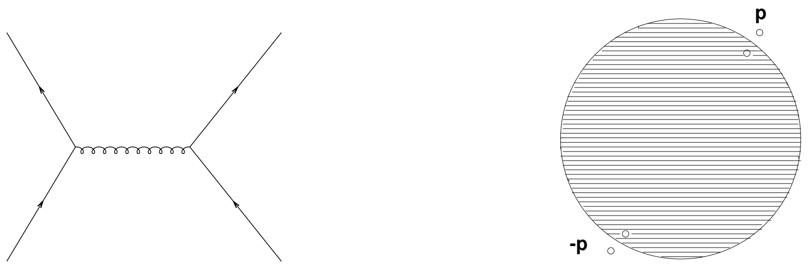


Figure 5: One gluon exchange interaction and Fermi Sea

- Color exchange interaction is attractive for $\bar{3}$ or 8.

$$\langle \psi_i(\vec{p})\psi_j(-\vec{p})\rangle \neq 0 \quad \text{or} \quad \langle \bar{\psi}_i(-\vec{p})\psi_j(\vec{p})\rangle \neq 0$$

- For $N_c = 3$, BCS is preferred to Overhauser:

Shuster, Son '99, Park, Rho, Wirzba, Zahed, '99.

- 2CS phase at an intermediate density ($\mu\Delta_0 < m_s^2$):

$$\left\langle \psi^a_{Li}(\vec{p})\psi^b_{Lj}(-\vec{p})\right\rangle = -\left\langle \psi^a_{Ri}(\vec{p})\psi^b_{Rj}(-\vec{p})\right\rangle = \epsilon_{ab}\epsilon^{ij3}\Delta$$

- Color-Flavor Locking (CFL) at $\mu\Delta_0 > m_s^2$:

$$\left\langle \psi^a_{Li}(\vec{p})\psi^b_{Lj}(-\vec{p})\right\rangle = -\left\langle \psi^a_{Ri}(\vec{p})\psi^b_{Rj}(-\vec{p})\right\rangle = k_1\delta^a_i\delta^b_j + k_2\delta^a_j\delta^b_i,$$

Alford, Rajagopal, Wilczek '98

- LOFF when $\mu_e(=\delta\mu) > \Delta_0$

$$\left\langle \psi^a_u(\vec{p}_u)\psi^b_d(\vec{p}_d)\right\rangle = \epsilon^{ab3}\Delta(\vec{q}), \quad \vec{p}_u + \vec{p}_d = 2\vec{q}$$

Larkin+Ovchinnikov '64, Fulde+Ferrel '64; Alford+Bowers+Rajagopal '01

- At the intermediate density gluons are screened:

$$\mathcal{L}_{\rm QCD}^{\rm eff} \ni \frac{G}{2}\bar{\psi}\psi\bar{\psi}\psi + \cdots, \tag{9}$$

BCS gap equation is given as

$$0 = \frac{\partial V_{\rm BCS}(\Delta)}{\partial \Delta} \quad = \quad \frac{\Delta}{G} - 4\int \frac{d^4k}{(2\pi)^4}\frac{\Delta}{k_0^2 + (\vec{k}\cdot\vec{v}_F)^2 + \Delta^2}$$

- Pole structure guarantees solution.

$$k_0 = \pm\sqrt{(\vec{k}\cdot\vec{v}_F)^2 + \Delta^2} \mp i\epsilon \tag{10}$$

The BCS gap

$$\Delta_0 = 2\bar{\mu}\exp\left(-\frac{\pi^2}{2G\bar{\mu}^2}\right). \tag{11}$$

is estimated to be $10 \sim 100$ MeV at the intermediate density.

- The Cooper-pair gap equation for up and strange quark under stress

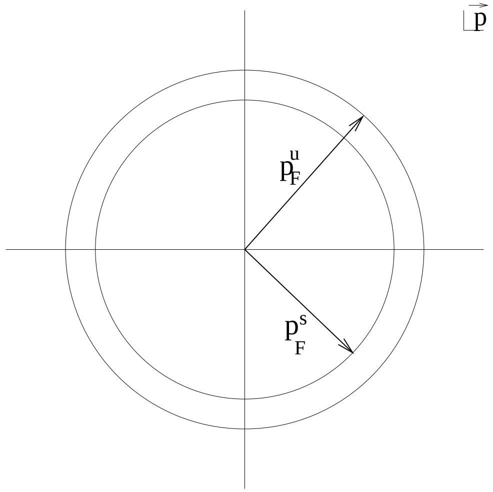


Figure 6: Fermi sea of up and strange quarks.

