Strong Coupling Lattice QCD at finite T and μ

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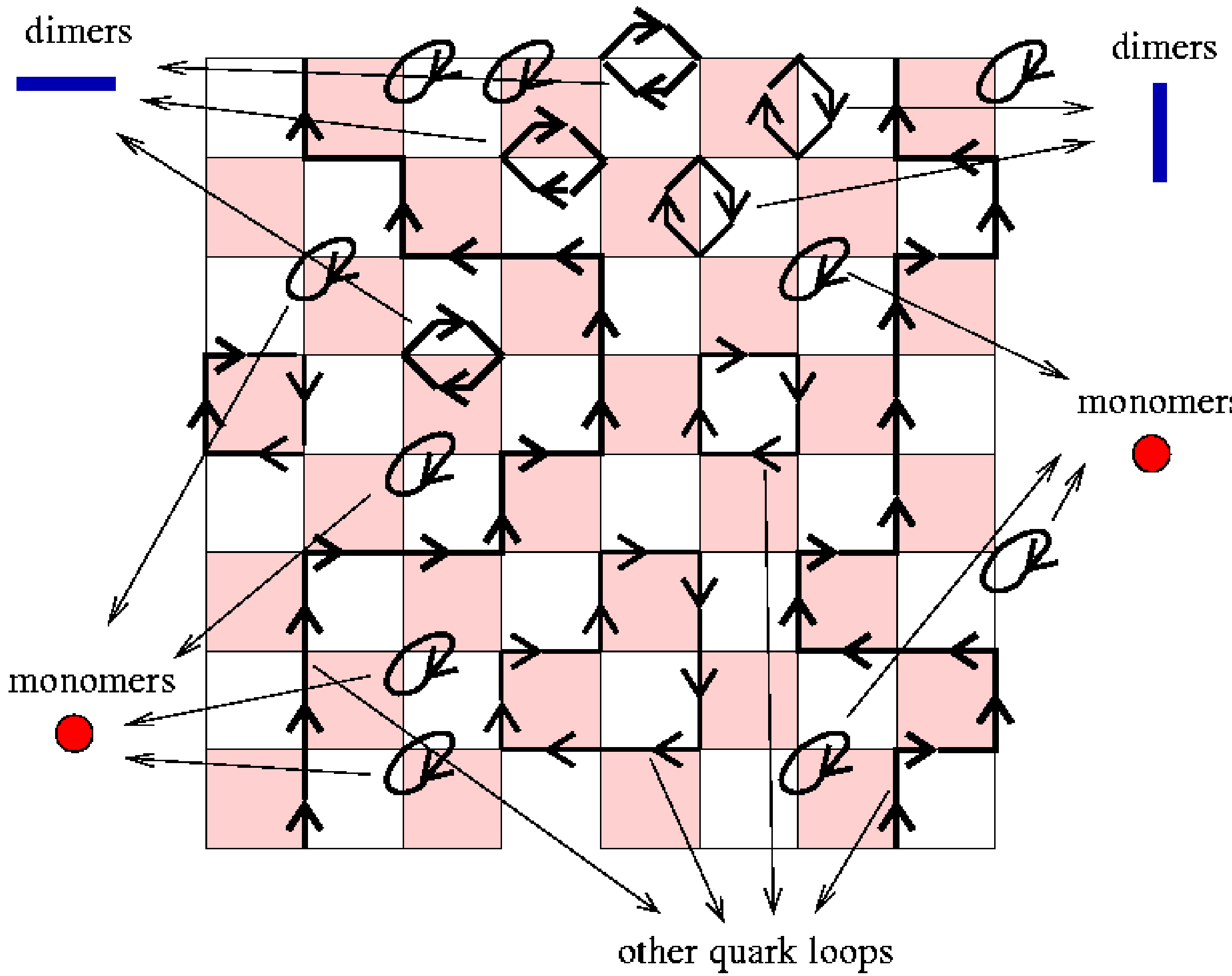
Motivation

- Current lattice QCD calculations become exceeding difficult in certain limits
  - chiral limit, continuum limit, thermodynamic limit…
- The difficulties are algorithmic:
  - Critical slowing down
  - Large fluctuations in certain observables
  - Sign problems
- Quantitative checks of universal behavior is lacking.
  - Universal critical exponents close to the phase transitions have not been computed with precision.
  - Signals for superfluidity/superconductivity at finite baryon densities (in QCD-like theories without sign problems) is only qualitative.
  - Most calculations are still performed far from the chiral limit

New Algorithms for Fermions

- Conventional algorithms integrate over the fermions
  - Work with a fermion determinant in a fixed background field
  - Zero modes of the “Dirac” operator create havoc!

- Ideally one would like a new representation of the partition function:
  - by (partially) integrating over gauge fields
  - with no sign problems
  - which can be solved by cluster algorithms


dimers
dimers
monomers
monomers
other quark loops

In the strong coupling limit with staggered fermions, this "ideal" algorithm has been discovered recently!

What can we solve with this new algorithm?

- ***U(N) gauge theory for any N***
- ***SU(N) gauge theory for any N at zero baryon chemical potential***
- ***SU(N) gauge theory for any even N at non-zero baryon chemical potential***

Strong Coupling QCD

- The QCD partition function on a lattice

$$Z = \int dU \; d\psi \; d\bar{\psi} \; \exp\left(-\frac{1}{g^2} S_g[U] \; + \; \bar{\psi} \; D[U;m] \; \psi \right)$$

- In the strong coupling limit $g \to \infty$.
  - A theory of interacting fermions
  - Contains confinement and chiral symmetry breaking
  - Maximal lattice artifacts

The chiral limit is interesting and unsolved!

