Quantum Monte Carlo (GFMC) Studies of Superfluid Fermi Gas

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Introduction

Pairing is a central problem of many-body physics.

Theories for weak coupling regime: BCS, Gorkov, etc.

Strong coupling limit still to be seen.

Fermi systems with attractive interaction through ${}^1S_0$ channel:

$\longrightarrow$ neutron matter at low density in the outer crust of neutron star.

$\longrightarrow$ proton fraction at low density at the inner core of neutron star.

$\longrightarrow$ dilute Fermi gas at low T, ^{6}Li, ^{40}K, etc.

interaction is ‘tunable’ through Feshbach resonance

temperature as low as $0.05T_F$ achieved

evidence of superfluidity ?

Superfluidity expected at low temperature $\longrightarrow$ nonzero energy gap Δ.

Importance for the evolution of neutron star.

Hamiltonian

Many-body Hamiltonian;

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{n=1}^{A}\nabla_n^2 + \sum_{i,j'} v(r_{ij'})$$

Denote spin ↑ particles by $i, j, k, \ldots$

Denote spin ↓ particles by $i', j', k', \ldots$

Spin ↑ and spin ↓ particles interact via potential of the form,

$$v(r_{ij'}) = -v_0 \frac{2\hbar^2}{m} \frac{\mu^2}{\cosh^2(\mu\, r_{ij'})}$$

Interaction is characterized by dimensionless quantity; ak_F

and effective range,

$$R_{eff} = 2\int_0^\infty (u^2_{asymp}(r) - u^2(r))dr$$

For example, $v_0 = 1$ corresponds to $ak_F = -\infty$ and $R_{eff} = \frac{2}{\mu}$.

Work with $\mu r_0 = 12$, where $(\frac{4\pi}{3} r_0^3 \rho = 1)$.

We checked with $\mu r_0 = 24$, and we want to approximate $\mu r_0 \longrightarrow \infty$.

then we have the short range limit; $R_{eff} << r_0$.

The unit of energy is the free Fermi gas energy per particle,

$$E_{FG} = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}$$

Green's Function Monte Carlo: Principle

First developed by M. Kalos, *et al* in the 1970s.

Let Ψ_i be the eigenstates of $\mathcal{H}$,

$$\Psi_i \mathcal{H} = E_i \Psi_i$$

The trial wave function can be expanded,

$$\Psi_V = \sum_i \alpha_i \Psi_i$$

We can project out the ground state Ψ_0 by evolution in imaginary time,

$$e^{-i\mathcal{H}t} \quad \longrightarrow \quad e^{-\mathcal{H}\tau} \quad , \quad \tau = it$$

We shift the energy by $E_T \approx E_0$ to control the norm, then,

$$\lim_{\tau\to\infty} e^{-(\mathcal{H}-E_T)\tau}\Psi_V = \lim_{\tau\to\infty} \sum_i \alpha_i e^{-(E_i-E_T)\tau}\Psi_i \;\longrightarrow\; \alpha_0 e^{-(E_0-E_T)\tau}\Psi_0$$

Estimate of the ground state energy is obtained from *mixed estimate*,

$$\Psi(\tau) = e^{-(\mathcal{H}-E_T)\tau}\Psi_V(\tau=0)$$

$$\langle\mathcal{H}\rangle_{mix} = \frac{\langle\Psi_V|\mathcal{H}|\Psi(\tau)\rangle}{\langle\Psi_V|\Psi(\tau)\rangle} = E_0 \quad , \quad \lim \tau \to \infty$$

E_T is updated to keep $\langle\Psi(\tau)|\Psi(\tau)\rangle$ constant $\longrightarrow$ *growth estimate*.

Green's Function

In general, for any value of τ the time evolution operator is not known.

Exception: system of free particles.

However, we can obtain small time propagator with controllable errors,

$$\Delta\tau = \tau/n \quad , \quad n = \text{total number of time steps}$$

$$e^{-(\mathcal{H}-E_T)\tau} = \prod e^{-(\mathcal{H}-E_T)\Delta\tau}$$

$$\Psi(\tau) = \left[\prod e^{-(\mathcal{H}-E_T)\Delta\tau}\right] \ \Psi_V(\tau \ = \ 0)$$

Let $\mathbf{R}$ be a vector in the configuration space of A particles,

$$\mathbf{R} = \{\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_A}\}$$

Green's function for short time $\Delta\tau$,

$$G(\mathbf{R}, \mathbf{R}') = \langle \mathbf{R} | e^{-(\mathcal{H}-E_T)\Delta\tau} | \mathbf{R}' \rangle$$

Green's function is used in order to advance one step in time,

$$\Psi^j(\mathbf{R}^j) = \int G(\mathbf{R}^j, \mathbf{R}^{j-1}) \Psi^{j-1}(\mathbf{R^{j-1}}) d\mathbf{R}^{j-1}$$

After n time steps we have,

$$\Psi^n(\mathbf{R}^n) = \int d\mathbf{R}^{n-1} \ldots d\mathbf{R}^0 G(\mathbf{R}^n, \mathbf{R}^{n-1}) \ldots G(\mathbf{R}^2, \mathbf{R}^1) G(\mathbf{R}^1, \mathbf{R}^0) \Psi_V(\mathbf{R}^0)$$

Short Time Green's Function

We need analytical expression for,

$$G(\mathbf{R},\mathbf{R}') = \langle \mathbf{R}| e^{-(\mathcal{H}-E_T)\Delta\tau} |\mathbf{R}'\rangle$$

where,

$$\mathcal{H} = \mathcal{T} + \mathcal{V} \quad \text{with} \quad \mathcal{T} = -\frac{\hbar^2}{2m}\sum_{i=1}^{A} \nabla_i^2 \quad , \quad \mathcal{V} = \sum_{i,j'} v(r_{ij'})$$

and $\quad [\mathcal{T},\mathcal{V}] \neq 0$

Short time Green's function is approximated as (Trotter & Feynman),

$$G(\mathbf{R},\mathbf{R}') \approx \langle \mathbf{R}| e^{-\frac{\mathcal{V}}{2}\Delta\tau} e^{-\mathcal{T}\Delta\tau} e^{-\frac{\mathcal{V}}{2}\Delta\tau} |\mathbf{R}'\rangle e^{E_T\Delta\tau}$$

with an error of $\mathcal{O}(\Delta\tau^3)$

Total error after n time steps is of the order,

$$\text{Total error} \sim n\Delta\tau^3 = \frac{\tau^3}{n^2}$$

Total error $\longrightarrow$ 0 for large n (check by doubling n).