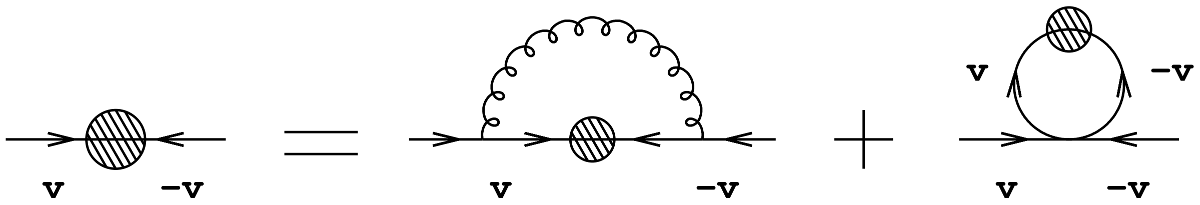
$$\Delta(p) = \int_l \frac{i\Delta(l)\, K(p-l)}{\left[(1+i\epsilon)l_0 - \vec{l}\cdot\vec{v}_u + \delta\mu^u\right]\left[(1+i\epsilon)l_0 + \vec{l}\cdot\vec{v}_s - \delta\mu^u\right] - \Delta^2}\,, \quad (12)$$

where $\delta\mu^u = \mu - \bar{p}$ and $\delta\mu^s = \mu - \sqrt{\bar{p}^2 + M_s^2}$. ($\bar{p}$ is the pairing momentum and K is kernel.) Gap is biggest if $\delta\mu^u = -\delta\mu^s$ or $\bar{p} = \mu - \frac{M_s^2}{4\mu}$.

- Pole structure changes and gap closes if $-\delta\mu^u\delta\mu^s > \Delta^2/4$ or $\Delta < M_s^2/(2\mu)$.

5. Higher Order Corrections

- At high density magnetic gluons are not screened. The long-range pairing force leads to the Eliashberg gap equation :



$$\Delta(p_0) \;=\; \frac{g_s^2}{36\pi^2}\int_{-\mu}^{\mu} dq_0 \frac{\Delta(q_0)}{\sqrt{q_0^2+\Delta^2}} \ln\left(\frac{\bar{\Lambda}}{|p_0-q_0|}\right) . \tag{13}$$

$\bar{\Lambda} = 4\mu/\pi \cdot (\mu/M)^5 e^{3/2\xi}.$ (ξ is a gauge parameter.)

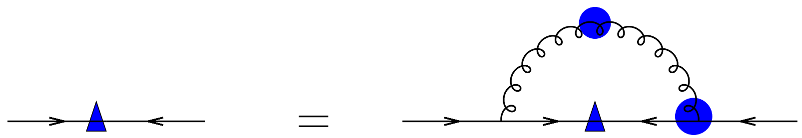
Son '98, DKH '99,DKH, Miransky, Shovkovy, Wijewardhana '99
Schäfer, Wilczek '99, Pisarski, Rischke '99

- Cooper pair gap at high density is

$$\Delta_0 = \frac{2^7\pi^4}{N_f^{5/2}} e^{3\xi/2+1} \cdot \frac{\mu}{g_s^5} \exp\left(-\frac{3\pi^2}{\sqrt{2}g_s}\right) . \tag{14}$$

- The gap equation in dense QCD takes a following form;

$$\Delta(p_0) \quad = \quad \frac{g_s^2}{c^2}\int \mathrm{d}q_0 \frac{\Delta(q_0)}{\sqrt{q_0^2+\Delta^2}}\left[(1+\eta)\ln\left(\frac{\mu}{|p_0-q_0|}\right)+\ln\, b+\zeta\right]. \tag{15}$$



- We use

$$T^a\, a(p)\left[(p_0+\mu)\gamma^0+b(p)\ \vec{\not p}\right]-a(p')\left[(p'_0+\mu)\gamma^0+b(p')\ \vec{\not p}'\right]T^a$$
$$=(p-p')_\mu\Lambda^\mu(p,p')T^a\ +\Gamma^a(p,p';-p-p'). \tag{16}$$

where

$$\Gamma^a(p,p';k)\delta(k+p+p')=\int_{z,x,y} e^{i(z\cdot k+x\cdot p+y\cdot p')}\left\langle\partial^\mu j_\mu^a(z)\psi(x)\bar\psi(y)\right\rangle. \tag{17}$$

- The one-loop vertex correction has two parts;

(a) (b)

Figure 7: The solid line denotes quarks and the curly lines gluons.

