Hadron Tomography

\[ \text{GPDs} \Rightarrow 3\text{-d images of the nucleon where } x - y \text{ plane in position space and } z \text{ axis in momentum space} \]

Matthias Burkardt

burkardt@nmsu.edu

New Mexico State University
Las Cruces, NM, 88003, U.S.A.
form factor ⇒ charge distribution in position space

Deeply virtual Compton scattering (DVCS)

→ Generalized parton distributions (GPDs)

Probabilistic interpretation of GPDs as Fourier transforms of impact parameter dependent PDFs

\[ H(x, 0, -\Delta^2_\perp) \rightarrow q(x, b_\perp) \]

\[ \tilde{H}(x, 0, -\Delta^2_\perp) \rightarrow \Delta q(x, b_\perp) \]

\[ E(x, 0, -\Delta^2_\perp) \rightarrow \perp \text{ distortion of PDFs when the target is transversely polarized} \]

Chromodynamik lensing and \( \perp \) single-spin asymmetries (SSA)

transverse distortion of PDFs

+ final state interactions

\[ \Rightarrow \perp \text{ SSA in } \gamma N \rightarrow \pi + X \]

Summary
plane wave states have uniform charge distribution

meaningful definition of $\rho(\vec{r})$ requires that state is localized in position space!

define localized state (center of mass frame)

\[
\left| \vec{R} = \vec{0} \right\rangle \equiv \mathcal{N} \int d^3\vec{p} |\vec{p}\rangle
\]

define charge distribution (for this localized state)

\[
\rho(\vec{r}) \equiv \left\langle \vec{R} = \vec{0} \right| j^0(\vec{r}) \left| \vec{R} = \vec{0} \right\rangle
\]
use translational invariance to relate to same matrix element that appears in def. of form factor

\[
\rho(\vec{r}) \equiv \left\langle \vec{R} = \vec{0} \right| j^0(\vec{r}) \left| \vec{R} = \vec{0} \right\rangle
\]

\[
= |\mathcal{N}|^2 \int d^3 \vec{p} \int d^3 \vec{p}' \langle \vec{p}' | j^0(\vec{0}) | \vec{p} \rangle
\]

\[
= |\mathcal{N}|^2 \int d^3 \vec{p} \int d^3 \vec{p}' \langle \vec{p}' | j^0(\vec{0}) | \vec{p} \rangle e^{i\vec{r} \cdot (\vec{p}' - \vec{p})},
\]

\[
= |\mathcal{N}|^2 \int d^3 \vec{p} \int d^3 \vec{p}' F \left( - (\vec{p}' - \vec{p})^2 \right) e^{i\vec{r} \cdot (\vec{p}' - \vec{p})}
\]

\[
\rho(\vec{r}) = \int \frac{d^3 \vec{q}}{(2\pi)^3} F(-\vec{q}^2) e^{i\vec{q} \cdot \vec{r}}
\]
Form Factors (relativistic)

- Lorentz invariance, parity, current conservation ⇒

\[
\langle p' | j^\mu(0) | p \rangle = \begin{cases} 
(p^\mu + p'^\mu) F(q^2) & \text{(spin 0)} \\
\bar{u}(p') \left[ \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q\nu}{2M} F_2(q^2) \right] u(p) & \text{(spin 1/2)}
\end{cases}
\]

with \( q^\mu = p^\mu - p'^\mu \).

- issue: “energy factors” spoil simple interpretation of form factors as FT of charge distributions
wave packet

\[ |\Psi\rangle = \int \frac{d^3p}{\sqrt{2E_{\vec{p}}(2\pi)^3}} \psi(\vec{p}) |\vec{p}\rangle, \]

\[ E_{\vec{p}} = \sqrt{M^2 + \vec{p}^2} \]

and covariant normalization \( \langle \vec{p}' | \vec{p} \rangle = 2E_{\vec{p}} \delta(\vec{p}' - \vec{p}) \)

Fourier transform of charge distribution in the wave packet

\[ \tilde{\rho}(\vec{q}) \equiv \int d^3x e^{-i\vec{q} \cdot \vec{x}} \langle \Psi | j^0(\vec{x}) | \Psi \rangle \]

\[ = \int \frac{d^3p}{\sqrt{2E_{\vec{p}}2E_{\vec{p}'}}} \Psi^* (\vec{p}' + \vec{q}) \Psi(\vec{p}) \langle \vec{p}' | j^0(\vec{0}) | \vec{p} \rangle \]

\[ = \frac{1}{2} \int d^3p \frac{E_{\vec{p}} + E_{\vec{p}'}}{\sqrt{E_{\vec{p}}E_{\vec{p}'}}} \Psi^* (\vec{p}' + \vec{q}) \Psi(\vec{p}) F(q^2). \]
Nonrelativistic case:

\[
\frac{E_{\vec{p}} + E_{\vec{p}'} }{2 \sqrt{E_{\vec{p}} E_{\vec{p}'}}} = 1 \quad \text{and} \quad q^2 = -\vec{q}^2
\]

\[\vec{p} + \vec{q} \approx \vec{p} \]

\[\rho(\vec{q}) = \int d^3 p \Psi^*(\vec{p} + \vec{q}) \Psi(\vec{p}) F(\vec{q}^2) \]

\[\rho(\vec{q}) = F(\vec{q}^2) \]
Form Factor vs. Charge Distribution (relativistic)

- Relativistic corrections (example rms radius):

\[
\tilde{\rho}(q^2) = 1 - \frac{R^2}{6}q^2 - \frac{R^2}{6} \int d^3p |\Psi(\vec{p})|^2 \frac{(\vec{q} \cdot \vec{p})^2}{E_{\vec{p}}^2}
\]

\[
+ \int d^3p \left| \vec{q} \cdot \vec{\nabla} \Psi(\vec{p}) \right|^2 - \frac{1}{8} \int d^3p |\Psi(\vec{p})|^2 \frac{(\vec{q} \cdot \vec{p})^2}{E_{\vec{p}}^4} \cdot
\]

\[ [R^2 \text{ defined as usual: } F(q^2) = 1 + \frac{R^2}{6}q^2 + O(q^4)] \]

- If one completely localizes the wave packet, i.e.

