Plan for Renormalization Group 2:

1. Recap of some important points from the slides shown yesterday (using 6.2a-2 and 2b, plus 6c exercises)

2. Local projection (visualizing potential changes in coordinate space) and flow to universal potentials (where does unresolved physics go?)

3. Alternative generators — Wegner and block diagonal (\text{Wang})

4. Operators and many-body contributions

5. Perturbativeness — Weinberg eigenvalues

6. Computational aspects
Recall of important points from the slides.

1. Nuclei would be at low resolution based on Fermi momenta in large nuclei.
2. Recall from exercises $p = \frac{2}{3\pi^2} k_f^3$ (for protons & neutrons).
   $\Rightarrow k_f = (3\pi^2 \rho(\rho))^\frac{1}{3}$ (see next page).

3. Density of heavy nuclei about constant $\Rightarrow k_f \approx 1.1 - 1.3 \text{ fm}^{-1}$.
4. So typical relative momentum of $\approx 1 \text{ fm}^{-1}$ ($\approx 200 \text{ MeV}$).
   In a large nucleus, even less in light nuclei.

But if the potential has a repulsive core, then there are strong high-momentum components ("short-range correlations").

$\Rightarrow \text{ slow-down convergence of many-body solution.}
   \text{ e.g., matrices get too big.}

\text{Low pass filter fails even for low energy.}

\text{Why? Because of quantum mechanics,}
T = V + V \sum_{H_0} \frac{1}{E - \omega} V + \cdots.
\Rightarrow \langle k_{1f} | k_f \rangle = \langle k_{1f} | V | k_f \rangle + \int \text{dk}^3 \frac{\langle k_{1f} | V | k_f \rangle \langle k_f | V | k_f \rangle}{(k^2 - k_f^2)/m} + \cdots.
\text{Low momentum}

\text{Solution? Unitary transformation to decouple! Use RG to do it.}
Aside: deriving \( g = \frac{2}{3\pi^2} k_f^3 \)

- For a non-interacting Fermi gas, imagine putting \( N \) particles in a box of side \( L \) \( \Rightarrow \ g = \frac{N}{L^3} \)

  - Let \( \nu \) be the spin-isospin degeneracy:
    \( \nu = 2 \) for neutrons only (spin up, spin down)
    \( \nu = 4 \) for symmetry matter (\( \uparrow \uparrow \), \( \downarrow \downarrow \), \( \uparrow \downarrow \), \( \downarrow \uparrow \))

Apply periodic boundary conditions \( \Rightarrow \) discrete momentum levels:

Then \( N = \sum_{n=1}^{n_{max}} \nu \cdot \frac{1}{n^3} \)

But pbc: \( e^{i(k_f x + k L)} = e^{ik_L} \) in each dimension

\( \Rightarrow k_f L = \frac{2\pi n}{1} \) \( n = 1, 2, 3, \ldots \) are allowed

\( \Rightarrow n = \frac{k_f}{2\pi} \) or \( \Delta N = \frac{1}{2\pi} \Delta k \) in each dimension

\( \nu \) large \( \Rightarrow \sum_{n} \approx \frac{1}{n^3} \times \frac{4}{2\pi} \int d^3 k = \frac{V}{\pi \rho^2} \int d^3 k \)

\( \Rightarrow N = \frac{V}{\pi \rho^2} \int d^3 k \cdot \nu \)

\( \text{volume of sphere in large } V \text{ limit} \)

\( \frac{N}{\nu} = g = \frac{1}{8\pi^2} \cdot \frac{4}{3} \pi k_f^3 \cdot \nu = \frac{V k_f^3}{\pi \rho^2} \)
November 11, 2013

Wednesday exercises review

1. (c) Why would we want to repeat nuclear structure
   calculations for different values of $\Delta$? (as $\Delta$):
      - Observables are supposed to be unchanged
      - Test if a quantity is an observable (example,
        clear demonstration that U-state probability in N-deuteron is not)
      - Determine if scale dependence of a quantity
        - Test for errors
        - Test approximations
          - We will see this particularly in considering many-body
            potentials and other operators.