- While second part negligible, (Fig. 7a) is related to the correction to the wavefunction renormalization constant as

$$(p - p')_\mu \Lambda^{(a)\mu} = a(p) \left[(p_0 + \mu)\gamma^0 + b(p) \not{\vec{p}} \right] - a(p') \left[(p'_0 + \mu)\gamma^0 + b(p') \not{\vec{p}}' \right] .$$

- The nonlocal gauge, where $a(p) = 1 = b(p)$, is found to be

$$\xi \approx \frac{\frac{2}{3} \ln \frac{(2\mu)^3}{M_0^2 \Delta + \pi M^2 |p_4 - q_4|/2}}{\ln \frac{(2\mu)^2}{|p_4 - q_4|^2}} \approx \frac{1}{3}. \tag{18}$$

6. Positivity of HDET and Vafa-Witten Theorem

Simple example in (1+1) dimensions

- Euclidean (1+1) action of non-relativistic fermions interacting with a gauge field A

$$S = \int d\tau dx\, \psi^*_\sigma \left[(-\partial_\tau + i\phi + \epsilon_F) - \epsilon(-i\partial_x + A) \right] \psi_\sigma \tag{19}$$

where $\epsilon(p)$ is the energy as a function of momentum, $\epsilon(p) \approx \frac{p^2}{2m} + \cdots$.

- Dispersion relation with chemical potential: $E(p) = \epsilon(p) - \epsilon_F$. Low energy modes have momentum near $\pm p_F$ ($\epsilon(\pm p_F) = \epsilon_F$).

- Near the Fermi points, the energy as a function of momentum,

$$E(p \pm p_F) \approx \pm v_F p, \quad v_F = \partial E / \partial p|_{p_F} \tag{20}$$

- Action (19) not obviously positive. Operator in brackets $[\ \cdots\ ]$ has complex eigenvalues.

- Assume gauge field has small amplitude and is slowly varying relative to scale p_F. Extract the slowly varying component of the fermion $\longrightarrow$ low energy effective theory involving quasiparticles and gauge fields with positive, semi-definite determinant.

- Extract quasiparticle modes $\psi_{L,R}$:

$$\psi(x,\tau) \;=\; \psi_L e^{+ip_F x} \;+\; \psi_R e^{-ip_F x} \quad , \tag{21}$$

Using $e^{\pm ip_F x}\, E(-i\partial_x + A)\, e^{\mp ip_F x}\, \psi(x) \approx \pm\, v_F(-i\partial_x + A)\psi(x)$, to obtain

$$\begin{aligned} S_{\rm eff} \quad &= \quad \int d\tau\, dx\, [\psi_L^\dagger(-\partial_\tau + i\phi + i\partial_x - A)\psi_L \\ &+ \quad \psi_R^*(-\partial_\tau + i\phi - i\partial_x + A)\psi_R]. \end{aligned} \tag{22}$$

- Introducing the Euclidean (1+1) gamma matrices $\gamma_{0,1,2}$ and $\psi_{L,R} = \frac{1}{2}(1 \pm \gamma_2)\psi$ we obtain a positive action:

$$S_{\rm eff} = \int d\tau dx\; \bar\psi\gamma^\mu(\partial_\mu + iA_\mu)\psi \;\equiv\; \int d\tau dx\; \bar\psi \not{D}\psi \quad . \tag{23}$$

- Since $(\partial_\mu + iA_\mu)$ is anti-Hermitian, the operator $\not{D}$ in (23) has purely imaginary eigenvalues. Since γ_2 anticommutes with $\not{D}$, the eigenvalues come in conjugate pairs: given $\not{D}\phi = \lambda\phi$, we have

$$\not{D}(\gamma_2\phi) = -\gamma_2\not{D}\phi = -\gamma_2\lambda\phi = -\lambda(\gamma_2\phi_n) \quad .$$

  Hence the determinant $\det \not{D} = \prod \lambda^* \lambda$ is real and positive semi-definite.