Staggered Fermions:

Dirac Operator:

$$\bar{\psi}\, \mathrm{D}[\mathrm{U};\mathrm{m}]\psi = -\sum_{x,\mu}\left\{\frac{\eta_\mu(x)}{2a_\mu}\left[\bar{\psi}(x)U_\mu(x)\psi(x+\hat{\mu}) - \bar{\psi}(x+\hat{\mu})U_\mu^\dagger(x)\psi(x)\right]\right\}$$

$$-m\sum_x \bar{\psi}(x)\psi(x)$$

$$\eta_1 = 1, \quad \eta_\mu = (-1)^{x_1+\ldots+x_{\mu-1}}, \mu = 2,3,\ldots,d$$

Lattice Spacing: $a_\mu = 1, \mu = 1,2,\ldots,d-1; a_d = \sqrt{\frac{1}{T}};$

By varying T one can study finite temperature effects

- One can integrate over gauge fields exactly
  - A single link U(N) or SU(N) group integral gives

$$\int [dU_\mu(x)] \exp\left\{\frac{\eta_\mu(x)}{2a_\mu}\left(\overline{\psi}(x)U_\mu(x)\psi(x+\hat{\mu})-\overline{\psi}(x+\hat{\mu})U_\mu^\dagger(x)\psi(x)\right)\right\}$$

$$= \sum_{b=0}^{N} \frac{(N-b)!}{\left(4a_\mu^2\right)^b b!\,N!}\left[\overline{\psi}(x)\psi(x)\overline{\psi}(x+\hat{\mu})\psi(x+\hat{\mu})\right]^b$$

$$+\frac{1}{N!}\left(\frac{\eta_\mu(x)}{2a_\mu}\right)^N\left[\left(\overline{\psi}(x)\psi(x+\hat{\mu})\right)^N+(-1)^N\left(\overline{\psi}(x+\hat{\mu})\psi(x)\right)^N\right]$$

- Integration over the Grassmann variables gives two types of contributions
  - Monomers and Dimers

$$\int \ldots d\psi(x)\, d\overline{\psi}(x) \ldots \left[\overline{\psi}(x)\psi(x)\right]^N \ldots$$

  - Baryon loops

$$\int \ldots d\psi(x+\hat{\mu})\, d\overline{\psi}(x+\hat{\mu})\, d\psi(x)\, d\overline{\psi}(x) \ldots \left[\overline{\psi}(x+\hat{\mu})\psi(x)\right]^N \ldots$$

Monomer-Dimer Representation

Rossi and Wolff, NPB248, 105, 1984

- The partition function of U(N) theory can be written as

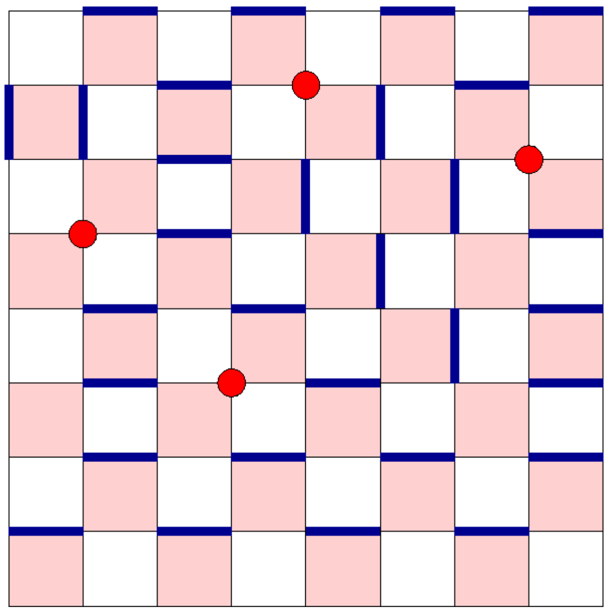
$$Z = \sum_{[n,b]} \prod_{x,\mu} \frac{(N-b_{x,\mu})!}{\left(4a_\mu^2\right)^{b_{x,\mu}} \left(b_{x,\mu}!\right) N!} \prod_x \frac{N!}{n_x!} m^{n_x}$$

  - Site variable: $n_x = 0,1,\ldots,N$
  - Bond variable: $b_{x,\mu} = 0,1,\ldots,N$
  - Constraints (due to fermions)

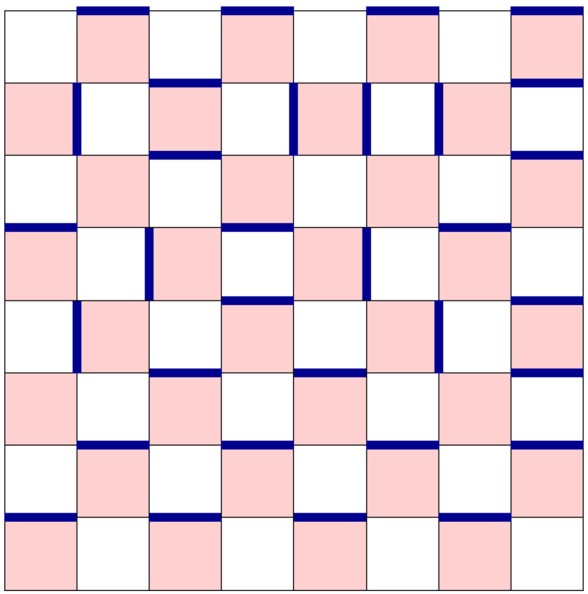
$$n_x + \sum_\mu \left(b_{x,\mu} + b_{x,-\mu}\right) = N$$

- Extension to SU(N) gauge fields
  - Involves baryonic loops
  - N odd the baryons are fermions → can lead to sign problems

U(1) monomer-dimer configurations



configuration with monomers

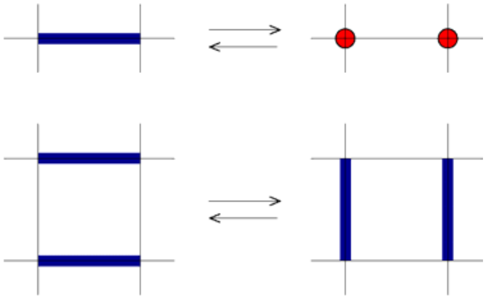


configuration without monomers

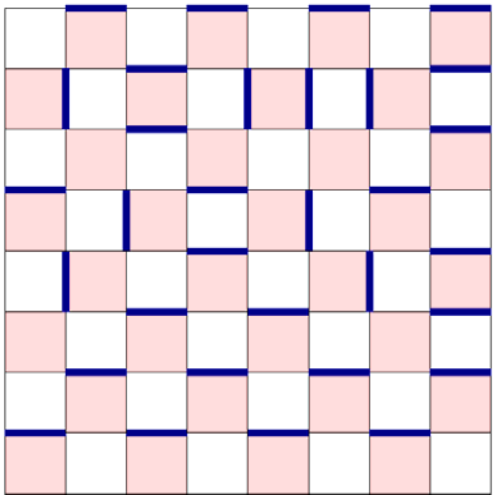
(For massless quarks)

