**Strongly Interacting or Dense Fermion and Bosons**

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- What happens when the scattering length $a$ goes to +/- 8? New scaling region for dense or strongly interacting particles: $|a| \rho^{1/3} \gg 1$

- Dense/Strongly Interacting Bose gas:
  - LOCV calculation
  - Similar scaling as for Fermions
  - Condensate depletion. BEC quenched when: $\rho a^3 \geq 0.6$

- Dense/Strongly Interacting Fermi gas:
  - Galitskii resummation vs. LOCV
  - New fundamental many-body parameter: $\beta = -E_{\text{int}} / E_{\text{kin}}$
  - Comparison to experiments near Feshbach resonances
  - FN-GFMC
  - Comparison to 6Li ENS data
  - E/N, Pairing gaps, sound speed,....
Dilute vs. dense or strongly interacting (unitary) limit

• Interaction energy per particle in the dilute limit:

\[ \frac{E_{\text{int}}}{N} = 2\pi \hbar^2 \rho a / m, \quad (\rho |a|^3 < 1) \]

for bosons and half that for fermions in two spin states.

• Interaction energy in the dense, strongly interacting or unitary limit:

\[ \frac{E_{\text{int}}}{N} \sim \hbar^2 \rho^{2/3} / m, \quad (\rho |a|^3 > 1) \]

where the constant is universal albeit spin dependent.

(It is assumed throughout that the range of interaction is small: \( R << |a|, r_0 \))
Near Feshbach resonances

- By tuning magnetic field, atoms can interact resonantly, so that $a \approx +/- 8$
- Expands hydrodynamically (Stringari et al.) either collisional (JILA, cond-mat/0305028), or as a superfluid (6Li) (Duke, Science 298(2002)2179, cond-mat/0304633)
- Molecule formation rate much slower than Cooper pair and BCS phase transition (MIT, cond-mat/0207046) (JILA, cond-mat/0311172)
- For $a<0$ a BEC collapses, whereas a degenerate Fermi gas does not!
LOCV calculation
Urbana/Nordita, PRL88(2002)210403

- LOCV invented by Pandharipande & Bethe for the strongly correlated 4He, 3He and nuclear liquid.
- Jastrow ansatz for the wave-function: \( \Psi = \prod_{i<j} f(r_{ij}) \)
- Determine corr.fct. \( f(r) \) variationally by minimizing: \( E/N = \langle \Psi | H | \Psi \rangle / \langle \Psi | \Psi \rangle \)
- To lowest order constrained variation (LOCV), \( f(r) \) is determined by two-body Schrödinger Eq. for: \( r < d \sim r_0 \), where coh.length \( d \) is of order the interparticle spacing \( r_0 \)
- Boundary conditions: \( f'(d) = 0 \) and \( (rf)'/rf = -1/a \) at \( r=0 \).
- Gives energy per particle
  \[ a = \frac{\kappa^{-1} \tan \kappa - 1}{\tan \kappa} \]
- Gives correct low (Lenz) density dilute limit

For \( \rho a^3 >> 1 \) : \( \kappa \tan \kappa = -1 \Rightarrow \kappa = 2.798.., 6.1212..,....., n\pi \)

Besides ~40 molecular states also one dimer state: \( \kappa = 1.1997 i \)

Higher orders in linked cluster expansion small
LOCV energy vs. scattering length

- **LOCV:**
  \[ energy = \frac{\hbar^2 \kappa^2}{2md^2} \]
  and \( d \approx \rho^{-1/3} \)

- **Dilute:**
  \[ energy = 2\pi \hbar^2 an / m \]

- **Molecule:**
  \[ energy = -\hbar^2 / 2ma^2 \]

**where**
\[
\begin{align*}
a & = \frac{\kappa^{-1} \tan \kappa - 1}{1 + \kappa \tan \kappa} \\
d & = 1/3 \\
K & \text{ vs. } a^1/3 \rho^{1/3}
\end{align*}
\]
85Rb JILA exp.