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  Hence the determinant $\det \not{D} = \prod \lambda^* \lambda$ is real and positive semi-definite.

- By considering only the low-energy modes near the Fermi points, we obtain an effective theory with desirable positivity properties.

- Note: interactions (background gauge field A) must not strongly couple the low-energy modes to fast modes which are far from the Fermi points – reasonable approximation in many situations: interactions among quasiparticles of primary interest.

- RECIPE: Near Fermi surface, modes have low energy and are slowly varying. Coupling these modes to slowly varying background field A leads to a positive effective theory.

- Slowly varying = relative to Fermi momentum p_F.

- QCD: strong coupling dynamics at scales $\sim \Lambda_{\rm QCD}$. By taking

$$\mu \sim p_F >> \Lambda_{\rm QCD}$$

we ensure that quark quasiparticles couple only to slowly varying, small amplitude background fields A.

Degenerate free NR fermions

- Consider an electron system, described by

$$\mathcal{L} = \psi^\dagger \left[i\partial_t - \epsilon(\vec{p}) \right] \psi + \mu \psi^\dagger \psi, \tag{24}$$

- The system suffer "*sign problem*", since the Euclidean determinant has complex eigenvalues,

$$M = -\partial_\tau - \epsilon(\vec{p}) + \mu. \tag{25}$$

- For modes near the Fermi surface, however, the sign problem becomes mild: The determinant becomes, if we integrate the fast modes,

$$M_{\rm EFT} = -\partial_\tau - \vec{v}_F \cdot \vec{l}, \quad M_{\rm EFT}(\vec{v}_F) M_{\rm EFT}(-\vec{v}_F) \leq 0, \tag{26}$$

if $\epsilon(\vec{p}) = \epsilon(-\vec{p})$.

Positivity of QCD at asymptotic density

- The quasiparticles are described by

$$\mathcal{L}_{eff} = \bar{\psi}_+ \gamma_{\|} \cdot D \psi_+(\vec{v}_F, x) - \psi_+^\dagger \frac{(\not{D}_\perp)^2}{2\mu} \psi_+(\vec{v}_F, x) + \cdots$$

- The leading term has a positive determinant:

$$M_{\rm eft} \,=\, \gamma_{\|}^E \cdot D(A) \,=\, \gamma_5 M_{\rm eft}^\dagger \gamma_5 \tag{27}$$

- Anomaly

$$i\not{D}_{\|} P_+ \psi = P_- i\not{D}_{\|} \psi \ . \tag{28}$$

The divergence of the quark current at one loop is

$$\langle \partial_\mu J^{a\mu}(\vec{v}_F, x) \rangle = g_s \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot x} p^\mu \Pi^{ab}_{\mu\nu}(p) A^{b\nu}_{\|}(-p) \neq 0\,. \tag{29}$$

- However, if we include $\psi_+(\vec{v}_F, x)$ and $\psi_+(-\vec{v}_F, x)$ the current is conserved:

$$\langle \partial_\mu J^\mu_a(\vec{v}_F, x) + \partial_\mu J^\mu_a(-\vec{v}_F, x) \rangle = 0. \tag{30}$$

- Under a gauge transformation, $U(x) = e^{i\vec{q}\cdot\vec{x}}$, the energy level shifts

$$E = \vec{l}\cdot\vec{v}_F \mapsto E = \vec{l}\cdot\vec{v}_F + \vec{q}\cdot\vec{v}_F. \tag{31}$$

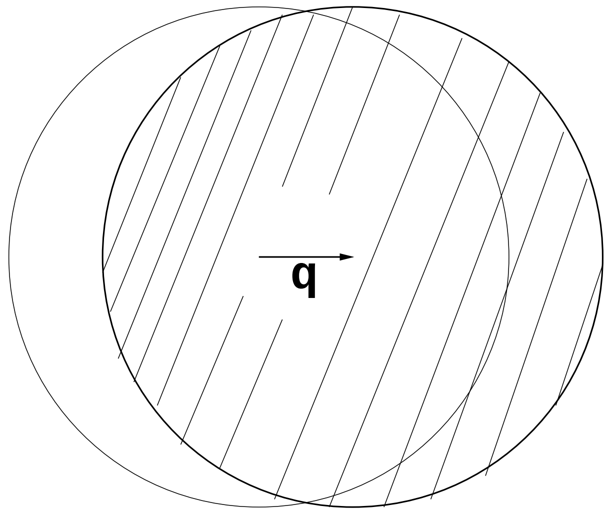


Figure 8: Spectral Flow

Vafa-Witten Theorem at High Density: CFL is exact.