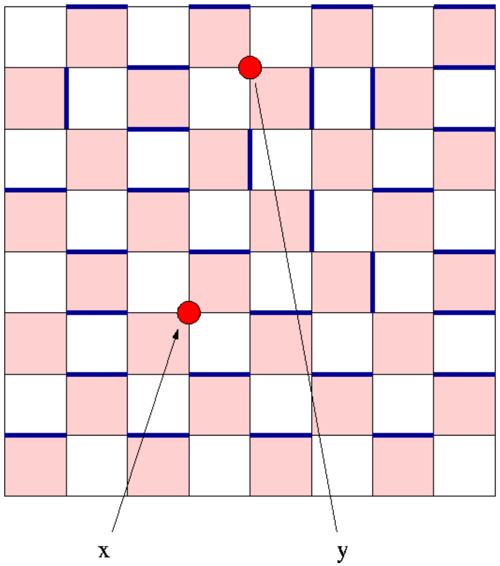
Algorithms for Monomer-Dimer Systems

- Need to update a system with constraints
- Previous approach based on local updates
  - Not suitable in the chiral limit
  - Impossible to update certain configurations
  - Very difficult to measure certain observables
    - Example: Monomer-Monomer correlations at zero mass



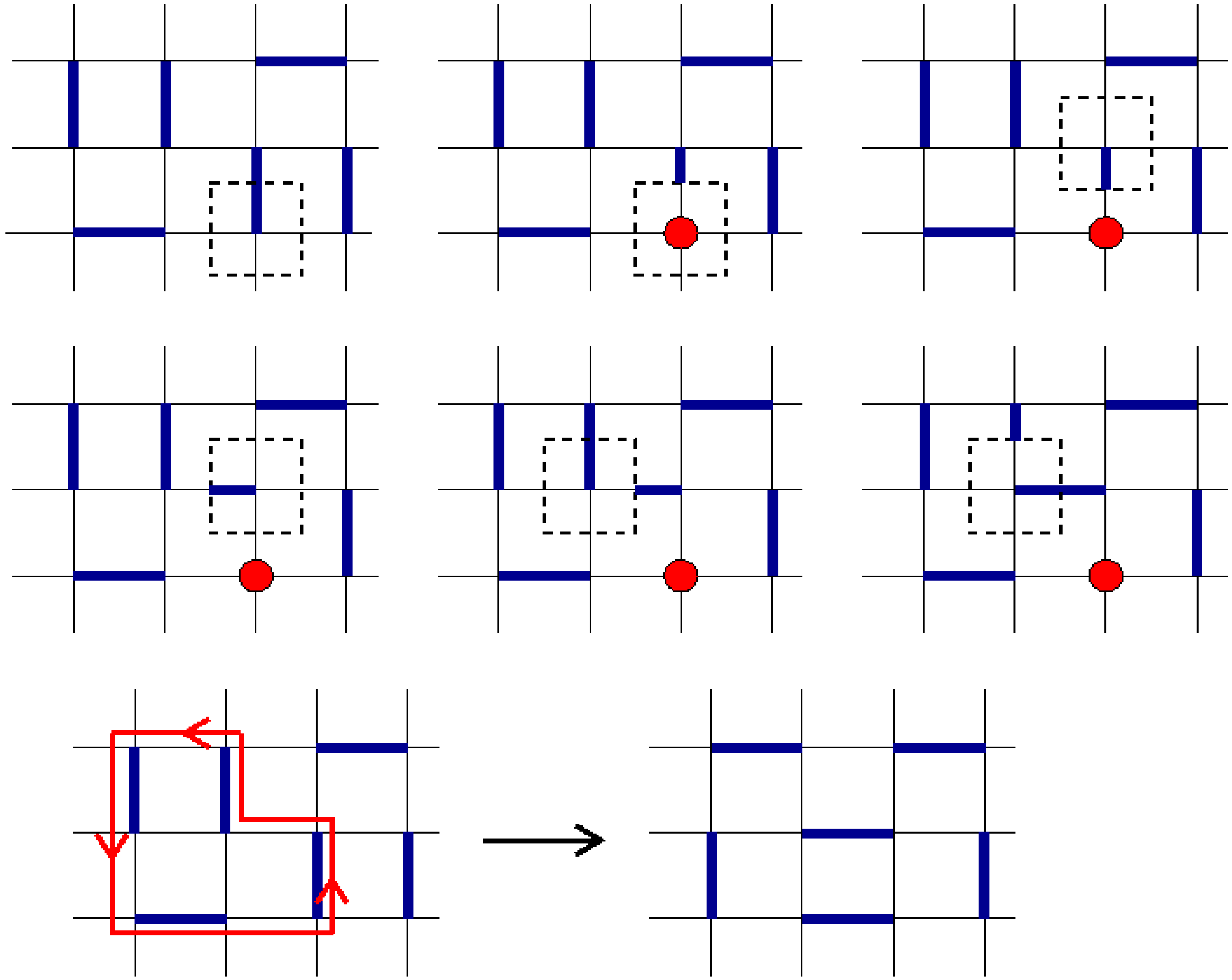
$$\left\langle \overline{\psi}(x)\psi(x)\,\overline{\psi}(y)\psi(y)\right\rangle$$





The "Directed Loop" Algorithm

Adams and SC, NPB662, 220, 2003



Features of the algorithm:

- Very efficient in the chiral limit
  - Can be extended to $m \neq 0$

- Easy to measure two point correlations of monomers
  - Example

$$\chi = \sum_x \left\langle \overline{\psi}(x)\psi(x)\,\overline{\psi}(0)\psi(0) \right\rangle = \frac{1}{4d} \left\langle S_{path}\left(1 + n_{monomer}\right) \right\rangle$$

- Extendable to U(N) and SU(N).
  - A very elegant algorithm for two-color (SU(2)) QCD at finite baryon densities!