\[ \int d^3p \left| \vec{q} \cdot \vec{\nabla} \Psi(\vec{p}) \right|^2 \rightarrow 0, \text{ then relativistic corrections diverge} \]

\[ (\Delta x \Delta p \sim 1) \]

\[
\frac{R^2}{6} \int d^3p |\Psi(\vec{p})|^2 \frac{(\vec{q} \cdot \vec{p})^2}{E_{\vec{p}}^2} \rightarrow \infty, \quad \frac{1}{8} \int d^3p |\Psi(\vec{p})|^2 \frac{(\vec{q} \cdot \vec{p})^2}{E_{\vec{p}}^4} \rightarrow \infty
\]
in rest frame, rel. corrections contribute $\Delta R^2 \sim \lambda_C^2 = \frac{1}{M^2}$

identification of charge distribution in rest frame with Fourier transformed form factor only unique down to scale $\lambda_C$

standard remedy: interpret $F(\vec{q})$ as Fourier transform of charge distribution in Breit “frame” $\vec{p}' = -\vec{p}$ (note: Breit “frame” is actually a different frame for each $\vec{q}$!)
infinite momentum frame: rel. corrections governed by $\frac{\vec{p} \cdot \vec{q}}{E_p^2}$ and $\frac{\vec{q}^2}{E_p^2}$

consider wave packet $\Psi(\vec{p}_\perp)$ in transverse direction, with
- sharp longitudinal momentum $P_z \to \infty$
- transverse size of wave packet $r_\perp$, with $R \gg r_\perp \gg \frac{1}{P_z}$

take purely transverse momentum transfer
\[
\tilde{\rho}(\vec{q}_\perp) = F(\vec{q}_\perp^2)
\]

form factor can be interpreted as Fourier transform of charge distribution w.r.t. impact parameter in $\infty$ momentum frame (without $\lambda_C$ uncertainties!)

impact parameter measured w.r.t. $\perp$ center of momentum
\[
R_\perp = \sum_{i \in q, g} x_i r_\perp^i
\]
Same Derivation in LF-Coordinates

- light-front (LF) coordinates

\[ p^+ = \frac{1}{\sqrt{2}} (p^0 + p^3) \quad p^- = \frac{1}{\sqrt{2}} (p^0 - p^3) \]

- form factor for spin \( \frac{1}{2} \) target (Lorentz invariance, parity, charge conservation)

\[
\langle p' | j^\mu (0) | p \rangle = \bar{u}(p') \left[ \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q^\nu}{2M} F_2(q^2) \right] u(p)
\]

with \( q^\mu = p^\mu - p'^\mu \).

- If \( q^+ = 0 \) (Drell-Yan-West frame) then

\[
\langle p', \uparrow | j^+ (0) | p, \uparrow \rangle = 2p^+ F_1(-q_{\perp}^2)
\]
\[ F(q^2_{\perp}) \rightarrow \rho(r_{\perp}) \text{ in LF-Coordinates} \]

- Define state that is localized in \( \perp \) position:

\[
|p^+, R_{\perp} = 0_{\perp}, \lambda\rangle \equiv \mathcal{N} \int d^2 p_{\perp} |p^+, p_{\perp}, \lambda\rangle
\]

Note: \( \perp \) boosts in IMF form Galilean subgroup \( \Rightarrow \) this state has

\[
R_{\perp} \equiv \frac{1}{p^+} \int dx^- d^2 x_{\perp} x_{\perp} T^{++}(x) = 0_{\perp}
\]

(cf.: working in CM frame in nonrel. physics)

- Define charge distribution in impact parameter space

\[
2p^+ \rho(b_{\perp}) \equiv \frac{1}{2p^+} \langle p^+, R_{\perp} = 0_{\perp} | j^+(0^-, b_{\perp}) | p^+, R_{\perp} = 0_{\perp} \rangle
\]
\[ F(q_\perp^2) \rightarrow \rho(r_\perp) \text{ in LF-Coordinates} \]

- use translational invariance to relate to same matrix element that appears in def. of form factor

\[
\rho(b_\perp) \equiv \frac{1}{2p^+} \langle p^+, R_\perp = 0_\perp | j^+(0^-, b_\perp) | p^+, R_\perp = 0_\perp \rangle = \frac{|N|^2}{2p^+} \int d^2 p_\perp \int d^2 p'_\perp \langle p^+, p'_\perp | j^+(0^-, b_\perp) | p^+, p_\perp \rangle e^{i\mathbf{q}_\perp \cdot b_\perp} = \frac{|N|^2}{2p^+} \int d^2 p_\perp \int d^2 p'_\perp F_1(-q_\perp^2) e^{i\mathbf{q}_\perp \cdot b_\perp} \]

\[ \rho(b_\perp) = \int \frac{d^2 q_\perp}{(2\pi)^2} F_1(-q_\perp^2) e^{i\mathbf{q}_\perp \cdot b_\perp} \]
Summary: Form Factor vs. Charge Distribution

- fixed target: FT of form factor = charge distribution in position space
- “moving” target:
  - nonrelativistically: FT of form factor = charge distribution in position space, where position is measured relative to center of mass
  - relativistic corrections usually make identification
    \[ F(q^2) \overset{FT}{\leftrightarrow} \rho(\vec{r}) \] ambiguous at scale \( \Delta R \sim \lambda_C = \frac{1}{M} \)
- exceptions:
  - Breit “frame”
  - \( \infty \) momentum frame (\( \rightarrow \) Galilean subgroup of \( \perp \) boosts)
  - Reference point: transverse center of longitudinal momentum
light-cone coordinates:

\[ x^+ = \left( x^0 + x^3 \right) / \sqrt{2} \]
\[ x^- = \left( x^0 - x^3 \right) / \sqrt{2} \]

DIS related to correlations along light–cone

\[ q(x_{Bj}) = \int \frac{dx^-}{2\pi} \langle P | \bar{q}(0^-, 0_) \gamma^+ q(x^-, 0_) | P \rangle e^{ix^- x_{Bj} P^+} \]

Probability interpretation!