- Cooper theorem says pairing at color anti-triplet channel. For three light flavors,

$$\left\langle \psi_{L}{}^{a}_{i\alpha}(\vec{v}_F, x)\psi_{L}{}^{b}_{j\beta}(-\vec{v}_F, x)\right\rangle = -\left\langle \psi_{R}{}^{a}_{i\alpha}\psi_{R}{}^{b}_{j\beta}\right\rangle = \epsilon_{ij}\epsilon^{abc}\epsilon_{\alpha\beta\gamma}K_c^{\gamma}(p_F)$$

- By the global color and flavor symmetry, $K_c^{\gamma} = \delta_c^{\gamma} K_{\gamma}$.

- The vacuum energy in the HDL approximation is given as

$$\begin{aligned} V(\Delta) &= -\mathrm{Tr}\ \ln S^{-1} + \mathrm{Tr}\ \ln \not{\partial} + \mathrm{Tr}\ (S^{-1} - \not{\partial})S + (\text{2PI diagrams}) \\ &= \frac{\mu^2}{4\pi}\sum_{i=1}^{9}\int \frac{d^2 l_{\parallel}}{(2\pi)^2}\left[\ln\left(\frac{l_{\parallel}^2}{l_{\parallel}^2 + \Delta_i^2(l_{\parallel})}\right) + \frac{1}{2}\cdot\frac{\Delta_i^2(l_{\parallel})}{l_{\parallel}^2 + \Delta_i^2(l_{\parallel})}\right] + h.o. \end{aligned}$$

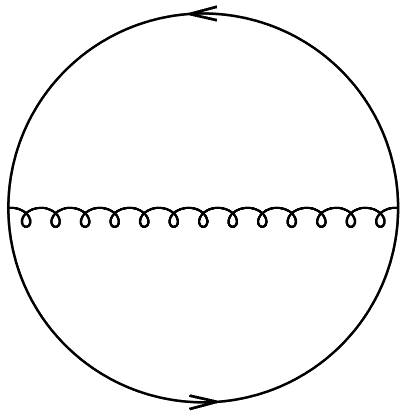


Figure 9: The 2PI vacuum energy diagram.

- The gluon energy is subleading, $V_g(\Delta) \sim M^2\Delta^2 \ln(\Delta/\mu) \sim g_s\mu^2\Delta^2$,

$$V(\Delta) \simeq -0.43\frac{\mu^2}{4\pi^2}\sum_i |\Delta_i(0)|^2$$

- Since $\Delta^{\alpha\beta}_{ab} = \epsilon_{\alpha\beta\gamma}\epsilon^{abc}\Delta_\gamma\delta^\gamma_c$,

$$V(\Delta) \simeq -0.43\frac{\mu^2}{4\pi^2}\,|\Delta_u|^2\,f(x,y),$$

  where $\Delta_d/\Delta_u = x$ and $\Delta_s/\Delta_u = y$. $f(x,y) \leq 13.4$ has a maximum at $x = 1 = y$.

- $K^\gamma_\alpha = K\delta^\gamma_\alpha$ and $SU(3)_V$ is unbroken

$$\Delta_i = \Delta_u \cdot (1, 1, 1, -1, 1, -1, 1, -1, -2).$$

- CFL is exact at asymptotic density:

  Vector current correlators fall off exponentially, if all quarks are gapped.

  $$\left\langle J_\mu^A(\vec{v}_F,x) J_\nu^B(\vec{v}_F,y)\right\rangle^A = -\mathrm{Tr}\,\gamma_\mu T^A S^A(x,y;\Delta)\gamma_\nu T^B S^A(y,x;\Delta),$$

  with $J_\mu^A(\vec{v}_F,x) = \bar{\psi}_+(\vec{v}_F,x)\gamma_\mu T^A \psi_+(\vec{v}_F,x)$.