Chiral Symmetry at Strong Couplings

- Theory contains a U(1) chiral symmetry in the chiral limit
- Previous work suggests:
  - Chiral symmetry is spontaneously broken at low temperatures
  - There is a 2^{nd} order chiral phase transition at finite temperatures
- In analogy with QCD there must be universal features of the chiral symmetry at strong couplings:
  - ***The low energy effective theory in the broken phase close to the chiral phase transition must be describable with 3d chiral perturbation theory.***
  - ***The chiral phase transition itself must belong to the 3d O(2) universality class.***

Observables

- Chiral condensate:

$$\langle \bar{\psi}\psi \rangle = \frac{1}{L^3}\frac{1}{Z}\frac{\partial}{\partial m} Z(T,m)$$

- Chiral susceptibility:

$$\chi = \frac{1}{L^3}\frac{1}{Z}\frac{\partial^2}{\partial m^2} Z(T,m)$$

- Helicity Modulus:

$$Y = \frac{1}{L^3}\left\langle \left\{ \left[\sum_x J_1(x)\right]^2 + \left[\sum_x J_2(x)\right]^2 + \left[\sum_x J_3(x)\right]^2 \right\} \right\rangle$$

- Pion Mass:

$$\lim_{|x_\mu|\to\infty} \sum_{x_\perp} \langle \sigma_x n_x n_0 \rangle = C e^{-M_\pi |x_\mu|} \ ; \ \sigma_x = (-1)^{x_1+\ldots+x_4}$$

Universal Predictions

- At a fixed temperature in the broken phase the low energy effective theory is given by

$$S_{eff} = \int d^3x \left[\frac{F^2}{2} \partial_\mu \vec{S} \bullet \partial_\mu \vec{S} + \Sigma \; \vec{h} \bullet \vec{S} \right]$$

$\vec{S}(x)$ is an O(2) vector field and $\left| \vec{h} \right| = m$.

- In three dimensions F^2 has dimensions of mass

$$F^2 \sim \left(T_c - T \right)^\nu \text{, } \nu = 0.67155(27) \text{ (3d XY model)}$$

- In order to avoid lattice artifacts in the chiral expansion

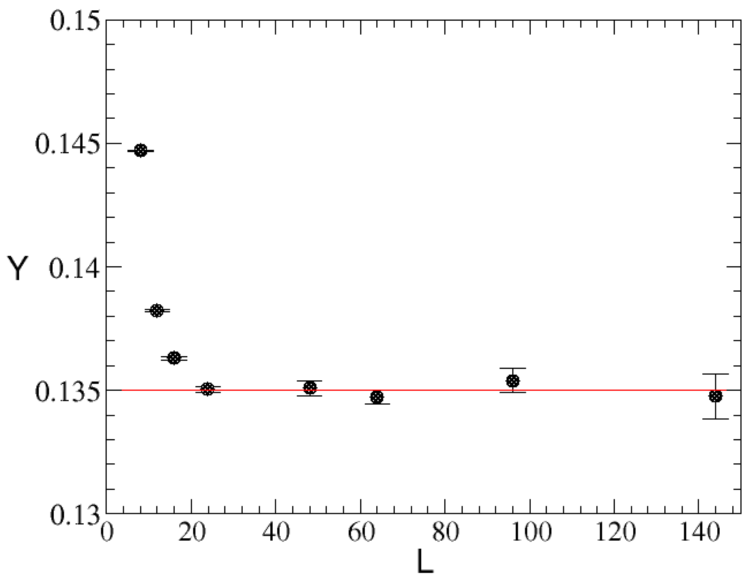
$$1 \ll \frac{1}{F^2} \ll \frac{1}{M_\pi}$$

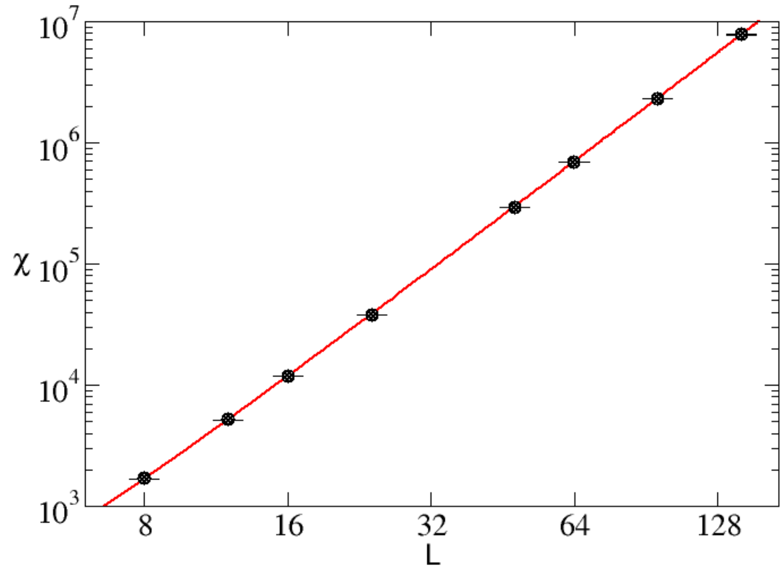
Finite Size Effects in the chiral limit

Hasenfratz and Leutwyler, NPB343, 241, 1990

$$\text{Lattice: } 4\times L^3;\ a_1 = a_2 = a_3 = 1;\ \ a_t = a_4 = \frac{1}{\sqrt{T}}, T = 7.0$$

$$F^2 = \lim_{V\to\infty} Y$$





$$\chi = \frac{\Sigma^2 L^3}{2}\left[1 + \frac{0.226}{F^2 L} + \frac{a}{L^2} + \ldots\right]$$