2. General equation is
   \[ \Delta H_5 = [\Delta H_5, H_5] = [\Delta G_5, H_5, H_5] \]
   - $\Delta T_{red}$ (or $T_{red}$) doesn't change by construction
   - What if we used $T = T_{red} + T_{red}$ for $G_5$ instead of $T_{red}$?
     (answer: $[\Delta G_5, H_5] = 0$, so no difference!)
   - Other choices for $G_5$: $[\Delta G_5, H_5] = 0$, so no difference!

3. $\langle k \mid \frac{d\Delta H_5}{ds} \mid k' \rangle = \frac{\delta N(k,k')}{\delta s} = -(k^2 - k'^2) V_5(k,k') + \frac{2}{\pi} \int_{k^2+k'^2}^{\infty} \frac{dV_5(k,k')}{\sqrt{k'^2+k^2}} \mid k'^2-k^2 \rangle$
   - If $-(k^2-k'^2) V_5(k,k')$ dominates then $V_5(k,k') \approx \frac{1}{2} \int_{k^2}^{\infty} V_5(k,k') dx$
   - Look at slides
Visualizing the softening of NN interactions

- In momentum space we associate softening of a potential with decreased coupling between high and low momenta:

\[ \langle k_{\text{high}} | V | k_{\text{low}} \rangle \rightarrow 0 \]

- But what does this do to our picture of potentials having strong short-range repulsion?

  - Visualizing is not so easy, because the potential becomes non-local, so it is a functional of \( r \) and \( r' \)

  - Note that the \(-\left(k^2 - k'^2\right) V(k, k')\) term in the SRG equation (not partial wave projected) can be written using

\[ R^2 - R'^2 = (E + k) \cdot (E - k') \rightarrow p \cdot q \]

as an explicit function of total momentum \( \vec{p} = (E + k) \) and not just momentum transfer \( \vec{q} = \vec{R} - \vec{R}' \) \( \rightarrow \) non-local

- Plan is use a local projection

  - The high-momentum tails of low-energy wavefunctions are suppressed by RG evolution which implies the wavefunction variation over short distances is small. So in the non-local Schrödinger equation:

\[ -\frac{1}{2m} \nabla^2 \psi(r) + \int d^3r' \frac{1}{|r - r'|} \psi(r') = E \psi(r) \]

\[ \rightarrow -\frac{1}{2m} \nabla^2 \psi(r) - \frac{1}{2} \int d^3r' \frac{\epsilon^2(r')}{|r - r'|} \psi(r') \]
7/11/2013

Define \( V_x(r) = \int d^3p' V_x(^3p', ^3p') \) as the local potential.

Kyle Wendt has developed this idea further, to apply beyond S-waves (which is all that survives the angular integral).

We'll sketch the result for the AV18 potential and look at the actual pictures later.

![Graph showing potential](image)

- S-wave: purely attractive local potential.
- Phase shift must fail to change sign.
- Non-local part at higher momentum.

![Graph showing potential](image)

- Tensor \( \Rightarrow \) highly suppressed.
- D-state probability changes greatly.
- But asymptotic D-S ratio unchanged.
- What about quad-cyclic moment?

Different potentials evolve to same in both momentum rep (at momentum below \( x \) and in local projection).

Where do you expect high energy contributions to go? \( \Rightarrow \)

\[ A_{1S0} \to A_{1S0} + A_{1D0} \]

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\( \Rightarrow \) see slides (same thing here!)
\[ \text{Wigner} \text{ } \text{chose } \text{for } \text{He} \text{-flow equation generator} \]
\[ \text{use } \text{the diagonal of } H_0 \text{ in whatever basis you are in. } \text{E.g., } H_0 = T_{\text{rel}} + V_0(k,k) \text{ diagonal} \]

- Let's consider the general case with \( H_{ij} = e_i \), where we are labeling the basis elements \( |i\rangle, |j\rangle \).
  - Note that these could be plane waves, harmonic oscillators, 2-particle or more,...

\[ \langle i | \frac{dH_0}{ds} | j \rangle = \frac{dH_{ij}}{ds} = \langle i | \left[ H_{ij}, H_0 \right] | j \rangle \]

\[ \text{insert } \sum_k \langle k | \text{ and we } H_0 | j \rangle = e_j \langle j |, \text{ etc.} \]

\[ \Rightarrow \frac{dH_{ij}}{ds} = \sum_k (e_i - e_k - e_k + e_j) H_{ik} H_{kj} \leq \text{simple matrix multiplication} \]