- The (anomalous) propagator with $SU(3)_V$-invariant IR regulator Δ is given as

  $$\langle x|\,\frac{1}{M}\,|y\rangle = \int_0^\infty \mathrm{d}\tau\,\langle x|\,e^{-i\tau(-iM)}\,|y\rangle$$

  where with $D = \partial + iA$

  $$M = \gamma_0 \begin{pmatrix} D\cdot V & \Delta \\ \Delta & D\cdot\bar{V} \end{pmatrix},$$

  Since the eigenvalues of M are bound from below by Δ, we have

  $$\left|\langle x|\,\frac{1}{M}\,|y\rangle\right| \le \int_R^\infty \mathrm{d}\tau\, e^{-\Delta\tau}\sqrt{\langle x|x\rangle}\sqrt{\langle y|y\rangle} = \frac{e^{-\Delta R}}{\Delta}\sqrt{\langle x|x\rangle}\sqrt{\langle y|y\rangle}.$$

- The current correlators fall off rapidly as $R \equiv |x-y| \to \infty$;

$$\left| \int \mathrm{d}A_+ \det M_{\rm eff}(A) e^{-S_{\rm eff}} \left\langle J^A_\mu(\vec{v}_F, x) J^B_\nu(\vec{v}_F, y) \right\rangle^{A_+} \right| \leq \int_{A_+} \left| \left\langle J^A_\mu(x) J^B_\nu(y) \right\rangle^{A_+} \right| \leq \frac{e^{-2\Delta R}}{\Delta^2} \int_{A_+} \sqrt{\langle x|x\rangle}\sqrt{\langle y|y\rangle}$$

- The (IR regulated) vector currents do not create a massless mode out of vacuum or Fermi sea.

- No Goldstone mode along the $SU(3)_V$ channel!

- For three light flavors $SU(3)_V$ has to be unbroken as in CFL.

- When $m \neq 0$, one has to take the $\mu \to \infty$ and $m_q \to 0$ limit carefully. If m_s goes to zero too slowly, Kaon condenses to break isospin.

Operator formalism

- We introduce an operator formalism for lattice

$$\vec{v} = \frac{-i}{\sqrt{-\nabla^2}} \frac{\partial}{\partial \vec{x}} \quad , \tag{32}$$

- The quasi-quarks near F.S. are defined as

$$\psi = \exp\left(+i\mu x \cdot v \; \alpha \cdot v\right) \psi_+ \quad , \tag{33}$$

- Now, the Lagrangian becomes with $X = \mu x \cdot v \; \alpha \cdot v$

$$\mathcal{L}_+ = \bar{\psi}_+ \gamma_{||}^{\mu} \left(\partial^\mu + i A_+^\mu \right) \psi_+ \; , \quad \left(A_+^\mu = e^{-iX} \; A^\mu \; e^{+iX} \right) \tag{34}$$

- Using $v \cdot \partial \, v \cdot \gamma = \partial \cdot \gamma$, we get

$$\gamma_{||}^{\mu} \partial^{\mu} = \gamma^{\mu} \partial^{\mu} \tag{35}$$

- Integrating out the fast modes (ψ_- and hard gluons), the slow modes have a positive measure in the leading order with an effective action

$$S_{\rm eff}(A) \approx \int \mathrm{d}^4 x_E \left(\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{M^2}{16\pi} \sum_{\vec{v}_F} A^a_{\perp\mu} A^a_{\perp\mu} \right) \geq 0, \tag{36}$$

where $A_\perp = A - A_\|$ and the Debye mass is $M = \sqrt{N_f/(2\pi^2)} g_s \mu$.

- QCD partition function now becomes

$$Z = \int \mathrm{d}A \mathrm{d}\psi \mathrm{d}\bar{\psi} e^{-S(A,\psi,\bar{\psi})} = \int \mathrm{d}A_+ \det\left(M_{\rm eff}\right) e^{-S_{\rm eff}(A_+)} \tag{37}$$

$$\det\ M_{\rm eff} \geq 0, \quad S_{\rm eff}(A_+) = \text{positive} \cdot [1 + O(1/\mu)].$$

7. Lattice Simulation

- Back to the original (not HDET) QCD partition function:

$$Z(\mu) = \int \mathrm{d}A_\mu \;\det(M)\; e^{-S(A_\mu)} \;.$$

M is the Dirac operator at finite density, **A** the usual gauge field.

- It is easy to show that at **A** = 0 (zero background gauge field), the Dirac determinant is real even at finite density.