$$F^2 = 0.1350(1)$$

$$\Sigma = 2.2648(10), a = 4.6(3); \chi^2/\text{d.o.f}=0.87$$

Quark Mass Dependence: Pion Decay Constant

SC and Strouthos, hep-lat/0401002

$$F_m = F\left[1 + a_1\sqrt{m} + a_2 m + a_3 m\sqrt{m} + \ldots\right]$$

$$F_m^2 = \lim_{V\to\infty} Y$$

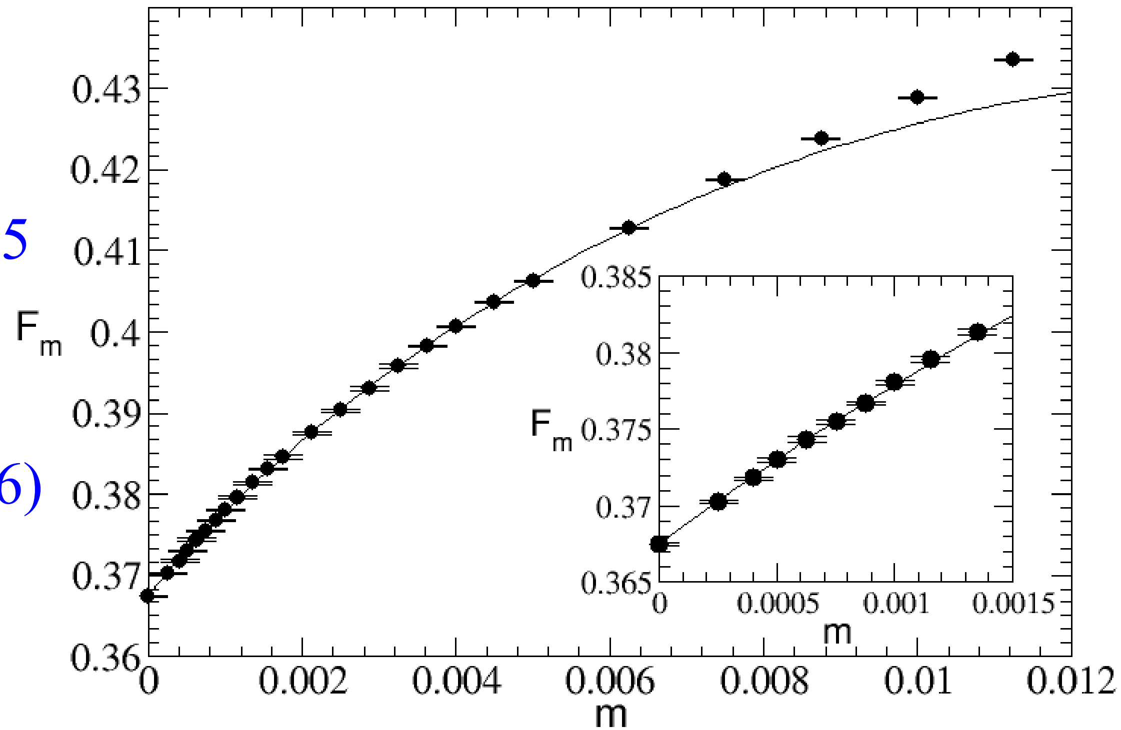
$a_1 = 0$ for O(2) symmetry

Fit range: $0 \le \mathrm{m} \le 0.00625$

$F = 0.36747(6)$

$a_2 = 34.0(7), a_3 = -182(6)$

χ^2 / d.o.f=1.2



Quark Mass Dependence: Chiral Condensate

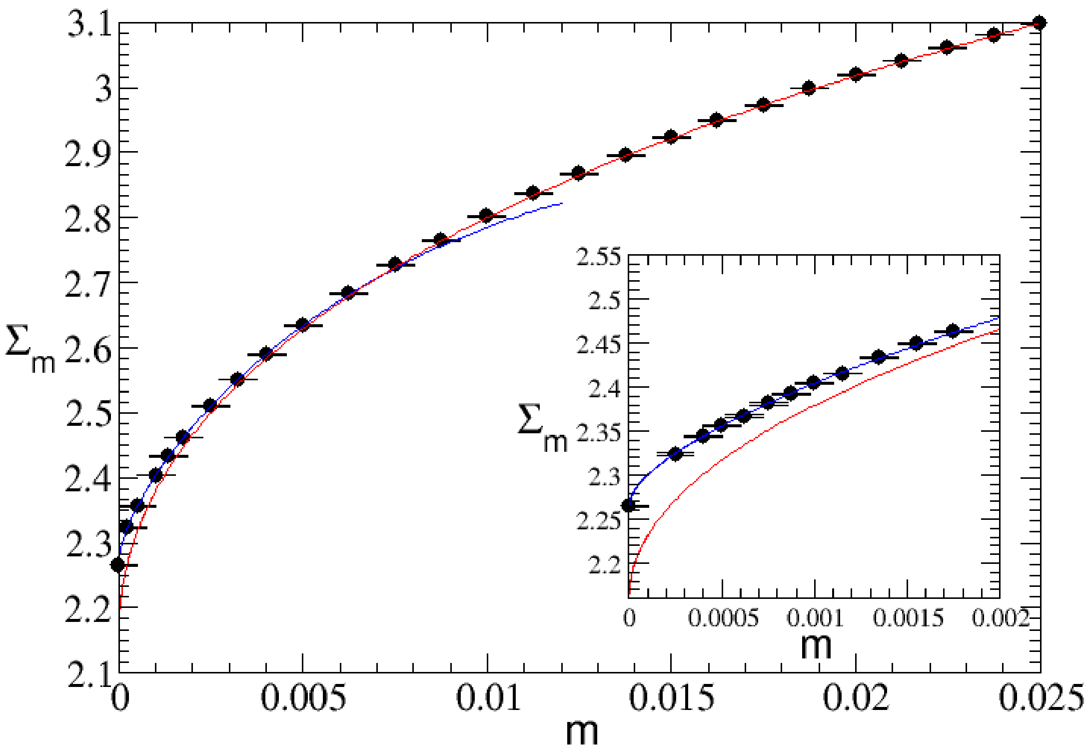
$$\Sigma_m = \Sigma\left[1 + b_1\sqrt{m} + b_2 m + b_3 m\sqrt{m} + \ldots\right]$$