As $\mathcal{V}|\mathbf{R}\rangle = V(\mathbf{R})|\mathbf{R}\rangle$ where $V(\mathbf{R}) = \sum_{i,j'} v(r_{ij'})$,

$$\begin{aligned} G(\mathbf{R},\mathbf{R}') &\approx e^{-V(\mathbf{R})\Delta\tau/2} \langle \mathbf{R}| e^{-\mathcal{T}\Delta\tau} |\mathbf{R}'\rangle e^{-V(\mathbf{R}')\Delta\tau/2} e^{E_T\Delta\tau} \\ &\approx e^{-(V(\mathbf{R})+V(\mathbf{R}')-2E_T)\frac{\Delta\tau}{2}} G_0(\mathbf{R},\mathbf{R}') \end{aligned}$$

where $G_0(\mathbf{R},\mathbf{R}')$ is the Green's function for A free particles,

$$G_0(\mathbf{R},\mathbf{R}') = \left[\frac{m}{2\pi\hbar^2\Delta\tau}\right]^{\frac{3}{2}A} \exp\left[\frac{-m(\mathbf{R}-\mathbf{R}')^2}{2\hbar^2\Delta\tau}\right]$$

Expectation Value

Exact expectation value for ground state is given by,

$$\langle \mathcal{H} \rangle_{exact} = \frac{\langle \Psi(\tau)|\mathcal{H}|\Psi(\tau)\rangle}{\langle \Psi(\tau)|\Psi(\tau)\rangle}$$

Instead, we use so-called mixed estimate,

$$\langle \mathcal{H} \rangle_{mix} = \frac{\langle \Psi_V|\mathcal{H}|\Psi(\tau)\rangle}{\langle \Psi_V|\Psi(\tau)\rangle}$$

$\mathcal{H}$ and time evolution operator commute

$\longrightarrow$ mixed estimate and exact estimate become equal for large τ,

$$\begin{aligned}
\langle \mathcal{H} \rangle_{mix} &= \frac{\langle \Psi_V|\mathcal{H}e^{-(\mathcal{H}-E_T)\tau}|\Psi_V\rangle}{\langle \Psi_V|e^{-(\mathcal{H}-E_T)\tau}|\Psi_V\rangle} \\
&= \frac{\langle \Psi_V|e^{-(\mathcal{H}-E_T)\tau/2}\mathcal{H}e^{-(\mathcal{H}-E_T)\tau/2}|\Psi_V\rangle}{\langle \Psi_V|e^{-(\mathcal{H}-E_T)\tau/2}e^{-(\mathcal{H}-E_T)\tau/2}|\Psi_V\rangle} \\
&= \frac{\langle \Psi(\tau/2)|\mathcal{H}|\Psi(\tau/2)\rangle}{\langle \Psi(\tau/2)|\Psi(\tau/2)\rangle} \rightarrow \langle \mathcal{H} \rangle_{exact} \qquad \text{for large } \tau
\end{aligned}$$

In terms of the Green's functions,

$$\langle \mathcal{H} \rangle_{mix} = \frac{\int d\mathbf{R}^n \ldots d\mathbf{R}^0 E_L(\mathbf{R}^n)\Psi_V^\dagger(\mathbf{R}^n)G(\mathbf{R^n},\mathbf{R^{n-1}})\ldots G(\mathbf{R^1},\mathbf{R}^0)\Psi_V(\mathbf{R}^0)}{\int d\mathbf{R}^n \ldots d\mathbf{R}^0 \Psi_V^\dagger(\mathbf{R}^n)G(\mathbf{R^n},\mathbf{R^{n-1}})\ldots G(\mathbf{R^1},\mathbf{R}^0)\Psi_V(\mathbf{R}^0)}$$

where,

$$E_L(\mathbf{R}^n) = \frac{\langle \Psi_V|\mathcal{H}|\mathbf{R}^n\rangle}{\langle \Psi_V|\mathbf{R}^n\rangle}$$

And,

$$\langle \mathcal{H} \rangle_{mix} \longrightarrow E_0 \qquad \text{for large enough} \quad n$$

Fermion Green's Function Monte Carlo

First introduced by J. B. Anderson in 1975.

Integrals evaluated stochastically (Monte Carlo),

$path_i = \{\mathbf{R}_i^n, \ldots, \mathbf{R}_i^0\}$

subscript (i) = path index

superscript (n) = time step

$P(\mathbf{R}_i^n, \ldots, \mathbf{R}_i^0)$ = probability used to sample the paths

N_t = total number of sampled paths

then,

$$\langle \mathcal{H} \rangle = \frac{\sum_{i=1}^{N_t} Np_i}{\sum_{i=1}^{N_t} Dp_i}$$

where we have defined,

$$Np_i = \frac{E_L(\mathbf{R}_i^n)\Psi_V(\mathbf{R}_i^n)G(\mathbf{R}_i^n, \mathbf{R}_i^{n-1}) \ldots G(\mathbf{R}_i^1, \mathbf{R}_i^0)\Psi_V(\mathbf{R}_i^0)}{P(\mathbf{R}_i^n, \ldots, \mathbf{R}_i^0)}$$

$$Dp_i = \frac{\Psi_V(\mathbf{R}_i^n)G(\mathbf{R}_i^n, \mathbf{R}_i^{n-1}) \ldots G(\mathbf{R}_i^1, \mathbf{R}_i^0)\Psi_V(\mathbf{R}_i^0)}{P(\mathbf{R}_i^n, \ldots, \mathbf{R}_i^0)}$$

The path probabilty (by Kalos) is,

$$P(path_i) = \left[\prod_{j=1,n} I(\mathbf{R}_i^j)G(\mathbf{R}_i^j, \mathbf{R}_i^{j-1})\frac{1}{I(\mathbf{R}_i^{j-1})}\right] I(\mathbf{R}_i^0)|\Psi_V(\mathbf{R}_i^0)|$$

We take 'Importance Function' $I(\mathbf{R}_i^j) = |\Psi_V(\mathbf{R}_i^j)|$,

then path probability becomes,

$$P(path_i) = |\Psi_V(\mathbf{R}_i^n)|G(\mathbf{R}_i^n, \mathbf{R}_i^{n-1}) \ldots G(\mathbf{R}_i^1, \mathbf{R}_i^0)|\Psi_V(\mathbf{R}_i^0)|$$

Denominator of the mixed estimate becomes,

$$\begin{aligned}\sum_{i=1}^{N_t} Np_i &= \sum_{i=1}^{N_t} \frac{\Psi_V(\mathbf{R}_i^n)\Psi_V(\mathbf{R}_i^0)}{|\Psi_V(\mathbf{R}_i^n)||\Psi_V(\mathbf{R}_i^0)|} \\ &= \sum_{i=1}^{N_t} \quad \{\ +1 \quad \text{or} \quad -1\ \}\end{aligned}$$

Positive domain : $\{\mathbf{R}^+\}$ such that $\Psi_V(\mathbf{R}^+) > 0$

Negative domain : $\{\mathbf{R}^-\}$ such that $\Psi_V(\mathbf{R}^-) < 0$

Nodal surface : $\{\mathbf{R}^{ns}\}$ such that $\Psi_V(\mathbf{R}^{ns}) = 0$

Neither positive domain nor negative domain is prefered by path probability.