We want to ask: what can we say about \( \frac{d}{ds} \sum_{ij} H_{ij}^2 \) ?
- This is the sum of the squares of the off-diagonal parts. Does it decrease?
  - The full sum is \( \sum_{ij} H_{ij}^2 = \sum_{ij} H_{ij} H_{ij} = Tr H_0^2 = \text{constant} \), the trace is invariant.

\[ \text{So } \frac{d}{ds} \sum_{ij} H_{ij}^2 = -\frac{d}{ds} \sum_{ij} H_{ij}^2 \leq -2 \sum_{i \neq k} e_i (e_i - e_k) \]

\[ \Rightarrow \text{except for degeneracies, off-diagonal elements } \sum_{i \neq k} H_{ik} \leq 0 \]
The use of $T_{\text{rel}}$ instead of $H_{\text{int}} = T_{\text{rel}} + H_{\text{coul}}$ is ok for nuclear physics, at least in the momentum basis, because

$$T_{\text{rel}} \approx (M^2)_ij \Rightarrow H_{\text{int}} \approx T_{\text{rel}}$$

It can fail, though. See Wendt et al., with large $\Lambda$ leading order forces,

$\Rightarrow$ good example of decoupling.
\\textbf{Th2a.7} \\

- $V_{low-k}$ RG equation - Bogner, Kuo, Schwemle (2001) \\

- based on requiring it half-off shell $T$ matrix to be invariant with a change in cutoff on the sum over intermediate states. \\

\[
\begin{align*}
T(k', k, k^2) &= V_{\text{low}}(k, k') + \frac{3}{\pi} \int_0^{\Lambda} V_{\text{low}}(k, p) T(p, k^2) \frac{p^2 dp}{k^2 - p^2} \\
&\quad \text{half-on-shell because } k, k^2 \text{ but } p \ll k
\end{align*}
\]

For all $k' < \Lambda$, with principal value in rad,
\[
\text{cutoff} = \frac{V_{\text{low}}(k', k)}{\frac{3}{\pi} \int_0^{\Lambda} \frac{V_{\text{low}}(k, p) T(p, k^2) dp}{k^2 - p^2}}
\]

Take $\frac{dr}{dx} = 0 \Rightarrow \frac{dV_{\text{low}}(k', k)}{dx} = \frac{3}{\pi} \int_0^{\Lambda} \frac{V_{\text{low}}(k', p) T(p, k^2) dp}{k^2 - p^2}
\text{ derivation is not immediate, (see Bogner et al.)}
\]

\(\aleph\), partial wave SRG equation (with $G_5 = T_{hi}$)
\[
\frac{d}{\Delta x} V_{\Delta}(k, k') = \left( -\frac{4}{\Lambda^2} \right) (k^2 - k'^2)^2 V_{\Delta}(k, k') + \frac{3}{\pi} \int_0^{\Lambda} (k^2 + k'^2 - 2kq) \frac{dV_{\Delta}(k, q)}{dq} d\phi
\]

\[
\frac{d}{\Delta x} V_{\Delta}(k, k') = \left( -\frac{4}{\Lambda^2} \right) (k^2 - k'^2)^2 V_{\Delta}(k, k') + \frac{3}{\pi} \int_0^{\Lambda} (k^2 + k'^2 - 2kq) \frac{dV_{\Delta}(k, q)}{dq} d\phi
\]

\[
\text{Compare LHS: } T \text{ matrix for } V_{\text{low}} \text{ but just potential for SRG}
\]

\(\Rightarrow\) SRG much easier for $A > 2$ (otherwise need $T$ matrix in all channels).
7/11/2013

"Can we get a Viola-like potential from the SAC flow equation by an appropriate choice of $G_s$? Yes!"

Use $\frac{dt}{ds} = [G_s, H_s]$, with $G_s$ of form $\begin{pmatrix} \frac{\partial H_P}{\partial H_s} & 0 \\ 0 & \frac{\partial H_s}{\partial H_P} \end{pmatrix}$.

Choose $\Lambda$ and then $G_s$ is the running Hamiltonian with the off-diagonal blocks defined by $\Lambda$ set equal to 0 so $\Lambda$ and $G$ are projection operators, $P + \Omega = I$.

Proof that this does what we want [Gubankova et al.]