No information about transverse position of partons!
\[ \int \frac{dx^-}{2\pi} e^{ix^-\hat{p}^+x} \left\langle p' \right| \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \right| p \right\rangle = H(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ u(p) + E(x, \xi, \Delta^2) \bar{u}(p') \frac{i\sigma^{+\nu} \Delta_\nu}{2M} u(p) \]

\[ \int \frac{dx^-}{2\pi} e^{ix^-\hat{p}^+x} \left\langle p' \right| \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ \gamma_5 q \left( \frac{x^-}{2} \right) \right| p \right\rangle = \tilde{H}(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ \gamma_5 u(p) + \tilde{E}(x, \xi, \Delta^2) \bar{u}(p') \frac{\gamma_5 \Delta^+}{2M} u(p) \]

where \( \Delta = p' - p \) is the momentum transfer and \( \xi \) measures the longitudinal momentum transfer on the target \( \Delta^+ = \xi(p^+ + p'^+) \).
\[ \int \frac{dx^-}{2\pi} e^{i x^- \vec{p}^+ x} \left( \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \right| p \right) = H(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ u(p) \]

\[ + E(x, \xi, \Delta^2) \bar{u}(p') \frac{i \sigma^{+ \nu} \Delta^\nu}{2M} u(p) \]

- \( x \) is mean long. momentum fraction carried by active quark
- In general no probabilistic interpretation since initial and final state not the same
- Instead: interpretation as transition amplitude
- \( \int dx H(x, \xi, \Delta^2) = F_1(\Delta^2) \) and \( \int dx E(x, \xi, \Delta^2) = F_2(\Delta^2) \)
- GPDs provide a decomposition of form factor w.r.t. the momentum fraction (in IMF) carried by the active quark
- Actually \( H = H(x, \xi, \Delta^2, q^2) \), but will not discuss \( q^2 \) dependence of GPDs today!
What is Physics of GPDs?

Definition of GPDs resembles that of form factors

\[
\left\langle p' \left| \hat{O} \right| p \right\rangle = H(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ u(p) + E(x, \xi, \Delta^2) \bar{u}(p') \frac{i \sigma^{+\nu} \Delta^\nu}{2M} u(p)
\]

with \( \hat{O} \equiv \int \frac{d x^-}{2\pi} e^{ix^- p^+ x^-} \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \)

relationship between PDFs and GPDs similar to relation between a charge and a form factor

If form factors can be interpreted as Fourier transforms of charge distributions in position space, what is the analogous physical interpretation for GPDs?
Form Factors vs. GPDs

\[
\begin{array}{|c|c|c|c|}
\hline
\text{operator} & \text{forward matrix elem.} & \text{off-forward matrix elem.} & \text{position space} \\
\hline
\bar{q} \gamma^+ q & Q & F(t) & \rho(\vec{r}^*) \\
\int \frac{dx^- e^{ixp^+ x^-}}{4\pi} \bar{q}\left(\frac{-x^-}{2}\right) \gamma^+ q\left(\frac{x^-}{2}\right) & q(x) & H(x, \xi, t) & ? \\
\hline
\end{array}
\]
### Form Factors vs. GPDs

<table>
<thead>
<tr>
<th>operator</th>
<th>forward matrix elem.</th>
<th>off-forward matrix elem.</th>
<th>position space</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{q}\gamma^+ q$</td>
<td>$Q$</td>
<td>$F(t)$</td>
<td>$\rho(\vec{r}^\ast)$</td>
</tr>
<tr>
<td>$\int \frac{dx^-}{4\pi} e^{ixp^+} \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right)$</td>
<td>$q(x)$</td>
<td>$H(x, 0, t)$</td>
<td>$q(x, b_\perp)$</td>
</tr>
</tbody>
</table>

$q(x, b_\perp) = \text{impact parameter dependent PDF}$
Impact parameter dependent PDFs

- define state that is localized in $\perp$ position:

$$ |p^+, R_\perp = 0_\perp, \lambda \rangle \equiv N \int d^2p_\perp |p^+, p_\perp, \lambda \rangle $$

Note: $\perp$ boosts in IMF form Galilean subgroup $\Rightarrow$ this state has $R_\perp \equiv \frac{1}{p^+} \int dx^- d^2x_\perp x_\perp T^{++}(x) = 0_\perp$

(cf.: working in CM frame in nonrel. physics)

- define impact parameter dependent PDF

$$ q(x, b_\perp) \equiv \int \frac{dx^-}{4\pi} \langle p^+, R_\perp = 0_\perp \mid \bar{q}(-\frac{x^-}{2}, b_\perp)\gamma^+ q(\frac{x^-}{2}, b_\perp) \mid p^+, R_\perp = 0_\perp \rangle e^{ixp^+ x^-} $$
Impact parameter dependent PDFs

- use translational invariance to relate to same matrix element that appears in def. of GPDs

\[
q(x, b_\perp) \equiv \int dx^- \langle p^+, R_\perp = 0_\perp | \bar{q}(-\frac{x^-}{2}, b_\perp)\gamma^+ q(\frac{x^-}{2}, b_\perp) | p^+, R_\perp = 0_\perp \rangle e^{ixp^+x^-} \\
= |\mathcal{N}|^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp | \bar{q}(-\frac{x^-}{2}, b_\perp)\gamma^+ q(\frac{x^-}{2}, b_\perp) | p^+, p_\perp \rangle e^{ixp^+x^-}
\]
Impact parameter dependent PDFs

- use translational invariance to relate to same matrix element that appears in def. of GPDs

\[ q(x, b_\perp) \equiv \int dx^- \langle p^+, R_\perp = 0_\perp \mid q(-x^-/2, b_\perp) \gamma^+ q(x^-/2, b_\perp) \mid p^+, R_\perp = 0_\perp \rangle e^{ixp^+x^-} \]

\[ = |N|^2 \int d^2p_\perp \int d^2p'_\perp \int dx^- \langle p^+, p'_\perp \mid q(-x^-/2, b_\perp) \gamma^+ q(x^-/2, b_\perp) \mid p^+, p_\perp \rangle e^{ixp^+x^-} \]

\[ = |N|^2 \int d^2p_\perp \int d^2p'_\perp \int dx^- \langle p^+, p'_\perp \mid q(-x^-/2, 0_\perp) \gamma^+ q(x^-/2, 0_\perp) \mid p^+, p_\perp \rangle e^{ixp^+x^-} \]

\[ \times e^{ib_\perp \cdot (p_\perp - p'_\perp)} \]
Impact parameter dependent PDFs