This means,

$$\sum_{i=1}^{N_t} \frac{\Psi_V(\mathbf{R}_i^n)\Psi(\mathbf{R}_i^0)}{|\Psi_V(\mathbf{R}_i^n)\Psi(\mathbf{R}_i^0)|} \longrightarrow \quad 0 \quad \text{for large } N_t$$

Similarly for the numerator, assuming that $E_L(\mathbf{R}_i^n) \approx E_0$,

$$\begin{aligned}\sum_{i=1}^{N_t} Dp_i &= \sum_{i=1}^{N_t} \frac{E_L(\mathbf{R}_i^n)\Psi_V(\mathbf{R}_i^n)\Psi(\mathbf{R}_i^0)}{|\Psi_V(\mathbf{R}_i^n)\Psi(\mathbf{R}_i^0)|} \\ &\approx E_0 \sum_{i=1}^{N_t} \frac{\Psi_V(\mathbf{R}_i^n)\Psi(\mathbf{R}_i^0)}{|\Psi_V(\mathbf{R}_i^n)\Psi(\mathbf{R}_i^0)|} \\ &\longrightarrow \quad 0 \quad \text{for large } N_t\end{aligned}$$

In the end, the so-called ‘Fermion Sign Problem’ of GFMC,

$$\langle \mathcal{H} \rangle \quad \longrightarrow \quad \frac{0}{0} \quad \text{for large } N_t$$

NOTE: Bose ground state wave function $\Psi_{Bose}(\mathbf{R}) \geq 0$, *hence no sign problem.*

Fixed Node GFMC

Fermion Sign Problem appears because $\Psi_V(\mathbf{R}_i^j)$ can change sign as the time step j increases.

A remedy; 'Fixed Node' GFMC.

Let $\Psi_V(\mathbf{R}_i^0) > 0$ for the initial time step.

Then constrain the path to the positive domain; $\Psi_V(\mathbf{R}_i^j) > 0$ for all j.

$$|e^{-(\mathcal{H}-E_T)\tau)}|_{FN}\ \Psi_V(\mathbf{R}^+,0) \longrightarrow \Psi(\mathbf{R}^+,\tau)$$

or

Let $\Psi_V(\mathbf{R}_i^0) < 0$ for the initial time step.

Then constrain the path to the negative domain; $\Psi_V(\mathbf{R}_i^j) < 0$ for all j.

$$|e^{-(\mathcal{H}-E_T)\tau)}|_{FN}\ \Psi_V(\mathbf{R}^-,0) \longrightarrow \Psi(\mathbf{R}^-,\tau)$$

When the constraint is imposed with the nodal surface of the exact $\Psi_0(\mathbf{R})$, we get the exact E_0,

$\longrightarrow \lim_{\tau\to\infty} \Psi(\mathbf{R}^\pm,\tau) = \Psi_0(\mathbf{R}^\pm)$

$\longrightarrow \quad \langle\mathcal{H}\rangle_{mix} \quad = \quad E_0$

Instead, when the nodal surface of $\Psi_V(\mathbf{R})$ is used to constrain,

$\longrightarrow \lim_{\tau\to\infty} \Psi(\mathbf{R}^\pm,\tau)$ is only $\approx \Psi_0(\mathbf{R}^\pm)$

$\longrightarrow \quad \langle\mathcal{H}\rangle_{mix} \quad \geq \quad E_0$

We minimize variationally Fixed Node $\langle\mathcal{H}\rangle_{mix}$.

Trial Wave Function Ψ_V

Ψ_V is used as:

⟶ Initial guess of $\Psi_0(\mathbf{R})$.

⟶ Importance function $I(\mathbf{R}) = |\Psi_V(\mathbf{R})|$.

⟶ Nodal surface of $\Psi_V(\mathbf{R})$ provides the fixed node constraint.

Assume that infinite matter is approximated as a cubic box $(L \times L \times L)$ with periodic boundary conditions.

The simplest nodal surface; free fermions,

$$\begin{aligned} \Psi_{JS}(\mathbf{R}) &= \prod_{i,j'} f_{ij'}(r_{ij'})\ |\Phi\rangle \\ |\Phi\rangle &= \text{ground state of free fermions} \end{aligned}$$

Jastrow correlation $f_{ij'}(r_{ij'})$ has no effects on the nodal structure.

More general nodal surface is given by the BCS variational wave function,

$$\begin{aligned} \Psi_{BCS} &= \prod_i (u_i + v_i a^\dagger_{\mathbf{k}_i\uparrow} a^\dagger_{-\mathbf{k}_i\downarrow})|0\rangle \\ |u_i|^2 + |v_i|^2 &= 1 \\ |0\rangle &= \text{vacuum} \end{aligned}$$

This common form of Ψ_{BCS} does not correspond to a definite number of particles N.

For N spin $\uparrow$ and N' $(= N)$ spin $\downarrow$ particles in the paired state the number conserving Ψ_{BCS} is given by,

$$\begin{aligned}\Psi_{BCS} &= \mathcal{A}[\phi(r_{11'})\phi(r_{22'})...\phi(r_{NN'})] \\ \phi(r) &= \sum_i \frac{v_i}{u_i} e^{i\mathbf{k}_i\cdot\mathbf{r}} = \sum_i \alpha_i e^{i\mathbf{k}_i\cdot\mathbf{r}} \quad , \quad \alpha_i \text{ can be taken as real}\end{aligned}$$

NOTE: $\Psi_{BCS} = \Psi_{JS}$ *when* $\alpha_i = \{1, 1, \ldots, 1, 1, 0, 0, 0, \ldots\}$.

Pair wave function $\phi(\mathbf{r})$ can be generalized,

$$\phi(\mathbf{r}) = \tilde{\beta}_{b,c}(r) + \sum_{i \le i_c} \alpha_i e^{i\mathbf{k_i}\cdot\mathbf{r}}$$

With u and d' unpaired $\uparrow$ and $\downarrow$ spin particles (Bouchaud, *et al*),

$$\Psi_{\rm BCS}(\mathbf{R}) = \mathcal{A}\left\{[\phi(r_{11'})...\phi(r_{NN'})]\,[\psi_{1\uparrow}(\mathbf{r}_{N+1})...\psi_{u\uparrow}(\mathbf{r}_{N+u})]\,\left[\psi_{1\downarrow}(\mathbf{r}_{(N+1)'})...\psi_{d\downarrow}(\mathbf{r}_{(N+d)'})\right]\right\}$$

$\psi_{i\uparrow}$ and $\psi_{j\downarrow}$ = single particle states.

