Bounds on the Strength of Order and Disorder Parameters in Quantum Spin Chains with Finite Abelian Symmetries

REU Final Presentation

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**Motivation: Condensed Matter Physics**

- **An objective of condensed matter**: to study emergent phenomena in quantum many-body systems.
- Examples of emergent phenomena:
  - Superconductors
  - Superfluids
  - Ferromagnets and antiferromagnets
  - Graphene
Motivation: 1D Systems Studied in Condensed Matter

- Serve as playgrounds for studying emergent phenomena
- Examples:
  - Thin wires
  - Optical lattices (top right)
  - Carbon nanotubes (two pictures below)
  - 1D arrays of interacting quantum dots, vortices, or other confined quantum systems
- These serve as inspirations for instructive toy models, such as quantum spin chains (the subject of this talk!)
Spin Chains: a Physical Origin

- Consider some electronic material:

- Can model the system’s dynamics and interactions by the following Hamiltonian:

\[ \hat{H} = \hat{K}_e + \hat{V}_{ee} + \hat{V}_{ei} + \hat{K}_i + \hat{V}_{ii} \]
Spin Chains: a Physical Origin

- Simplifying assumptions to make the Hamiltonian more tractable:
  1. Electron-ion attraction term $V_{ei}$ is a spatially-periodic lattice potential experienced by the valence electrons (see figure). Also, ignore lattice distortions.

  2. This periodic potential $V_{ei}$ is an array of deep quantum wells (each well corresponds to an ion site), so that there is one valence electron localized at each well/ion/lattice site.

- Such assumptions result in a Hamiltonian telling us that the spins of nearest-neighbor electrons either have a tendency to align or anti-align, thereby explaining ferromagnetism and anti-ferromagnetism.
Definition: A spin chain Hilbert space is a Hilbert space that is equal to the tensor product of the finite-dimensional inner-product (or Hilbert) spaces corresponding to each site.

To keep things simple, all the finite-dimensional Hilbert spaces of the lattice sites have the same dimension, which we denote by $d$.

Intuition for tensor product: If each site contains a spin-1/2 particle like the electron, then $d=2$, so if the total number of sites is $L$, the dimension of the whole spin chain Hilbert space is $2^L$, such that a general state in spin chain is a superposition of $2^L$ possible configuration states.
Spin Chains: Ising Model

- Consider the following Hamiltonian, called the **1D Ising Hamiltonian** and defined on a spin chain with *infinitely-many lattice sites*

\[
H = - \sum_i \sigma_i^z \sigma_{i+1}^z
\]

- Right away, we notice that this Hamiltonian only includes spin degrees of freedom:
  - \( \sigma_i^z \) is the operator that corresponds to the measurement of the *z-component of the spin of the electron at the i\(^{th}\) lattice site*.

- In the basis of spin up and spin down states for the z-component of spin, the **Pauli operators are represented by the following matrices**:

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
The ground state of the 1D Ising Hamiltonian is *two-fold degenerate*, i.e. the ground state subspace is two-dimensional and is spanned by two states:

- All spin up in z-orientation
- All spin down in z-orientation

\[ H = - \sum_i \sigma_i^z \sigma_{i+1}^z \]
But suppose we introduce a **transverse magnetic field** in the x-direction. Then we add another term to the Ising Hamiltonian, leading us to the **transverse field Ising model**:

\[
H = - \sum_i \sigma_i^z \sigma_{i+1}^z - B \sum_i \sigma_i^x
\]

|B| is the strength of the magnetic field, and as we increase this strength, the spins go from being spin up or down with a z-orientation to being more and more oriented in the x-direction of the magnetic field.

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Transverse Field Ising Model

- Basis of ground states when \( B = -\infty \)
- Basis of ground states when \( B = 0 \)
- Basis of ground states when \( B = \infty \)
Hence, the two-fold spin-up-and-down degeneracy of the ground state should disappear when $|B|$ is large enough, causing all the spins to be oriented toward the magnetic field.

The loss of ground state degeneracy happens at $|B| = 1$ and corresponds to the transition from the ferromagnetic phase to the paramagnetic phase (as shown in the phase diagram below).
But how can we know where the critical points occur? Magnetization

- **Magnetization**: In electrodynamics, a pseudovector field that represents the density of the magnetic dipole moment, i.e. evaluates **magnetic dipole per unit volume** at every position.