- use translational invariance to relate to same matrix element that appears in def. of GPDs

\[
q(x, b_\perp) \equiv \int dx^- \langle p^+, R_\perp = 0_\perp \mid \bar{q}(\frac{-x^\perp}{2}, b_\perp) \gamma^+ q(\frac{-x^\perp}{2}, b_\perp) \mid p^+, R_\perp = 0_\perp \rangle e^{ixp^+ x^-} \\
= \mid N \mid^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp \mid \bar{q}(\frac{-x^\perp}{2}, b_\perp) \gamma^+ q(\frac{-x^\perp}{2}, b_\perp) \mid p^+, p_\perp \rangle e^{ixp^+ x^-} \\
= \mid N \mid^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp \mid \bar{q}(\frac{-x^\perp}{2}, 0_\perp) \gamma^+ q(\frac{-x^\perp}{2}, 0_\perp) \mid p^+, p_\perp \rangle e^{ixp^+ x^-} \\
\times e^{ib_\perp \cdot (p_\perp - p'_\perp)} \\
= \mid N \mid^2 \int d^2 p_\perp \int d^2 p'_\perp H \left( x, 0, -(p'_\perp - p_\perp)^2 \right) e^{ib_\perp \cdot (p_\perp - p'_\perp)}
\]

\[
\rightarrow q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} H(x, -\Delta_\perp^2) e^{-ib_\perp \cdot \Delta_\perp}
\]
Impact parameter dependent PDFs

\[ q(x, b_{\perp}) = \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} H(x, -\Delta^2_{\perp}) e^{-ib_{\perp} \cdot \Delta_{\perp}} \]

(\(\Delta_{\perp} = p'_{\perp} - p_{\perp}, \xi = 0\))

- \(q(x, b_{\perp})\) has physical interpretation of a density

\[ q(x, b_{\perp}) \geq 0 \quad \text{for} \quad x > 0 \]
\[ q(x, b_{\perp}) \leq 0 \quad \text{for} \quad x < 0 \]
Discussion: \[ GPD \leftrightarrow q(x, b_\perp) \]

- GPDs allow simultaneous determination of longitudinal momentum and transverse position of partons

\[
q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} H(x, 0, -\Delta^2_\perp) e^{-i b_\perp \cdot \Delta_\perp}
\]

- \( q(x, b_\perp) \) has interpretation as density (positivity constraints!)

\[
q(x, b_\perp) \sim \langle p^+, 0_\perp | b^\dagger(xp^+, b_\perp) b(xp^+, b_\perp) | p^+, 0_\perp \rangle \\
= | b(xp^+, b_\perp) | p^+, 0_\perp \rangle|^2 \geq 0
\]

\( \leftarrow \) positivity constraint on models
Nonrelativistically such a result not surprising!
Absence of relativistic corrections to identification

\[ H(x, 0, -\Delta^2_\perp) \xrightarrow{FT} q(x, b_\perp) \]

due to Galilean subgroup in IMF

\( b_\perp \) distribution measured w.r.t. \( R_{CM}^{CM} \equiv \sum_i x_i r_{i,\perp} \)
width of the \( b_\perp \) distribution should go to zero as \( x \to 1 \), since
the active quark becomes the \( \perp \) center of momentum in that limit!
\( H(x, t) \) must become \( t \)-indep. as \( x \to 1 \).

very similar results for impact parameter dependent polarized quark distributions (nucleon longitudinally polarized)

\[ \Delta q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} \tilde{H}(x, 0, -\Delta^2_\perp) e^{-ib_\perp \cdot \Delta_\perp} \]
Use intuition about nucleon structure in position space to make predictions for GPDs:

- large $x$: quarks from localized valence ‘core’,
- small $x$: contributions from larger ‘meson cloud’

→ expect a gradual increase of the $t$-dependence ($\perp$ size) of $H(x, 0, t)$ as $x$ decreases

- small $x$, expect transverse size to increase

very simple model: $H_q(x, 0, -\Delta^2_{\perp}) = q(x) e^{-a\Delta^2_{\perp}(1-x) \ln \frac{1}{x}}$. 
Other topics

- QCD evolution
- extrapolating to $\xi = 0$
So far: only unpolarized (or long. polarized) nucleon

In general, use (\(\Delta^+ = 0\))

\[
\int \frac{dx^-}{4\pi} e^{ip^+x^-x} \left\langle P+\Delta, \uparrow \left| \bar{q}(0) \gamma^+ q(x^-) \right| P, \uparrow \right\rangle = H(x,0,-\Delta^2_\perp)
\]

\[
\int \frac{dx^-}{4\pi} e^{ip^+x^-x} \left\langle P+\Delta, \uparrow \left| \bar{q}(0) \gamma^+ q(x^-) \right| P, \downarrow \right\rangle = -\frac{\Delta x - i\Delta y}{2M} E(x,0,-\Delta^2_\perp).
\]

Consider nucleon polarized in \(x\) direction (in IMF)

\(|X\rangle \equiv |p^+, \mathbf{R}_\perp = 0_\perp, \uparrow\rangle + |p^+, \mathbf{R}_\perp = 0_\perp, \downarrow\rangle\).

\(\leftrightarrow\) unpolarized quark distribution for this state:

\[
q_X(x,b_\perp) = q(x,b_\perp) - \frac{1}{2M} \frac{\partial}{\partial b_y} \int \frac{d^2\Delta_\perp}{(2\pi)^2} E(x,-\Delta^2_\perp) e^{-ib_\perp \cdot \Delta_\perp}
\]
The physics of $E(x, 0, -\Delta^2_\perp)$

$q_X(x, b_\perp)$ in transversely polarized nucleon is transversely distorted compared to longitudinally polarized nucleons!