$$\longrightarrow \quad \Psi_{BCS} \;=\; \text{Determinant of}$$

$$\begin{pmatrix} \phi(r_{11'}) & \phi(r_{12'}) & \ldots & \phi(r_{1(N+d)'}) & \psi_{1\uparrow}(\mathbf{r}_1) & \psi_{2\uparrow}(\mathbf{r}_1) \\ \phi(r_{21'}) & \phi(r_{22'}) & \ldots & \phi(r_{2(N+d)'}) & \psi_{1\uparrow}(\mathbf{r}_2) & \psi_{2\uparrow}(\mathbf{r}_2) \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\ \phi(r_{(N+u)1'}) & \phi(r_{(N+u)2'}) & \ldots & \phi(r_{(N+u)(N+d)'}) & \psi_{1\uparrow}(\mathbf{r}_{N+u}) & \psi_{2\uparrow}(\mathbf{r}_{N+u}) \\ \psi_{1\downarrow}(\mathbf{r}_{1'}) & \psi_{1\downarrow}(\mathbf{r}_{2'}) & \ldots & \psi_{1\downarrow}(\mathbf{r}_{(N+d)'}) & 0 & 0 \\ \psi_{2\downarrow}(\mathbf{r}_{1'}) & \psi_{2\downarrow}(\mathbf{r}_{2'}) & \ldots & \psi_{2\downarrow}(\mathbf{r}_{(N+d)'}) & 0 & 0 \\ \psi_{3\downarrow}(\mathbf{r}_{1'}) & \psi_{3\downarrow}(\mathbf{r}_{2'}) & \ldots & \psi_{3\downarrow}(\mathbf{r}_{(N+d)'}) & 0 & 0 \end{pmatrix}$$

Variational parameters: b, c, and $\{\alpha_0, \alpha_1, \ldots, \alpha_{i_c}\}$.

STRATEGY:

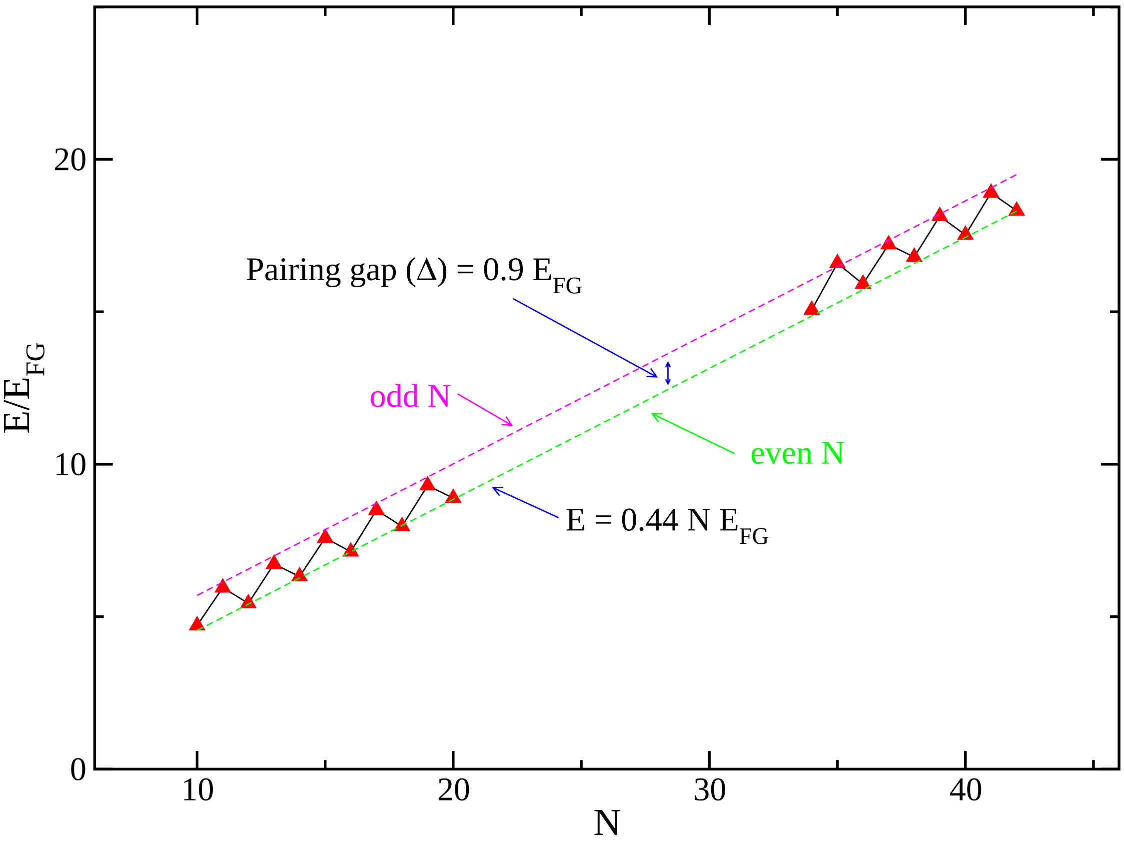
Find the optimal set of parameters by minimizing fixed node GFMC energy

$\longrightarrow$ *best overlap with the nodal structure of the true ground state.*