$$\Sigma_m = \lim_{V\to\infty}\left\langle\overline{\psi}\psi\right\rangle$$

$$\Sigma = 2.2642(10)$$

$$b_1 = 1.36(4), b_2 = 15.2(7)$$

$$\chi^2 / \mathrm{d.o.f} = 1.1$$



Quark Mass Dependence: Pion mass

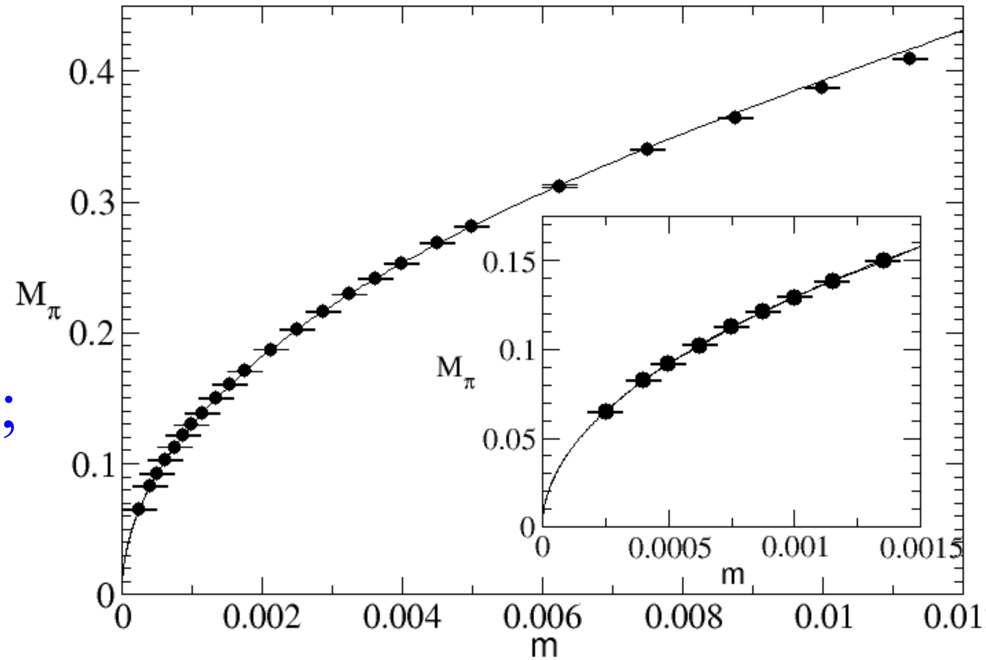
$$M_\pi = \sqrt{\frac{\Sigma}{F^2}}\,\sqrt{m}\left[1 + c_1\sqrt{m} + c_2 m + c_3 m\sqrt{m} + \ldots\right]$$

Chiral Ward Indentity

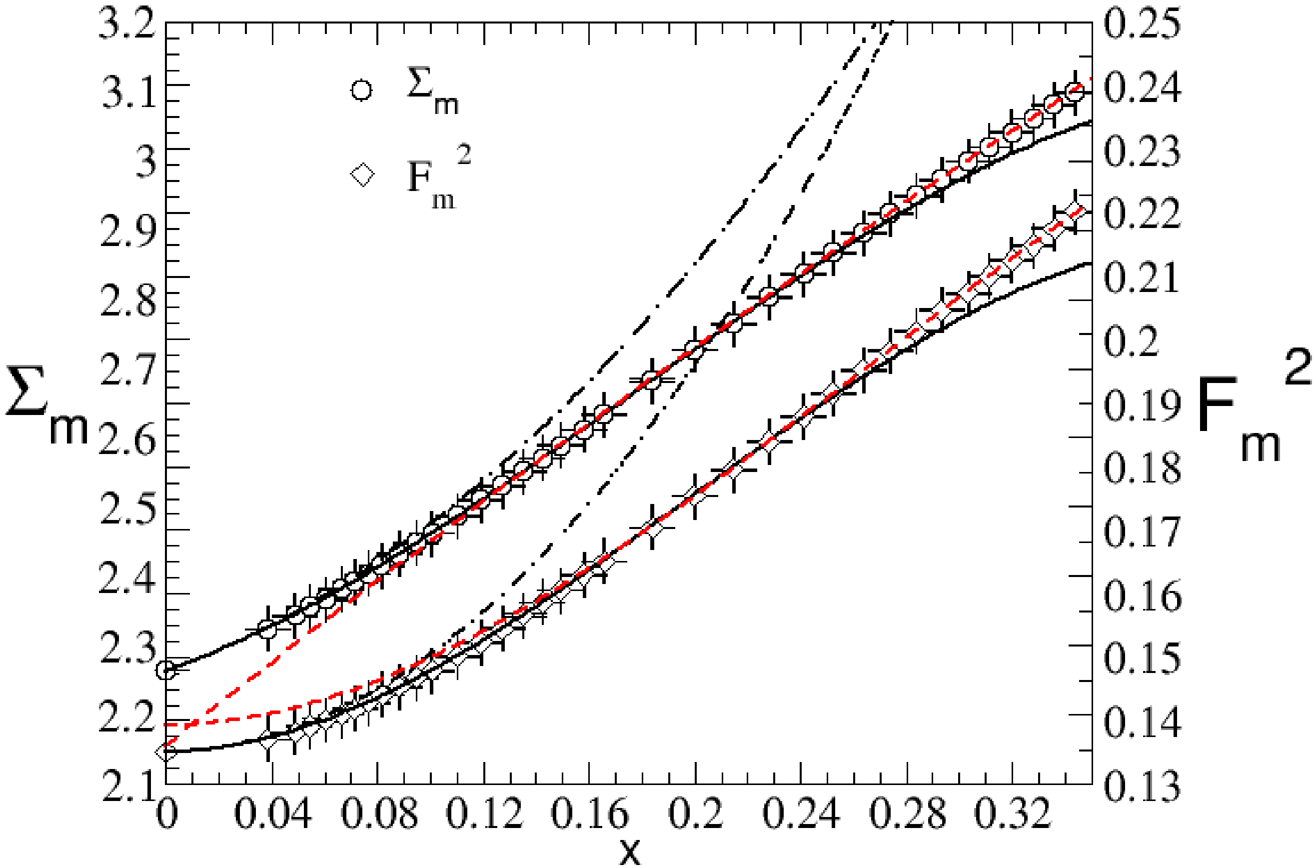
$$c_1 = \frac{b_1}{2} - a_1 \approx 0.68$$