- Scattering length near Feshbach resonance induced in the BEC at frequency \( \omega (a(B)) \)
- Coherent atom-molecule transitions at radio frequency \( \omega (B) \) in LOCV

\[
\nu = \frac{1}{\hbar} \left( \frac{2E}{N} + \frac{\hbar^2}{ma} \right) = \frac{\hbar}{2\pi m} (\kappa^2 d^{-2} + a^{-2})
\]

- At Feshbach resonance LOCV

\[
E / N = \hbar^2 \kappa_i^2 / 2md^2 = 13.33\hbar^2 \rho^{2/3} / m
\]

- Inserting the 85Rb density in the JILA exp. \( \sim 2 \times 10^{13} \) gives \( \omega (B) = 5 \text{kHz} \)
  at Feshbach resonance
- Minimum in \( \omega (B) \) found
- Atom-molecule transitions overdamped between 155-156G
Condensate Fraction

- Dilute limit:
  \[
  \frac{\rho^0}{\rho} = 1 - \frac{4}{\sqrt{3\pi}} \left(\frac{a}{r_0}\right)^{3/2}
  \]

- LOCV:
  \[
  \frac{\rho^0}{\rho} = 1 - \rho \int [1 - f(r)]^2 d^3r = \frac{d^3}{r_0^3} \left[ \frac{6}{\kappa^3} (\sin \kappa - \kappa \cos \kappa) - 1 \right]
  \]

BEC quenched for:
  \[
  \rho \geq 0.6a^{-3}
  \]

\[
T_0 = 3.22 \hbar^2 \rho^{2/3} / m
\]
The dense/strongly interacting Bose gas

- The energy per particle in a dilute Bose gas with repulsive interactions is

\[
\frac{E}{N} = 2\pi \frac{\hbar^2 a}{m} \rho \left(1 + \frac{128}{15} \left(\frac{\rho a}{\pi}\right)^{1/2} + \ldots \right), \quad (\rho |a|^3 << 1)
\]

- The dense/strongly interacting repulsive Bose gas also has the new scaling as the Fermi liquid:

\[
\frac{E}{N} = 13.33 \hbar^2 \rho^{2/3} / m, \quad (\rho a^3 >> 1)
\]

- Similar scaling in the two- and three-body problem in a harmonic oscillator trap (S.Jonsell et al., PRL 88 (2002)50401)
- Chemical potential, sound speed and collective modes are similar to those in Fermi gases and liquids
Strongly interacting Fermions

• Consider an uniform Fermi gas with density: $(\text{? components or spin states})$
  \[ \rho = \nu \frac{k_F^3}{6\pi^2} \]

• A dense/strongly interacting Fermi gas enters a new scaling region when: \(? |a|^3 > 1\)
  \[ a \rightarrow a_{\text{eff}} \approx k_F^{-1} \approx \rho^{-1/3} \]

• Energy per particle in a dilute vs. dense/strongly int. liquid:
  \[ E/N = (3/5)E_F + (\nu - 1)\pi a \rho / m, \quad (\rho \mid a \mid^3 < 1) \]
  \[ E/N = (3/5)E_F(1 - \beta), \quad (\rho \mid a \mid^3 > 1) \]

• Universal parameter (only spin dependent)

• First studied for a neutron gas/nuclear matter in ’99, where the NN $^1S_0$ scattering lengths are:
  \[ a_{nn} \approx -18 \text{ fm}, \quad a_{np} \approx +5 \text{ fm} \]
Galitskii´s integral equations (MBX’99)

- Galitskii’s ladder resummation for the scattering amplitude:
  \[
  \Gamma(p, p', P) = \Gamma_0(p, p', P) + m \sum_k \Gamma_0(p, k, P) \left[ \frac{N(P, k)}{\kappa^2 - k^2} - \frac{1}{\kappa^2 - k^2} \right] \Gamma(k, p', P)
  \]

- Energy per particle for \( N \) components:
  \[
  \frac{E}{N} = \frac{3}{5} E_F + \frac{\nu (\nu - 1)}{N} \sum_{|p \pm P| \leq P_F} \Gamma(p, p, P)
  \]
  \[
  \Gamma_0 = 4\pi a / m
  \]
  \[
  E = E_F \left[ \frac{3}{5} + (\nu - 1) \frac{2}{3\pi} k_F a + (\nu - 1) \frac{4(11 - 2 \ln 2)}{35\pi^2} (k_F a)^2 + \ldots \right]
  \]

- Dilute limit:
  \[
  \frac{E}{N} = E_F \left[ \frac{3}{5} - \frac{35(\nu - 1)}{9(11 - 2 \ln 2)} \right] = \frac{3}{5} E_F \left[ 1 - (\nu - 1) \beta \right], \quad \beta = 0.67
  \]
  \[
  \Delta \approx 0.5 E_F
  \]

- Dense limit:
  \[
  \Gamma \approx 1 / k_F m
  \]
  Higher orders suppressed by phase space (Bethe-Brueckner).