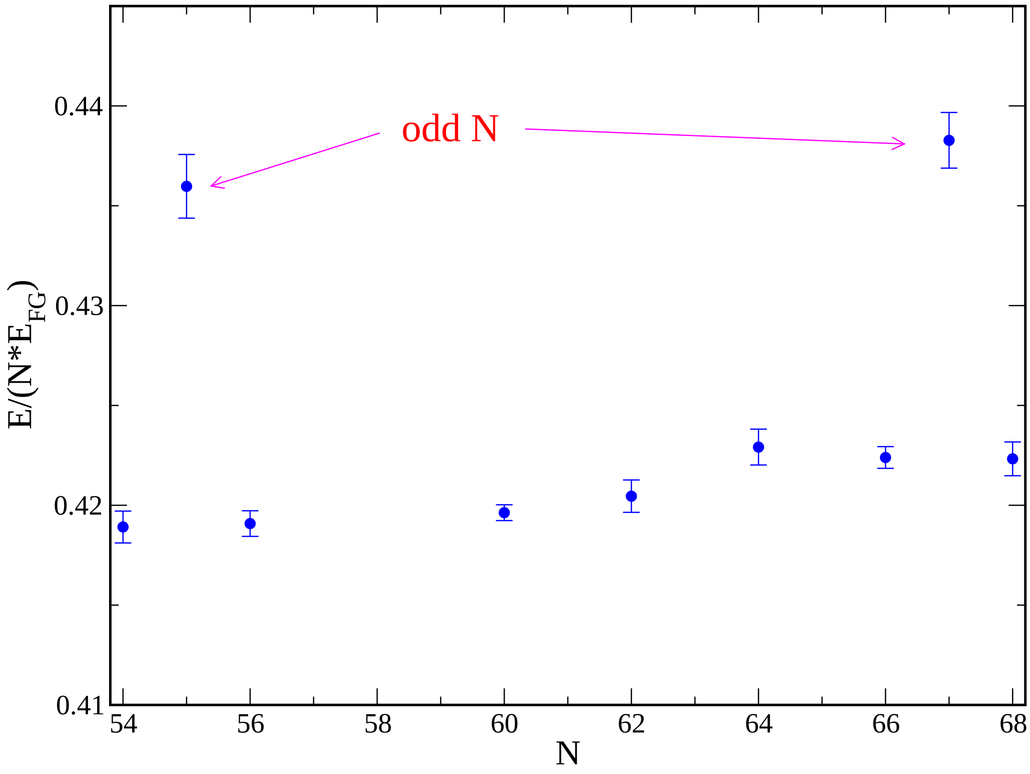
Gap Energy

Gap energy is obtained from,

$$\Delta(2n+1) = E(2n+1) - \frac{1}{2}(E(2n) + E(2n+2))$$



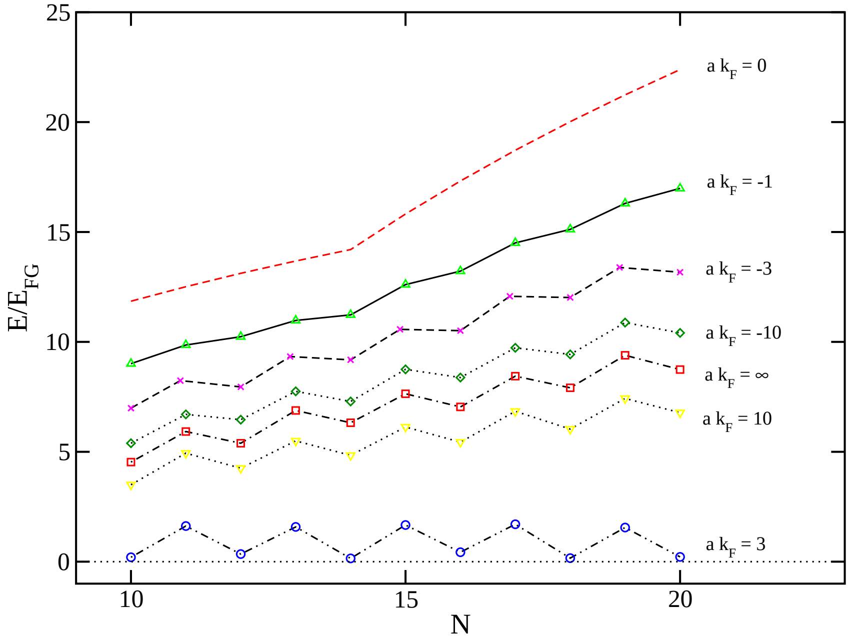
Result for $ak_F = -\infty$

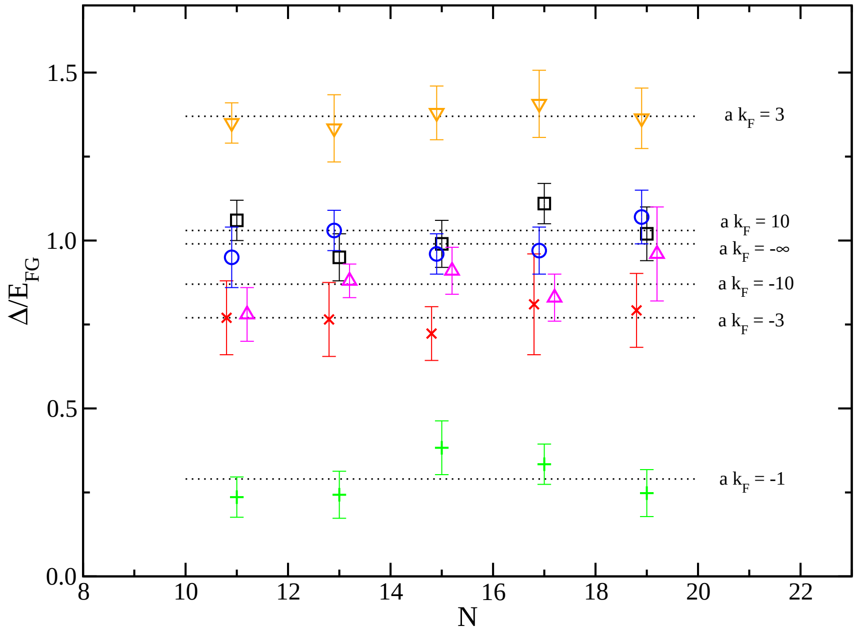


Result for $ak_F = -\infty$; Energy per particle for large N.