$$c_2 = -25.8(8),\ c_3 = 149(14);$$

$$\chi^2/\mathrm{d.o.f} = 0.5$$



Chiral Extrapolations



$$x = \frac{M_\pi}{4\pi F^2}$$

3d O(2) Critical Behavior

SC and Jiang, PRD58 (Rapid), 091501, 2003

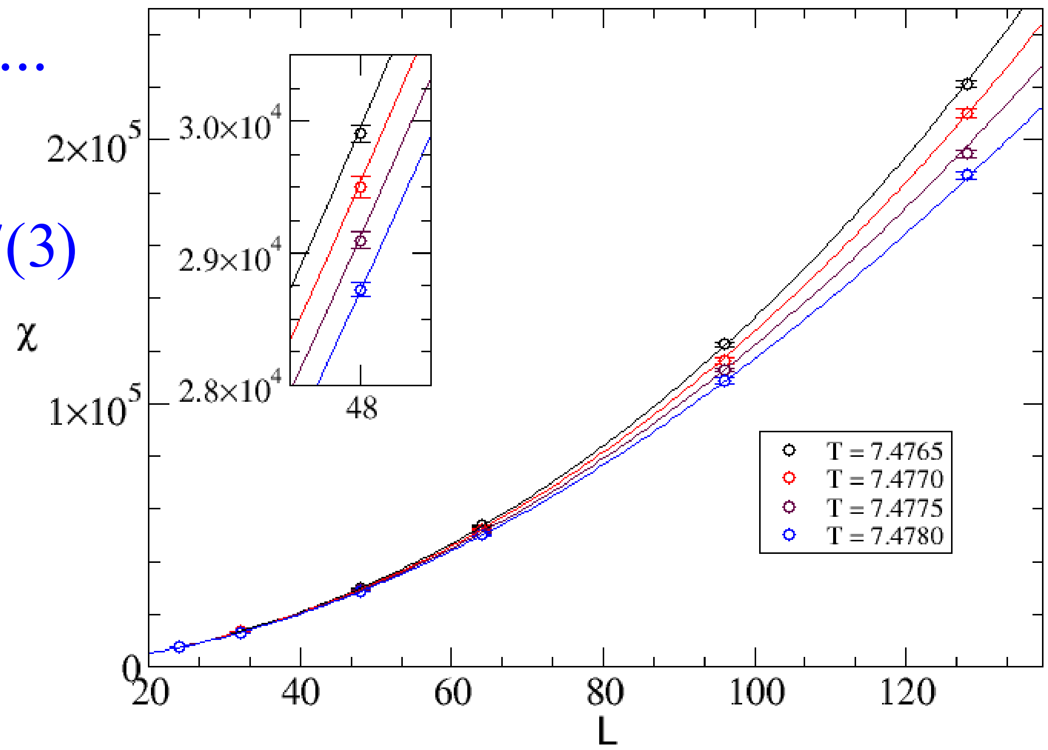
$$\chi = L^{2-\eta} g\left(tL^{1/\nu}\right), \quad t = \left(\frac{T}{T_c} - 1\right)$$