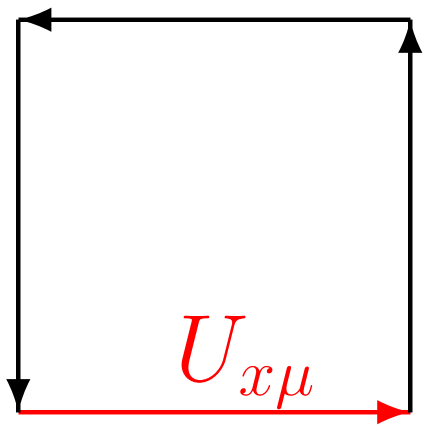
- Now, consider small amplitude, slowly varying background gauge fields **A** whose magnitude and derivatives $\partial\mathbf{A}$ are small relative to μ. (e.g. $\mu >> \Lambda_{\rm QCD} \sim \mathbf{A}\;,\; \partial\mathbf{A}$, or $\mathbf{F}_{\mu\nu}, \mathbf{D}_\mu$.)

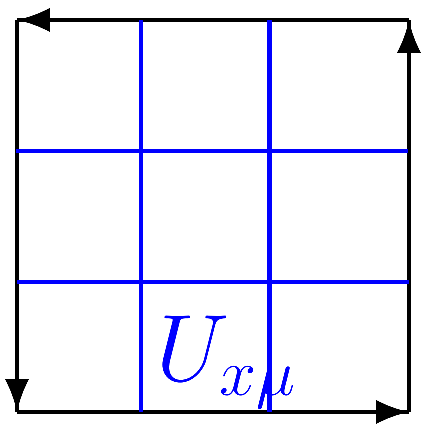
- Expand about the FS. Integrate out heavy modes (antiquarks, quarks far from the FS). These modes contribute to $\det(M)$, but their contribution is suppressed by $1/\mu$. Find

$$\det(M) \;=\; [\mathrm{real},\mathrm{positive}]\left(1 \;+\; \mathcal{O}(\frac{\mathbf{F}}{\mu^2})\right)$$

- How to enforce small amplitude, slowly varying gauge field **A**?
- Use two lattices with different spacings $a_{\rm det}$, $a_{\rm gauge}$. Compute determinant on lattice with spacing $a_{\rm det} \sim \mu^{-1} << a_{\rm gauge}$.
- Determinant is a function of plaquettes $\{\mathbf{U}_{\mathbf{x}\mu}\}$ which are obtained by interpolation from the plaquettes on the coarser a_{gauge} lattice.
- Interpolation: link variables $\mathbf{U}_{\mathbf{x}\mu} \in SU(3)$. Connect any two points g_1, g_2 on the group manifold:

$$g(t) = g_1 + t(g_2 - g_1)\ ,\ 0 \leq t \leq 1$$





- Use leading real, positive part of determinant for importance sampling.
- Nontrivial check on analytic results at asymptotic density. Extrapolate to intermediate density?

8. Conclusion

- We have constructed an effective theory (HDET) for dense QCD.
  1. It deals with relevant modes only and is effectively (1+1) dim'nal.
  2. Marginal Cooper-pair operator, Screening mass, $\cdots$ naturally arise at one-loop.
  3. Systematic expansion in $1/\mu$ and α_s.
  4. It has a consistent power counting rule.
- With this effective theory, we calculate the properties of varous phases in dense QCD: Cooper-pair gap, Critical temperature and density ...
  1. At high density, the superconducting gap takes at two-loop

$$\Delta_0 = \frac{2^7\pi^4}{N_f^{5/2}} e^{1.5} \cdot \frac{\mu}{g_s^5} \exp\left(-\frac{3\pi^2}{\sqrt{2}g_s}\right).$$

  2. Critical phenomena at $T_C = 0.57\Delta_0$ or $\mu_C \simeq 0.22\mathrm{GeV}$.

- Quark matter under stress.
  1. Crystalline superconductor if $\Delta_0/\sqrt{2} < (\mu_d - \mu_u)/2 < 0.74\Delta_0$
  2. Kaon condensation if $m_s > m^{1/3}\Delta_0^{2/3}$.
- The HDET has a positive measure.
  1. Therefore, lattice QCD is positive at asymptotic density. Lattice simulation should be possible for dense QCD
  2. Vafa-Witten theorem applies: Vector symmetries except $U(1)_B$ are unbroken in QCD at asymptotic density: CFL is exact!
- The lattice simulation is possible with HDET or two-lattice QCD.