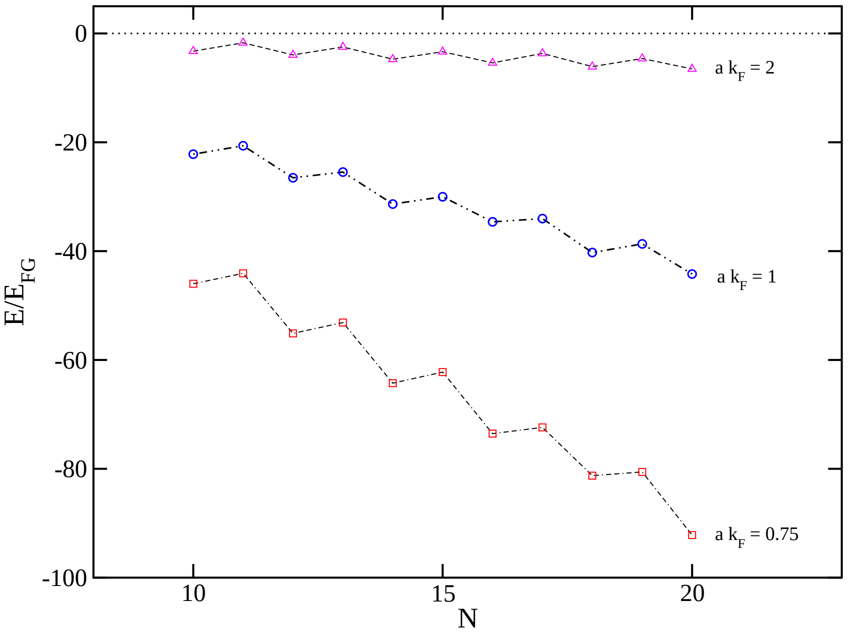
$$E/N = 0.422(4) E_{FG}$$

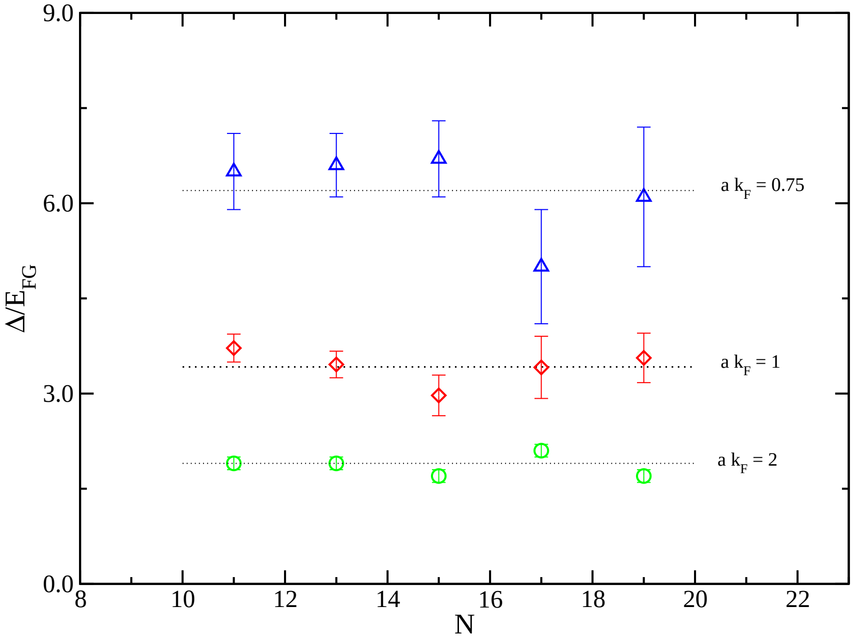
Results: $E > 0$ for various ak_F



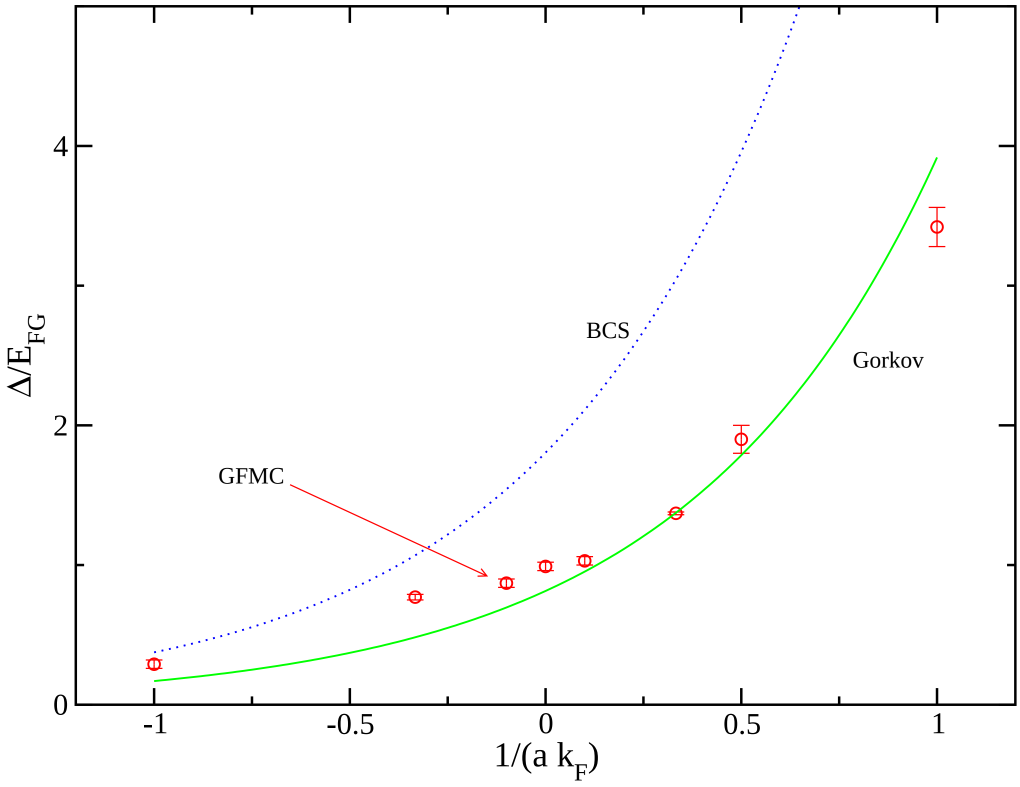


Results: $E < 0$ *for various* ak_F





Comparison of Gap: GFMC, BCS and Gorkov



$$\Delta_{BCS} = \frac{8}{e^2}\ \epsilon_F\ e^{\pi/(2k_F a)}$$

$$\Delta_{Gorkov} = \left(\frac{2}{e}\right)^{7/3} \epsilon_F\ e^{\pi/(2k_F a)}$$

Comparison with LOCV

LOCV was first used in 1970s to study neutron matter.

LOCV is used to study unstable Bose gases.

Comparison of results: ‘cosh’ potential with δ potential.

Given a Hamiltonian of the form,

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{n=1}^{A}\nabla_n^2 + \sum_{i,j'} v(r_{ij'})$$

Use trial wave function of the form,

$$\begin{aligned} |\Psi\rangle &= \prod_{i,j'} f_{ij'}|\Phi\rangle \\ |\Phi\rangle &= \text{ground state of free fermions} \end{aligned}$$

Do the cluster expansion of $\langle\mathcal{H}\rangle$.

Include up to two-body terms.

‘healing distance’ d is defined such that,

$$f(|r > d|) = 1 \quad \text{and} \quad \frac{df}{dr}(|r = d|) = 0$$

After Euler-Lagrange minimization of energy, we have LOCV equation,

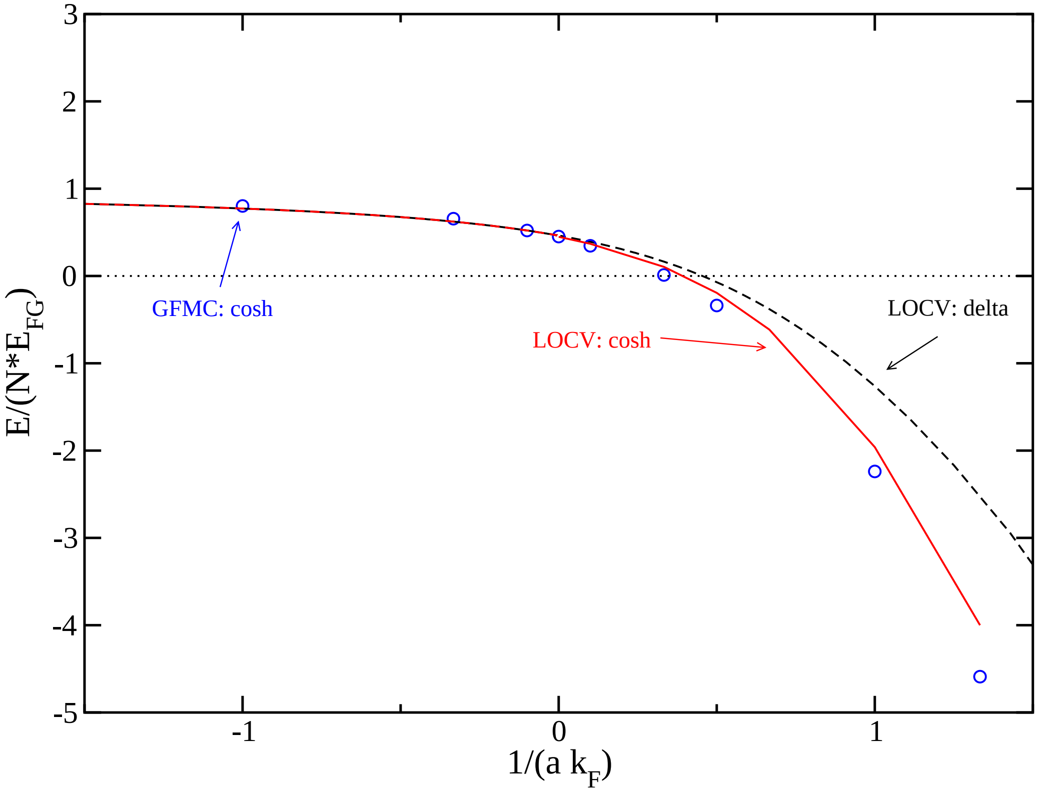
$$-\frac{\hbar^2}{m}\nabla^2 f(r) + v(r)f(r) = \lambda f(r)$$