- Spin can be thought of as a measurement proportional to the magnetic dipole moment, so $\sigma_i^z$ also corresponds to measurement of the $z$-component of the magnetic dipole moment at the $i^{\text{th}}$ lattice site.

- Since our system is discrete, can treat “per lattice site” as “per unit volume,” so that the **measurement by** $\sigma_i^z$ **essentially means the measurement of the z-component of magnetization**.

- **Net magnetization**: Average magnetization over the whole system.
Observations about Net Magnetization in Transverse Ising

- One can make two key observations about the ground states $|\Omega\rangle$ of the transverse field Ising model:

  1. For any $i^{th}$ lattice site, $\lim_{j \to \infty} \langle \Omega | \sigma_i^z \sigma_j^z | \Omega \rangle \neq 0$ for $|B| < 1$ and $= 0$ otherwise.

  2. $\lim_{|i-j| \to \infty} \langle \Omega | \prod_{k=i+1}^j \sigma_k^x | \Omega \rangle \neq 0$ for $|B| > 1$ and $= 0$ otherwise.

- #1 informs us that for $|B| < 1$, the ground states have a correlation between a measurements of the $z$-component of magnetization at any $i^{th}$ lattice site and an analogous measurement at a distance infinitely far away.

- This signifies that there must be a net magnetization of the spins in the $z$-direction at least in the case that $|B| < 1$.

- #2 means that for $|B| > 1$, spin flips applied to a ground state at almost every subinterval of the spin chain must result in a state that overlaps with (i.e. not orthogonal to) that ground state.
The Two Observations Combine into a Full Picture

- With spin flips most often not resulting in states orthogonal to $|\Omega\rangle$ for $|B| > 1$, we can deduce that the net magnetization is zero for $|B| > 1$.

- This makes the **ferromagnetic phase**, which is characterized by nonzero net magnetization, be exactly $|B| < 1$.

- In addition, this also makes $|B| > 1$ the **paramagnetic phase**, since we see zero net magnetization and the spins like to orient in the direction of the transverse magnetic field as $|B| \to \infty$. 

![Diagram showing ferromagnetic and paramagnetic phases with critical points at -1, 0, and 1.](image)
Can such analysis be generalized to any spin-chain model?

- We were lucky that in the Ising model, we could analyze net magnetization to identify the collective behavior of spins under different magnetic regimes.

- But magnetization is a very specific kind of observable and it would be nice to be able to identify phases for other more complicated Hamiltonians defined over the spin-chain.

- This provides a motivation for generalizing the measurement procedure via order and disorder parameters.
Spin Chains: Heads-Up

- We will focus our attention on two types of spin chains:
  - **Infinite:**
    - ![Infinite Spin Chain Diagram]
  - **Finite spin chains with L spins arranged in a circle** (thereby periodic boundary conditions):
    - ![Finite Spin Chain Diagram]

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Hamiltonians: Heads-Up

- Our Hamiltonians of interest correspond to nearest neighbor interactions that have a bounded norm:

\[ H = \sum_{i=1}^{L} H_{i,i+1}, \]

such that

\[ \text{supp}(H_{i,i+1}) \subset \{i, i + 1\} \text{ and } \|H_{i,i+1}\| \leq 1. \]

- Assume that Hamiltonian \( H \) is symmetric with respect to some Abelian group \( G \) of finite order \( n \).

- A representation of \( G \) as a direct product of cyclic groups is:

\[ G = \langle S^{(1)} \rangle \times \langle S^{(2)} \rangle \times ... \times \langle S^{(m)} \rangle \]

- Further assume that \( G \) is an onsite symmetry, i.e. can write for any generator:

\[ S^{(j)} = \prod_{i=1}^{L} S^{(j)}_{i}, \]

where \( \text{supp}(S^{(j)}_{i}) = \{i\} \).