- Mean displacement of flavor $q$ ($\perp$ flavor dipole moment)

$$d_y^q \equiv \int dx \int d^2 b_\perp q_X(x, b_\perp) b_y = \frac{1}{2M} \int dx E_q(x, 0, 0) = \kappa^p_q$$

with $\kappa^p_u/d \equiv F^{u/d}_2(0) = \mathcal{O}(1 - 2) \Rightarrow d_y^q = \mathcal{O}(0.2 \text{ fm})$

- CM for flavor $q$ shifted relative to CM for whole proton by

$$\int dx \int d^2 b_\perp q_X(x, b_\perp) x b_y = \frac{1}{2M} \int dx x E_q(x, 0, 0)$$

\[\text{\leftarrow not surprising to find that second moment of } E_q \text{ is related to angular momentum carried by flavor } q\]
Comparison of a non-rotating sphere that moves in \( z \) direction with a sphere that spins at the same time around the \( z \) axis and a sphere that spins around the \( x \) axis. When the sphere spins around the \( x \) axis, the rotation changes the distribution of momenta in the \( z \) direction (adds/subtracts to velocity for \( y > 0 \) and \( y < 0 \) respectively). For the nucleon, the resulting modification of the (unpolarized) momentum distribution is described by \( E(x, 0, -\Delta_{\perp}^2) \).
simple model for $E_q(x, 0, -\Delta^2_\perp)$

- For simplicity, make ansatz where $E_q \propto H_q$

  $$E_u(x, 0, -\Delta^2_\perp) = \frac{\kappa_u^p}{2} H_u(x, 0, -\Delta^2_\perp)$$
  $$E_d(x, 0, -\Delta^2_\perp) = \kappa_d^p H_d(x, 0, -\Delta^2_\perp)$$

  with $H_q(x, 0, -\Delta^2_\perp) = q(x)e^{-a\Delta^2_\perp(1-x)\ln\frac{1}{x}}$ and

  $$\kappa_u^p = 2\kappa_p + \kappa_n = 1.673 \quad \kappa_d^p = 2\kappa_n + \kappa_p = -2.033.$$  

- Satisfies: $\int dx E_q(x, 0, 0) = \kappa_q^p$

- Model too simple but illustrates that anticipated distortion is very significant since $\int dx E_q \sim \kappa_q$ known to be large!
Example: left-right asymmetry in semi-inclusive $\gamma p \rightarrow \pi^+ + X$ on a $\perp$ polarized target ($\vec{p}_\gamma \propto \vec{e}_z$, $\vec{S}_p \propto \vec{e}_x$, asymmetry $\propto \vec{e}_y$)

Sivers mechanism: left-right asymmetry due to $\perp$ asymmetry of $\perp$-momentum dependent PDFs $f(x, k_\perp)$

$$f(x, k_\perp) = \int \frac{dy^- d^2y_\perp}{16\pi^3} e^{-ixp^+y^-+ik_\perp \cdot y_\perp} \langle p | \bar{q}(0, y^-, y_\perp)\gamma^+ q(0) | p \rangle.$$  

gauge invariance $\rightarrow$ include Wilson line!

Naively, $f(x, k_\perp) = f(x, -k_\perp)$, due to time-reversal invariance, i.e. with above definition, Sivers asymmetry vanishes identically $[\vec{p}_q \cdot (\vec{p}_p \times \vec{S}_p)$ is T-odd]
single-spin asymmetry

However, Brodsky et al. ⇒ Sivers asymmetry possible due to FSI!

Formal argument: include FSI in eikonal approximation

define \( f(x, k_\perp) \) gauge invariantly

\[
f(x, k_\perp) = \int \frac{dy^- d^2y_\perp}{16\pi^3} e^{-ixp^+y^- + ik_\perp \cdot y_\perp} \langle p | \bar{q}(0, y^- , y_\perp) W_{y\infty}^\dagger \gamma^+ W_{0\infty} q(0) | p \rangle.
\]

\( W_{y\infty} = P \exp \left( -ig \int_{y^-}^{\infty} dz^- A^+(y^+ , z^- , y_\perp) \right) \) indicates a Wilson-line operator going from point \( y \) to infinity (FSI!).

Wilson line not invariant under \( T \)

Sivers asymmetry possible \([f(x, k_\perp) \neq f(x, -k_\perp)]\)
single-spin asymmetry

Presence of phase factors in definition of Sivers distribution do explain why SSA can be nonzero.

does not obviously explain:

- why these “phase factors” give rise to such large ‘stable’ polarization effects? (example: $\perp$ polarization in hyperon production)
- which sign should one expect in which reaction?
Physical origin of SSA

- mean $\perp$ momentum of active quark in semi-inclusive DIS contains term

$$\langle k_\perp \rangle \sim \int dy^- e^{-ixp^+y^-} \langle p | \bar{q}(0, y^-, y_\perp) W_{y^\infty}^+ \partial_\perp W_{0^\infty} \gamma^+ q(0) | p \rangle$$

- Physics of this term (for simplicity abelian case):

  - this term simplifies as $\langle k_\perp \rangle \sim ... - g \int_{z^-}^\infty dy^- \partial_\perp A^+(y^-, z_\perp)$ which has semi-classical interpretation as impulse experienced by the active quark on its way out from $\perp$ position $z_\perp$.

- mean $\perp$ momentum obtained as correlation between PDF and transverse impulse $I_{\perp}(z_\perp) = g \int_{z^-}^\infty dy^- \partial_\perp A^+(y^-, z_\perp)$

- physics of this correlation $\rightarrow$ switch to impact parameter representation
Connection with \( \perp \) distortion of \( q(x, b_\perp) \)

- use simple potential model to estimate
  \[
  I_\perp(z_\perp) \equiv \partial_\perp \int dy^- A^+(y^-, z_\perp) = \text{mean } \perp \text{ impulse that the FSI exert on active quark on its way out as function of the separation from the CM}
  \]
use simple potential model to estimate
\[ I_\perp(zT) \equiv \partial_\perp \int dy^- A^+(y^-, z_\perp) = \text{mean } \perp \text{ impulse that the FSI exert on active quark on its way out as function of the separation from the CM} \]
$\gamma p \rightarrow \pi X$ in Breit frame

- $u, d$ distributions in $\perp$ polarized proton have left-right asymmetry in $\perp$ position space (T-even!)
- Attractive FSI deflects active quark towards the center of momentum
- FSI converts left-right position space asymmetry of leading quark into right-left asymmetry in momentum
- Compare: convex lens that is illuminated asymmetrically
Summary

- DVCS allows probing GPDS

\[ \int \frac{dx^-}{2\pi} e^{ixp^+x^-} \left\langle p' \left| \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \right| p \right\rangle \]

- GPDs resemble both PDFs and form factors: defined through matrix elements of light-cone correlation, but \( \Delta \equiv p' - p \neq 0 \).