Allow correlation between closest pair by imposing constraint,

$$\frac{\rho}{2}\int_0^d f^2(r)d^3r = 1$$

The energy per particle is given by,

$$E_{LOCV} = E_{FG} + \frac{\lambda}{2}$$

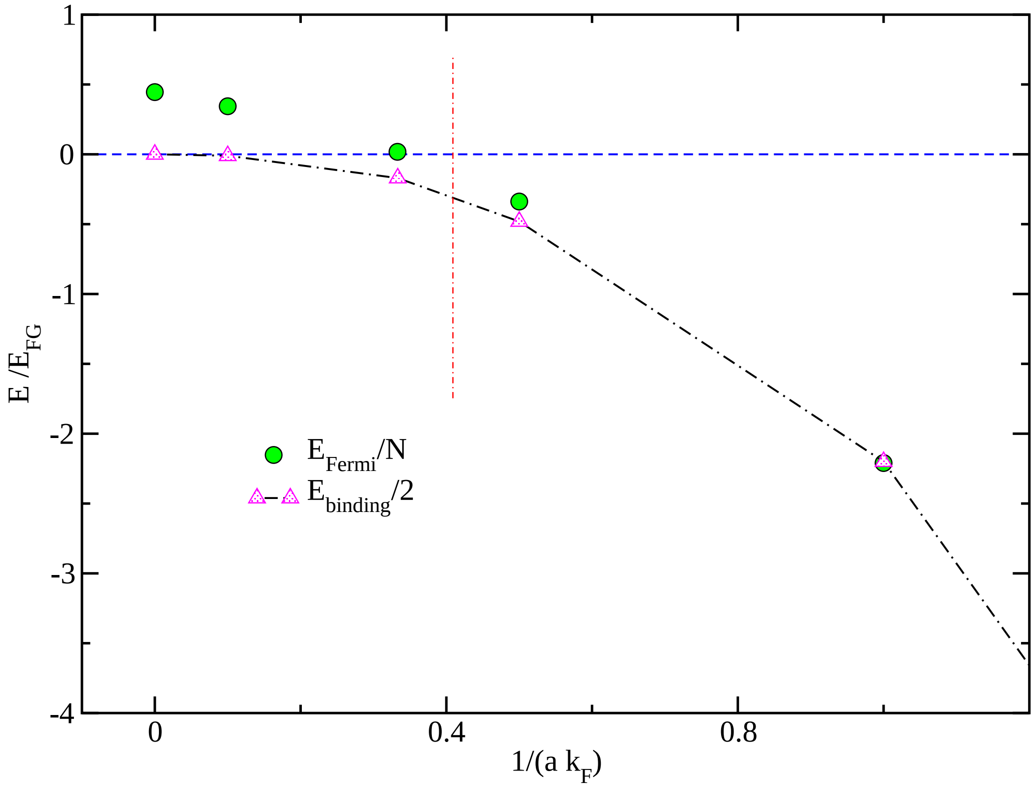


1. ‘cosh’ and δ potentials are almost the same for $\frac{1}{ak_F} < 0$, but they differ when $\frac{1}{ak_F} > 0$.

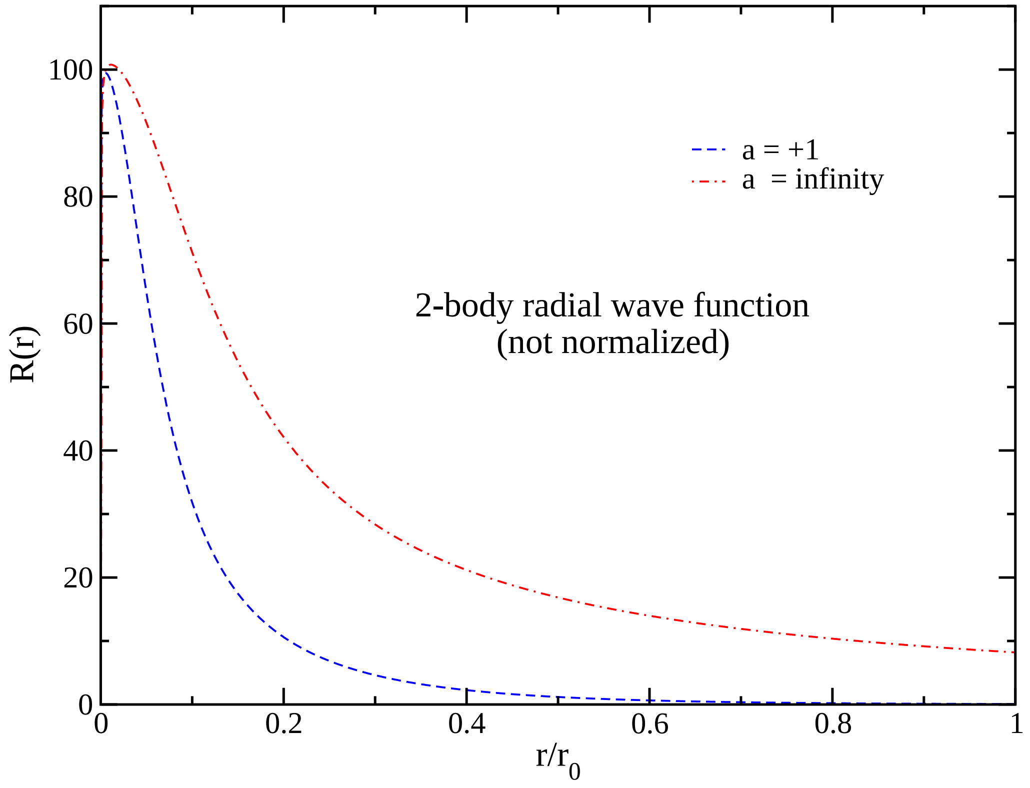
2. LOCV energies with ‘cosh’ potential are in good agreement with GFMC energies.