- **Generic assumption:** Trivial subspace (i.e. eigenspace of \( G \) with eigenvalue 1) has a non-degenerate lowest energy state.
Definition of Order Parameter

**Definition**: A collection of operators \( \{O_i : i \in X \} \) with \( X \subset 1, 2, ..., L \) is called a \((\delta, \ell, k)\) order parameter for state \( |\psi\rangle \) if and only if

1. \( O_i \) transforms under a nontrivial irreducible representation of \( G \) such that generator \( S^{(k)} \) is not in the kernel of the representation.
2. \( O_i \) is supported on \([i - \ell, i + \ell] \).
3. \( \langle \psi | O_i^\dagger O_j |\psi\rangle \geq \delta \) for all \( i, j \in X \) with \( |i - j| \geq 2\ell \).
4. \( ||O_i|| \leq 1 \).

- **Please note**: This is a generalization of Michael Levin’s definition provided in:
- While Levin’s definition was just for \( \mathbb{Z}_2 \) symmetry, mine is for all finite abelian groups.
Definition of Disorder Parameter

**Definition:** A collection of operators \( \bigcup_{g \in G - \{1\}} \{ O_i^{(g)} : i \in X \} \) with \( X \subset 1, 2, ..., L \) is a called a \( (\delta, \ell) \) disorder parameter for state \( |\psi\rangle \) if and only if

1. \( O_i^{(g)} \) transforms under some irreducible representation of \( G \) (can be trivial or nontrivial).

2. \( O_i^{(g)} \) is supported on \([i - \ell, i + \ell]\).

3. \( |\langle \psi | O_i^{(g)\dagger} O_j^{(g)} \prod_{p=i+1}^j g_p |\psi\rangle \geq \delta \) for all \( i, j \in X \) with \( |i - j| \geq 2\ell \).

4. \( \| O_i^{(g)} \| \leq 1 \).

- **Please note:** This is a generalization of Michael Levin’s definition provided in:  
- While Levin’s definition was just for \( \mathbb{Z}_2 \) symmetry, mine is for all finite abelian groups.
One of the main results by Michael Levin that I generalized to all finite Abelian groups

**Theorem:** Let $|\psi\rangle$ be an eigenstate of the Abelian symmetry group $G$ of order $n$. For any given $\delta \in [0, 1]$ and every pair of disjoint intervals $I_1$ and $I_2$, the state $|\psi\rangle$ is either $\delta/(n - |\langle S^{(i)} \rangle|)$ weakly-ordered on $I_1, I_2$ with respect to generator $S^{(i)}$ or $(1-\delta)/n$ weakly-disordered on the complementary intervals $J_1, J_2$.

- Note: This result is for the finite circular spin-chains.

- Using general results like the one above, I aim to prove some theorems regarding how the strengths $\delta$ of order and disorder parameters are bounded from below for spin chains with any Abelian symmetries.

- We are seeking to prove results that are analogous to those in Michael Levin’s paper: M. Levin, *Constraints on order and disorder parameters in quantum spin chains*, arXiv:1903.09028

- Most important goal: To relate Levin’s constraints on order and disorder parameters to the classification of phases of the spin chain using group cohomologies $H^2(G, U(1))$
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References

Spin Chains: a Physical Origin

- Consider two valence electrons inhabiting neighboring quantum wells. Due to tunneling, their wavefunctions overlap. The overall wavefunction for the two electrons must be anti-symmetric.

- The two electrons electrostatically repel, so it would be more energetically favorable if the spatial part of their overall wavefunction was anti-symmetric.

- Such spatial anti-symmetry enforces a symmetric alignment of the electron spins, so there is an energetic benefit for nearest-neighbor spins to align.
General Topological Picture of Phases

A real vector space formed by Hermitian operators acting on some Hilbert space. For our purposes, the Hilbert space is vector space of all possible states in some quantum system.

A subspace formed by Hermitian operators that can be physically realized as Hamiltonians describing localized interactions.

Disconnected subset consisting of local Hamiltonians that are gapped. Gapped means there is a nonzero difference between energies of the ground state and first excited state and the ground state is nondegenerate.
Suppose we wanted to consider systems with a certain kind of symmetry.

A real vector space formed by Hermitian operators.

Subspace of Hamiltonians that describe localized interactions.

Hamiltonians obeying the symmetry.