- \( t \)-dependence of GPDs at \( \xi = 0 \) (purely \( \perp \) momentum transfer) \( \Rightarrow \) Fourier transform of impact parameter dependent PDFs \( q(x, b_\perp) \)

\[ q(x, b_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} H(x, 0, -\Delta_\perp^2) e^{-ib_\perp \cdot \Delta_\perp} \]

\[ \Delta q(x, b_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} \tilde{H}(x, 0, -\Delta_\perp^2) e^{-ib_\perp \cdot \Delta_\perp} \]

- \( q(x, b_\perp), \Delta q(x, b_\perp) \) have probabilistic interpretation, e.g. \( q(x, b_\perp) > 0 \) for \( x > 0 \)
Summary

\( \frac{\Delta_\perp}{2M} E(x, -\Delta_\perp^2) \) describes how the momentum distribution of unpolarized partons in the \( \perp \) plane gets transversely distorted when is nucleon polarized in \( \perp \) direction.

(attractive) final state interaction converts \( \perp \) position space asymmetry into \( \perp \) momentum space asymmetry
/simple physical explanation for sign of left-right asymmetry in semi-inclusive DIS

Similar mechanism also applicable to many other semi-inclusive events, such as transverse polarizations in hyperon production.


extrapolating to $\xi = 0$

- bad news: $\xi = 0$ not directly accessible in DVCS since long. momentum transfer necessary to convert virtual $\gamma$ into real $\gamma$
- good news: moments of GPDs have simple $\xi$-dependence (polynomials in $\xi$)
  $\Rightarrow$ should be possible to extrapolate!

even moments of $H(x, \xi, t)$:

$$H_n(\xi, t) \equiv \int_{-1}^{1} dx x^{n-1} H(x, \xi, t) = \sum_{i=0}^{\left[ \frac{n-1}{2} \right]} A_{n,2i}(t)\xi^{2i} + C_n(t)$$

$$= A_{n,0}(t) + A_{n,2}(t)\xi^2 + ... + A_{n,n-2}(t)\xi^{n-2} + C_n(t)\xi^n,$$
\[ \int_{-1}^{1} dx xH(x, \xi, t) = A_{2,0}(t) + C_2(t)\xi^2. \]

- For \( n^{th} \) moment, need \( \frac{n}{2} + 1 \) measurements of \( H_n(\xi, t) \) for same \( t \) but different \( \xi \) to determine \( A_{n,2i}(t) \).
- GPDs \( \xi = 0 \) obtained from \( H_n(\xi = 0, t) = A_{n,0}(t) \)
- similar procedure exists for moments of \( \tilde{H} \)

back
QCD evolution

So far ignored! Can be easily included because

- For $t \ll Q^2$, leading order evolution $t$-independent
- For $\xi = 0$ evolution kernel for GPDs
  same as DGLAP evolution kernel

likewise:

- impact parameter dependent PDFs evolve such that different $b_\perp$
do not mix (as long as $\perp$ spatial resolution much smaller than $Q^2$)
above results consistent with QCD evolution:

\[
H(x, 0, -\Delta_\perp^2, Q^2) = \int d^2 b_\perp q(x, b_\perp, Q^2) e^{-i\vec{b}_\perp \Delta_\perp}
\]

\[
\tilde{H}(x, 0, -\Delta_\perp^2, Q^2) = \int d^2 b_\perp \Delta q(x, b_\perp, Q^2) e^{-i\vec{b}_\perp \Delta_\perp}
\]

where QCD evolution of \(H, \tilde{H}, q, \Delta q\) is described by DGLAP and is independent on both \(b_\perp\) and \(\Delta_\perp^2\), provided one does not look at scales in \(b_\perp\) that are smaller than \(1/Q\).
suppression of crossed diagrams

Flow of the large momentum $q$ through typical diagrams contributing to the forward Compton amplitude. a) ‘handbag’ diagrams; b) ‘cat’s ears’ diagram. Diagram b) is suppressed at large $q$ due to the presence of additional propagators.
density interpretation for \( q(x, b_\perp) \)

- express quark-bilinear in twist-2 GPD in terms of light-cone ‘good’ component \( q(+) \equiv \frac{1}{2} \gamma^- \gamma^+ q \)

\[
q' \gamma^+ q = q'_+ \gamma^+ q(+) = \sqrt{2} q'_{(+)} q(+).
\]

- expand \( q(+) \) in terms of canonical raising and lowering operators

\[
q(+) (x^-, x_\perp) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \int \frac{d^2 k_\perp}{2\pi} \sum_s \left[ u(+) (k, s) b_s (k^+, k_\perp) e^{-ikx} + v(+) (k, s) d_s^\dagger (k^+, k_\perp) e^{ikx} \right],
\]
Density interpretation for $q(x, b_\perp)$

with usual (canonical) equal light-cone time $x^+$ anti-commutation relations, e.g.

$$\{ b_r(k^+, k_\perp), b_s^\dagger(q^+, q_\perp) \} = \delta(k^+ - q^+)\delta(k_\perp - q_\perp)\delta_{rs}$$

and the normalization of the spinors is such that

$$\bar{u}_{(+)}(p, r)\gamma^+ u_{(+)}(p, s) = 2p^+\delta_{rs}.$$  

Note: $\bar{u}_{(+)}(p', r)\gamma^+ u_{(+)}(p, s) = 2p^+\delta_{rs}$ for $p^+ = p'^+$, one finds for $x > 0$

$$q(x, b_\perp) = \mathcal{N}' \sum_s \int \frac{d^2k_\perp}{2\pi} \int \frac{d^2k'_\perp}{2\pi} \langle p^+, 0_\perp | b_s^\dagger(xp^+, k'_\perp) b_s(xp^+, k_\perp) | p^+, 0_\perp \rangle \times e^{ib_\perp \cdot (k_\perp - k'_\perp)}.$$
density interpretation for \( q(x, b_{\perp}) \)