$$\chi = g_0 L^{1.962} + g_1 t L^{3.451} + \ldots$$

$$g_0 = 14.69(2), g_1 = -9.7(3)$$

$$T_c = 7.47739(3)$$

$$\chi^2 / \text{d.o.f} = 0.74$$



$$\Sigma = A\left[T_c - T\right]^{\beta}$$

$\beta = 0.3485(2)$ in the 3d XY model

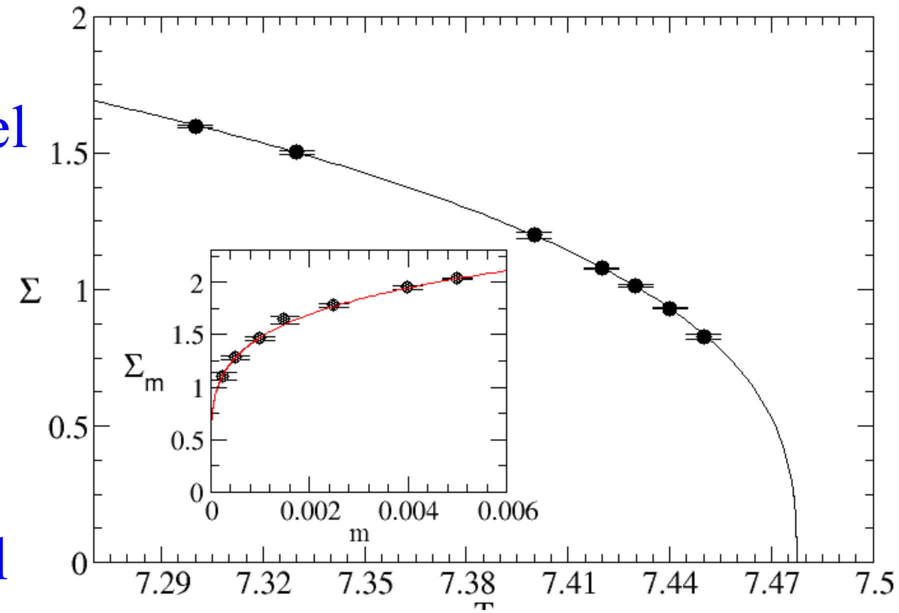
$$A = 2.92(2), \beta = 0.348(2)$$

$$\chi^2 / \text{d.o.f} = 0.53$$

$$\Sigma_m = Bm^{\frac{1}{\delta}}$$

$\delta = 4.780(2)$ in the 3d XY model

$$B = 6.18(2), \chi^2 / \text{d.o.f} = 0.88$$

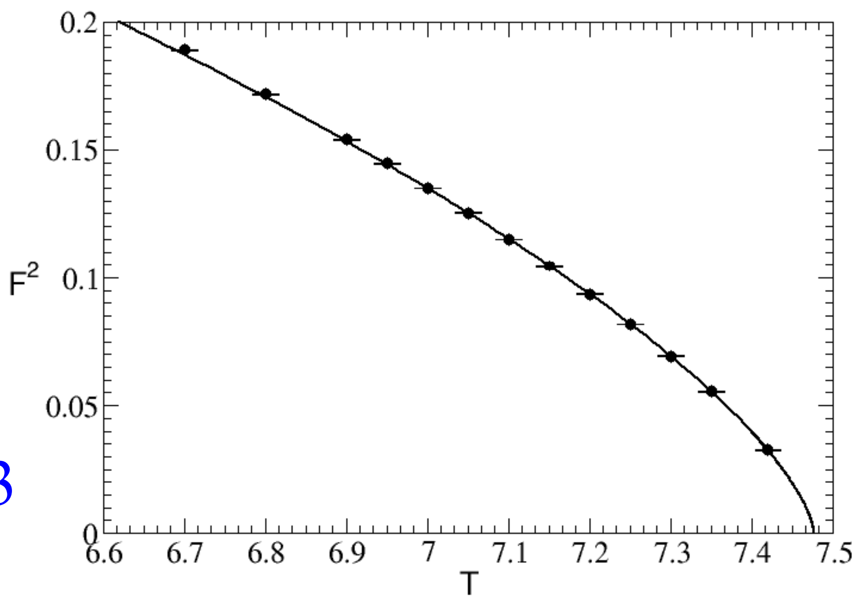


$$F^2 = C\left[T_c - T\right]^{\nu}$$

$\nu = 0.67155(27)$ in the 3d XY model

Fit range $7.05 \leq T \leq 7.42$

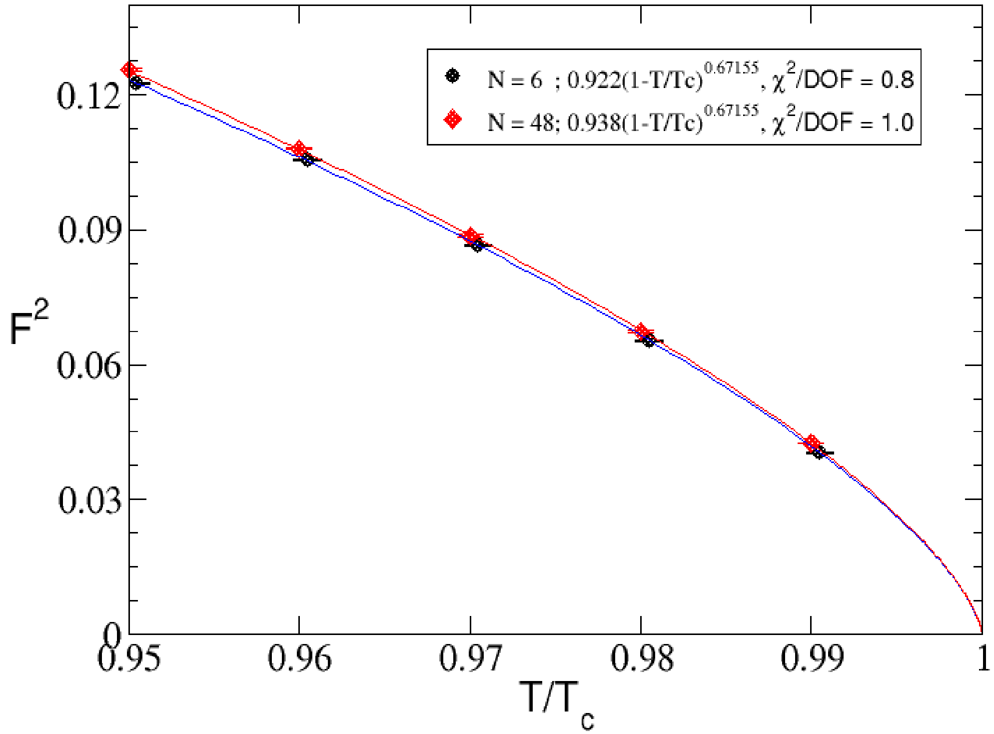
$$C = 0.2217(1), \chi^2 / \text{d.o.f} = 1.03$$



Universality at large N

- Large N leads to mean field critical exponents
- Any finite N should still yield the correct universal critical exponents
  - Critical region shrinks as N becomes large

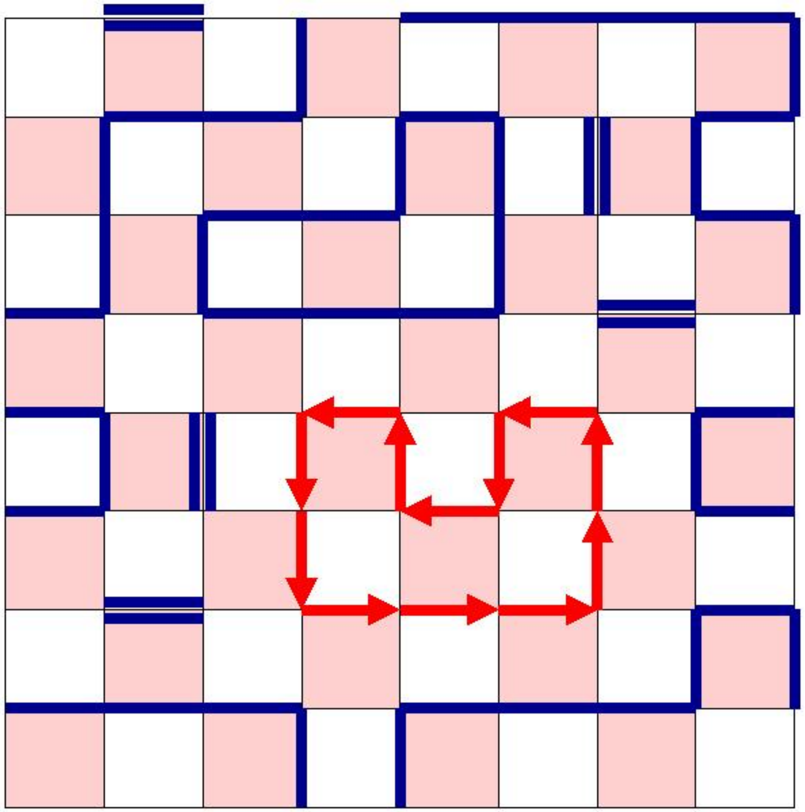
Kogut, Stephanov, Strouthos
PRD58,096001, 1998



Critical region does not appear to shrink as N becomes large!

2-Color QCD at Strong Coupling

- Can study the theory at finite T and μ
- Can approach the chiral limit
- Interesting phase diagram
  - Symmetry U(1) X SU(2) at zero chemical potential.
  - Broken to U(1) X U(1) at non-zero chemical potential.
- Perhaps interesting from a condensed matter physics view point
  - A model for superconductivity + antiferromagnetism?



Conclusions

- Strong coupling QCD with staggered fermions can be solved elegantly
  - Many questions can now be answered
- Universal predictions in the chiral limit do emerge
  - however extremely small quark masses are needed
  - Chiral extrapolations from larger masses may be unreliable
- Extension of the new methods?
  - Wilson quarks and domain-wall (overlap) quarks
  - other four-fermi models
  - weak coupling QCD?