- Unstable towards collapse when: \( \nu \geq 3 \quad \Rightarrow \quad \text{´Ferminova´} \)
Bosenovae, Ferminovae & Supernovae

- BEC with attractive interactions collapse and subsequently explodes leaving a cold core (Bosenovae)
- Collapse/implosion also seen in other physical systems as fission bombs, sonoluminoscense and supernovae
- Traps with Fermi atoms are unstable towards molecule formation but do not collapse directly due to the mean field
- Bosenovae offer tabletop “simulations” of Supernova explosions though energies are much smaller
Lowest Order Constrained Variation

- LOCV invented by Pandharipande & Bethe for the strongly correlated nuclear liquid. Jastrow ansatz for the wave-function with periodic boundary condition.
- LOCV as for bosons when finite momenta are ignored
- Correcting for exchange by changing a factor: \( \nu \rightarrow \nu - 1 \)
- For: \( k_F | a | \gg 1 \), it gives the energy per particle for \( \nu = 2 \):
  \[
  \frac{E}{N} = \frac{3}{5} E_F (1 - \beta), \quad \beta = |\kappa_0| (2/3\pi)^{2/3} 5/6 = 0.43...
  \]
  for the dimer state:
  \[
  a \rightarrow -\infty
  \]
The dense/strongly interacting Fermi gas

- The new universal many-body parameter in dense or strongly interacting Fermi liquids with two spin states is:

\[ E / N = (3/5) E_F (1 - \beta), \quad \beta = -E_{int} / E_{kin} \]

where:

- \( \beta = 0.67 \), Galitskii approx.
- \( \beta = 0.43 \), LOCV approx.
- \( \beta = 0.67 \), Pade’ approx. (Baker, MBX99)
- \( \beta = 5/9 \), EFT (Steele, nucl-th/0010066)
- \( \beta = 0.26+/-0.07 \), \(^6\)Li exp. (Duke, cond-mat/0212499)
- \( \beta \sim 0.7+/-0.2 \), \(^{40}\)K exp. (JILA, cond-mat/0302246)
- \( \beta = 0.3-0.4 \), \(^6\)Li exp. (ENS, cond-mat/0303079)
- \( \beta = 0.56+/-0.01 \), FN-GFMC calc. (Carlson et al., physics/0303094)
ENS exp. with 6Li

- Measuring expansion energies of a 6Li gas near B=855G (Bourdell et al.)

\[ \beta = -\frac{E_{int}}{E_{kin}} \]

- Agrees with LOCV prediction
  - except just below resonance
- Plateau due to molecule formation?
Trapped Bose vs. Fermi atoms

- Hamiltonian:

\[ H = \sum_{i=1}^{n} \left( \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2}m\omega^2 r_i^2 \right) + \frac{4\pi \hbar^2 a}{m} \sum_{i<j} \delta^3(r_i - r_j) \]
Shopping list for experiments

• More experiments near Feshbach resonances for bosons and fermions
• Several densities to check:
  • Measure \[ \beta = -\frac{E_{\text{int}}}{E_{\text{kin}}} \]
  • Dependence on spin states: \[ \nu \neq 2 \]
• Molecule formation rates, BCS-BEC cross over
• Lower temperatures
• Superfluid fermions
• Measure gaps vs. density and scattering length
• .............

\[ E_{\text{int}} \propto \rho^{2/3} \]
Summary

- Trapped Fermi & Bose atoms near Feshbach resonance provides new table top test ground for studying strongly interacting/dense systems $|a|^3 > 1$ in the unitary limit:
- Experimental confirmation of unitary limit
- Detailed agreement with $^6$Li and $^{85}$Rb data near Feshbach resonances
- New scaling laws and universal many-body parameters
  - Bosons: $E/N = 13.33 \hbar^2 \rho^{2/3}/m, \quad (\rho a^3 >> 1)$
  - Fermions: $E/N = (3/5) E_F (1 - \beta), \quad \beta = -E_{int}/E_{kin} \sim 0.5$
  where $\beta$ depends on spin only.
- Pairing gap: $\Delta \equiv 0.54 E_F$
- Fermi gases particular relevant for BCS pairing in general and for solids, nuclei and neutron stars in particular - more on Thursday!
- Ferminovae for $a \to -\infty$ when $\nu \geq 3$
- Cold atomic systems are perfect playgrounds because parameters can be controlled and varied:
  $\rho, a, N, T, \nu, m, ...$