- **Switch to mixed representation:**
  - **momentum** in longitudinal direction
  - **position** in transverse direction

\[
\tilde{b}_s(k^+, x_{\perp}) \equiv \int \frac{d^2k_{\perp}}{2\pi} b_s(k^+, k_{\perp}) e^{ik_{\perp} \cdot x_{\perp}}
\]

\[
q(x, b_{\perp}) = \sum_s \langle p^+, 0_{\perp} | \tilde{b}_s^+(xp^+, b_{\perp}) \tilde{b}_s(xp^+, b_{\perp}) | p^+, 0_{\perp} \rangle.
\]

\[
= \sum_s \left| \tilde{b}_s(xp^+, b_{\perp}) | p^+, 0_{\perp} \rangle \right|^2 \\
\geq 0.
\]
express quark-bilinear in twist-2 PDF in terms of light-cone ‘good’ component \( q(+) = \frac{1}{2} \gamma^- \gamma^+ q \)

\[
\bar{q}' \gamma^+ q = \bar{q}'(+) \gamma^+ q(+) = \sqrt{2} q'(+) q(+) .
\]

expand \( q(+) \) in terms of canonical raising and lowering operators

\[
q(+) (x^-, x_\perp) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \int \frac{d^2k_\perp}{2\pi} \sum_s \left[ u(+) (k, s) b_s(k^+, k_\perp) e^{-ikx} + v(+) (k, s) d_s^\dagger(k^+, k_\perp) e^{ikx} \right] ,
\]
density interpretation for $q(x)$

with usual (canonical) equal light-cone time $x^+$ anti-commutation relations, e.g.

$$\{ b_r(k^+, k_\perp), b_s^+(q^+, q_\perp) \} = \delta(k^+ - q^+ \, \delta(k_\perp - q_\perp) \delta_{rs}$$

and the normalization of the spinors is such that

$$\bar{u}(+)(p, r) \gamma^+ u(+)(p, s) = 2p^+ \delta_{rs}.$$ 

(Note: $\bar{u}(+)(p', r) \gamma^+ u(+)(p, s) = 2p^+ \delta_{rs}$ for $p^+ = p'^+$)

- Insert in

$$q(x) = \int \frac{dx^-}{2\pi} \langle p | \bar{q}(0^-, 0_\perp) \gamma^+ q(x^-, 0_\perp) | p \rangle e^{ix^-xp^+}$$

Hadron Tomography – p.53/70
density interpretation for $q(x)$

- one finds for $x > 0$

$$q(x) = \mathcal{N}' \sum_s \int \frac{d^2k_{\perp}}{2\pi} \int \frac{d^2k_{\perp}'}{2\pi} \langle p | b_s^\dagger(xp^+, k_{\perp}') b_s(xp^+, k_{\perp}) | p \rangle$$

$$= \mathcal{N}' \sum_s \left| \int \frac{d^2k_{\perp}}{2\pi} b_s(xp^+, k_{\perp}) | p \rangle \right|^2 \geq 0.$$

- antiquarks ($x < 0$) yield $q(x) < 0$

$\iff$ usually define positive antiquark distribution

$$\bar{q}(x) \equiv -q(-x) \quad (x > 0)$$
Boosts in nonrelativistic QM

\[ \vec{x}' = \vec{x} + \vec{v}t \quad t' = t \]

purely kinematical (quantization surface \( t = 0 \) inv.)

1. boosting wavefunctions very simple

\[ \Psi_v(\vec{p}_1, \vec{p}_2) = \Psi_0(\vec{p}_1 - m_1 \vec{v}, \vec{p}_2 - m_2 \vec{v}). \]

2. dynamics of center of mass

\[ \vec{R} \equiv \sum \limits_i x_i \vec{r}_i \quad \text{with} \quad x_i \equiv \frac{m_i}{M} \]

decouples from the internal dynamics
Relativistic Boosts

\[ t' = \gamma \left( t + \frac{v}{c^2} z \right), \quad z' = \gamma (z + vt) \quad x'_\perp = x_\perp \]

generators satisfy Poincaré algebra:

\[
\begin{align*}
[P^\mu, P^\nu] &= 0 \\
[M^{\mu\nu}, P^\rho] &= i \left( g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu \right) \\
[M^{\mu\nu}, M^{\rho\lambda}] &= i \left( g^{\mu\lambda} M^{\nu\rho} + g^{\nu\rho} M^{\mu\lambda} - g^{\mu\rho} M^{\nu\lambda} - g^{\nu\lambda} M^{\mu\rho} \right)
\end{align*}
\]

rotations: \( M_{ij} = \varepsilon_{ijk} J_k \), boosts: \( M_{i0} = K_i \).
introduce generator of ↓ ‘boosts’:

\[ B_x \equiv M^+x = \frac{K_x + J_y}{\sqrt{2}} \quad B_y \equiv M^+y = \frac{K_y - J_x}{\sqrt{2}} \]

Poincaré algebra \( \implies \) commutation relations:

\[ [J_3, B_k] = i\epsilon_{kl} B_l \quad [P_k, B_l] = -i\delta_{kl} P^+ \]
\[ [P^-, B_k] = -iP_k \quad [P^+, B_k] = 0 \]

with \( k, l \in \{x, y\} \), \( \epsilon_{xy} = -\epsilon_{yx} = 1 \), and \( \epsilon_{xx} = \epsilon_{yy} = 0 \).
Together with $[J_z, P_k] = i\varepsilon_{kl} P_l$, as well as

$$
\begin{align*}
[P^-, P_k] &= [P^-, P^+] = [P^-, J_z] = 0 \\
[P^+, P_k] &= [P^+, B_k] = [P^+, J_z] = 0.
\end{align*}
$$

Same as commutation relations among generators of nonrel. boosts, translations, and rotations in x-y plane, provided one identifies

- $P^-$ → Hamiltonian
- $\mathbf{P}_\perp$ → momentum in the plane
- $P^+$ → mass
- $L_z$ → rotations around z-axis
- $\mathbf{B}_\perp$ → generator of boosts in the plane,

back to discussion
many results from NRQM carry over to \( \perp \) boosts in IMF, e.g.