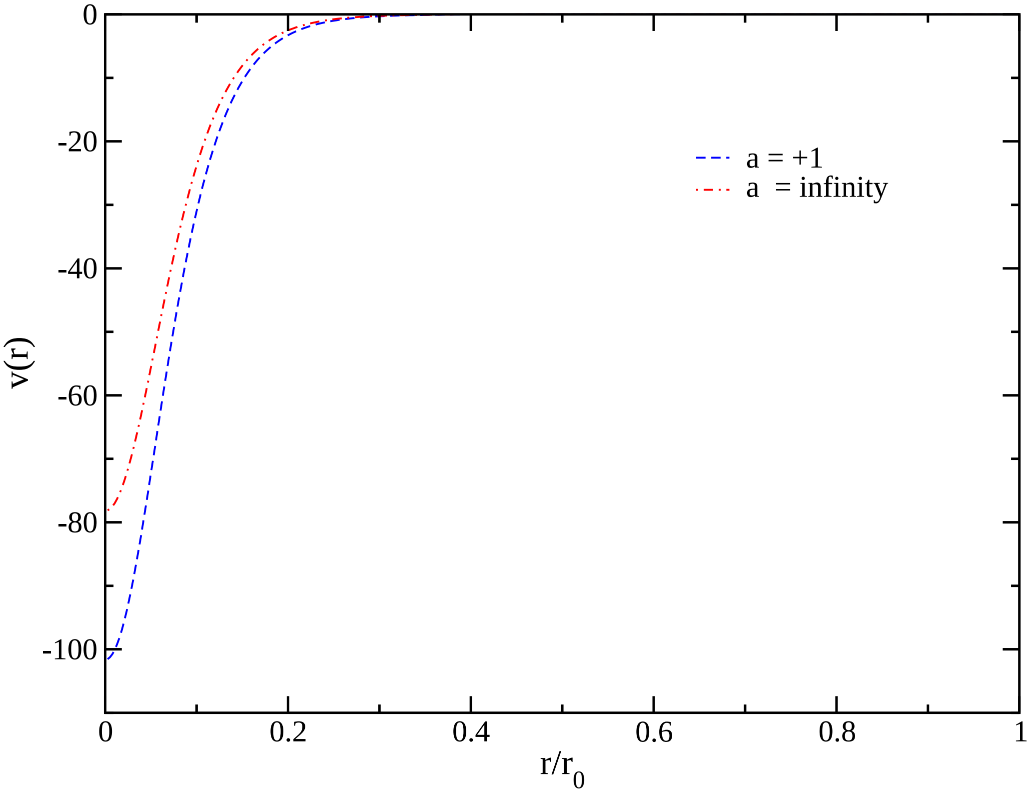
3. How to calculate Δ with LOCV ?

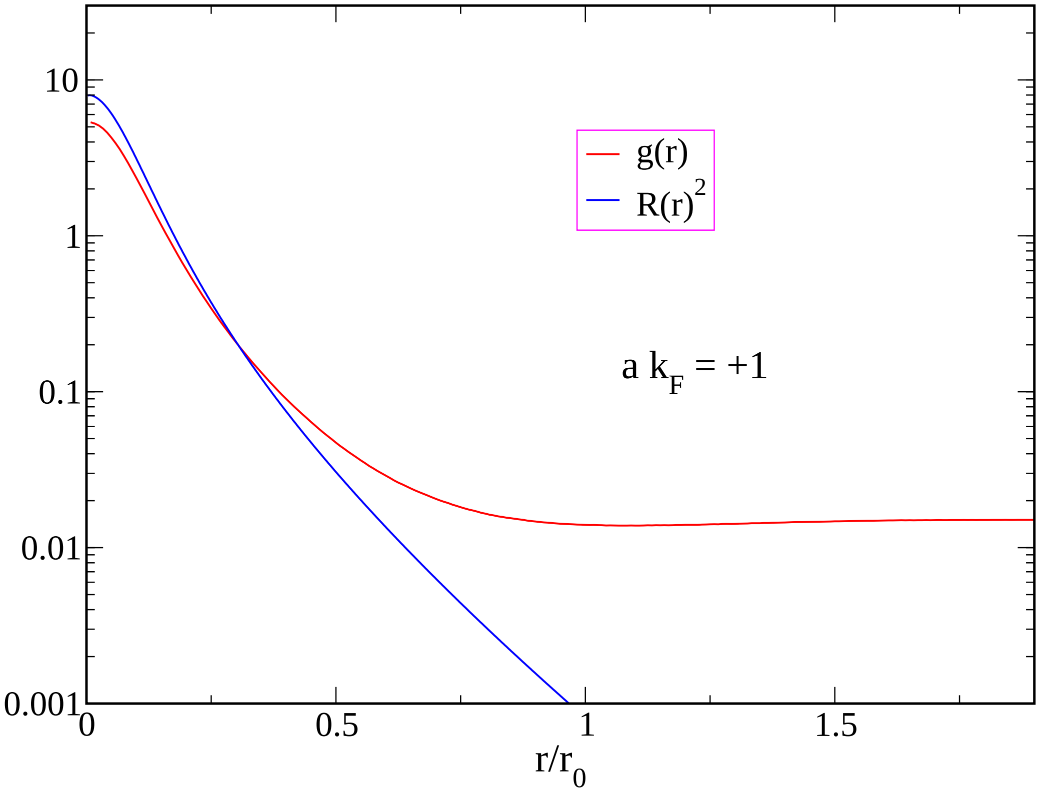
Bosonization



$1/ak_F$	Δ	E_{GFMC}/N	$E_{pair}/2$	$\sqrt{< r^2 >}/r_0$	R_{eff}/r_0	$R_{eff}/\sqrt{< r^2 >}$
0	0.9	0.44	0	∞	0.17	
0.1	1.0	0.34	-0.01	3.69	0.16	
0.3	1.4	0.02	-0.17	1.21	0.16	
0.5	1.9	-0.34	-0.48	*0.74*		*0.20*
1.0	3.4	-2.20	-2.22	*0.38*		*0.37*
1.3	6.2	-4.60	-4.63	*0.28*		*0.43*
2.0	14.4	-12.8	-12.9	*0.19*		*0.53*


R(r)
100
80
60
40
20
0
0
0.2
0.4
0.6
0.8
1
r/r_0
a = +1
a = infinity
2-body radial wave function
(not normalized)


v(r)
0
-20
-40
-60
-80
-100
0
0.2
0.4
0.6
0.8
1
r/r_0
a = +1
a = infinity



2-body radial solution $R(r)^2$ vs pair correlation function $g(r)$.

(both normalized)