\( \Psi_{\Delta \perp}(x, k_\perp) = \Psi_{0 \perp}(x, k_\perp - x \Delta_\perp) \)

\( \Psi_{\Delta \perp}(x, k_\perp, y, l_\perp) = \Psi_{0 \perp}(x, k_\perp - x \Delta_\perp, y, l_\perp - y \Delta_\perp) \)

Transverse center of momentum \( \mathbf{R}_\perp \equiv \sum_i x_i \mathbf{r}_\perp,i \) plays role similar to NR center of mass, e.g. \( \int d^2 p_\perp |p^+, p_\perp \rangle \) corresponds to state with \( \mathbf{R}_\perp = 0_\perp \).
Center of Momentum

- **field theoretic definition**

\[ p^+ R_\perp \equiv \int dx^- \int d^2 x_\perp T^{++}(x) x_\perp = M^{+\perp} \]

- \( M^{+\perp} = B^{\perp} \) generator of transverse boosts

- **parton representation:**

\[ R_\perp = \sum_i x_i r_{\perp,i} \]

\( (x_i = \text{momentum fraction carried by } i^{th} \text{ parton}) \)
Poincaré algebra:

\[
\begin{align*}
[P^\mu, P^\nu] &= 0 \\
[M^{\mu\nu}, P^\rho] &= i \left( g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu \right) \\
[M^{\mu\nu}, M^{\rho\lambda}] &= i \left( g^{\mu\lambda} M^{\nu\rho} + g^{\nu\rho} M^{\mu\lambda} - g^{\mu\rho} M^{\nu\lambda} - g^{\nu\lambda} M^{\mu\rho} \right)
\end{align*}
\]

rotations: \( M_{ij} = \varepsilon_{ijk} J_k \), boosts: \( M_{i0} = K_i \).
introduce generator of ↓ ‘boosts’:

\[ B_x \equiv M^{+x} = \frac{K_x + J_y}{\sqrt{2}} \quad B_y \equiv M^{+y} = \frac{K_y - J_x}{\sqrt{2}} \]

Poincaré algebra \implies commutation relations:

\[
\begin{align*}
[J_3, B_k] &= i\varepsilon_{kl} B_l \\
[P^-, B_k] &= -iP_k \\
[P^+, B_k] &= 0
\end{align*}
\]

with \( k, l \in \{x, y\} \), \( \varepsilon_{xy} = -\varepsilon_{yx} = 1 \), and \( \varepsilon_{xx} = \varepsilon_{yy} = 0 \).
Together with $[J_z, P_k] = i\varepsilon_{kl} P_l$, as well as

$$
[P^-, P_k] = [P^-, P^+] = [P^-, J_z] = 0
$$

$$
[P^+, P_k] = [P^+, B_k] = [P^+, J_z] = 0.
$$

Same as commutation relations among generators of nonrel. boosts, translations, and rotations in x-y plane, provided one identifies

- $P^-$ $\rightarrow$ Hamiltonian
- $P_\perp$ $\rightarrow$ momentum in the plane
- $P^+$ $\rightarrow$ mass
- $L_z$ $\rightarrow$ rotations around $z$-axis
- $B_\perp$ $\rightarrow$ generator of boosts in the plane,
many results from NRQM carry over to \( \perp \) boosts in IMF, e.g.

\[ \Psi_{\Delta_{\perp}}(x, k_{\perp}) = \Psi_{0_{\perp}}(x, k_{\perp} - x \Delta_{\perp}) \]

\[ \Psi_{\Delta_{\perp}}(x, k_{\perp}, y, l_{\perp}) = \Psi_{0_{\perp}}(x, k_{\perp} - x \Delta_{\perp}, y, l_{\perp} - y \Delta_{\perp}) \]

Transverse center of momentum \( R_{\perp} \equiv \sum_i x_i r_{\perp,i} \) plays role similar to NR center of mass, e.g. \( |p^+, R_{\perp} = 0_{\perp} \rangle \equiv \int d^2 p_{\perp} |p^+, p_{\perp} \rangle \) corresponds to state with \( R_{\perp} = 0_{\perp} \).
Proof that $B_\perp |p^+, R_\perp = 0_\perp \rangle = 0$

Use

$$e^{-i \mathbf{v}_\perp \cdot \mathbf{B}_\perp} |p^+, \mathbf{p}_\perp, \lambda \rangle = |p^+, \mathbf{p}_\perp + p^+ \mathbf{v}_\perp, \lambda \rangle$$

$$\Longleftrightarrow$$

$$e^{-i \mathbf{v}_\perp \cdot \mathbf{B}_\perp} \int d^2 \mathbf{p}_\perp |p^+, \mathbf{p}_\perp, \lambda \rangle = \int d^2 \mathbf{p}_\perp |p^+, \mathbf{p}_\perp, \lambda \rangle$$

$$\Longleftrightarrow$$

$$B_\perp \int d^2 \mathbf{p}_\perp |p^+, \mathbf{p}_\perp, \lambda \rangle = 0$$
 Ansatz: \( H_q(x, 0, -\Delta_{\perp}^2) = q(x)e^{-a\Delta_{\perp}^2(1-x)\ln\frac{1}{x}}. \)

\[
\iff q(x, b_{\perp}) = q(x)\frac{1}{4\pi a(1-x)\ln\frac{1}{x}}e^{-\frac{b_{\perp}^2}{4a(1-x)\ln\frac{1}{x}}} 
\]
$q(x, b_\perp)$
The boost operator in NRQM:

$$\vec{R} = \frac{1}{M} \sum_i m_i \vec{r}_i$$

satisfies ($H = \sum_l \frac{\vec{p}_l^2}{2m_l} + V(\vec{r}_i)$)

$$\left[ R_k, P_l \right] = -i\delta_{kl} \quad \quad \left[ \vec{R}, H \right] = i\vec{P}$$
Physical Meaning of \( x_{Bj} = \frac{Q^2}{2p \cdot q} \)

- Go to frame where \( q_\perp = 0 \), i.e.
  \[
  Q^2 = -q^2 = -2q^+ q^- \\
  2p \cdot q = 2q^- p^+ + 2q^+ p^-
  \]

- Bjorken limit: \( q^- \to \infty \), \( q^+ \) fixed

\[
  x_{Bj} = -\frac{q^+ q^-}{q^- p^+ + q^+ p^-} \to -\frac{q^+}{p^+}
  \]
Physical Meaning of $x_{Bj} = \frac{Q^2}{2p \cdot q}$

- $x_{Bj} = -\frac{q^+}{p^+}$

- LC energy-momentum dispersion relation

$$k^- = \frac{m^2 + k^2_{\perp}}{2k^+}$$

$\leftrightarrow$ struck quark with $k'^- = k^- + q^- \to \infty$ can only be on mass shell if $k'^+ = k^+ + q^+ \approx 0$

$\leftrightarrow$

$$k^+ = -q^+ \quad \Rightarrow \quad x \equiv \frac{k^+}{p^+} = x_{Bj}$$

$\leftrightarrow$ $x_{Bj}$ has physical meaning of light-cone momentum fraction carried by struck quark before it is hit by photon