A measure of off-diagonal coupling is $\Omega H_s P$

so this is the part that does the coupling. Note

$\Rightarrow \exists \frac{n}{n} \langle \frac{n}{n} (\Omega H_s P)^{+} (\Omega H_s P) \rangle = \text{Tr} [P H_s Q H_s P] > 0$ (since $\Omega^2 = \Omega$, $P^2 = P$)

$\Rightarrow \text{Tr} [P H_s Q H_s P] > 0$

Now check how this changes with $s$ using $\frac{dt}{ds} H_s = [H_s, H_s]$

$\Rightarrow \frac{dt}{ds} \text{Tr} [P H_s Q H_s P] = \text{Tr} [P Q S (Q H_s Q H_s P - Q H_s H_s P)]$

$\Rightarrow \text{Tr} [P H_s Q H_s Q H_s (Q H_s P) P]$

you are invited to prove it.

$\Rightarrow -2 \text{Tr} [(Q H_s P)^{+} (Q H_s P)] \leq 0$

$\Rightarrow$ the off-diagonal $Q H_s P$ block will decrease (or not increase) as $s$ increases.

Two examples: $G_s = \text{Tr} \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \Rightarrow$ goes to $\left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$

$G_s = \left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \Rightarrow$ goes to $\left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$

Does it always evolve to the pattern of $G_s$? See pictures!
21/11/2003

SR6 Flow of Operators

- We'll have more to say about operators in a future lecture, just some basic ideas here.

- When we transform \( H_0 = U(s) H U_s^+ \), the wave functions also get transformed: \( \left| \psi_n^{s0} \right> = U(s) \left| \psi_n \right> \), so that energies are unchanged: \( E_n = \left< \psi_n^{s0} | H_0 | \psi_n^{s0} \right> = \left< \psi_n | H | \psi_n \right> \) .

- So this means that any operator \( O \) must be transformed:

\[
O_s = U(s) O U_s^+ 
\]

- We can calculate this directly by constructing \( U(s) \) and

- What we do this by first evolving \( H \to H_0 \), then finding all the eigenstates \( \left| \psi_n \right> \) of \( H \) and \( \left| \psi_n^{s0} \right> \) of \( H_0 \).

- Then we have: \( U(s)_n = \sum \left< \psi_n^{s0} | \psi_n \right> \left| \psi_n \right> \).

- In a basis like momentum space, this would give us the matrix element: \( \left< k' | U(s) | k \right> = \sum \left< \psi_n^{s0} | \psi_n \right> \left< \psi_n \right| \psi_n^{s0} \rangle \left| k' \right> \).

- just an outer product.

- This works fine in practice but there are two other ways:

  i) Evolve \( O_s \) with its own flow equation

  ii) Evolve \( U(s) \) " " " " " " " " and then use \( O_s = U(s) O U_s^+ \)

What are the equations? \( \Rightarrow \) you do that for exercises!
How do we know that SRG evolution of operators (including its Hamiltonian) must generate many-body terms?

- In exercises: Play about physics, here: formal discussion.

Consider 2nd quantization. This is defined with two ingredients:

1. A single-particle basis (e.g., plane waves in a box or HO wells).
2. A reference state that serves as the "vacuum".

Examples:
- Could be the actual vacuum
- Or a filled core (Fermi sea or a closed shell)

Kinetic energy: \( \hat{T} = \sum_i \frac{p_i^2}{2m} a_i^\dagger a_i \)

Two-body potential: \( \hat{V} = \sum_{ijkl} \delta_{ijkl} a_i^\dagger a_j^\dagger a_k a_l \)

3-body potential: \( \frac{1}{3!} \sum_{ijklm} V_{ijklm} a_i^\dagger a_j^\dagger a_k^\dagger a_l a_m a_n \)

These operators have anti-commutation relations:

\[ \{ a_i, a_j^\dagger \} = a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}, \quad \{ a_i, a_j \} = \{ a_i^\dagger, a_j^\dagger \} = 0 \]

Claim:
\[
\frac{d\hat{\rho}}{ds} = \sum_{g=1}^{-\infty} \left[ \frac{\hat{L}}{g} \right]_{g=1}^{\infty} + \sum_{g=1}^{-\infty} \left[ \frac{\hat{L}}{g} \right]_{g=1}^{\infty} + \sum_{g=1}^{-\infty} \left[ \frac{\hat{L}}{g} \right]_{g=1}^{\infty}
\]

And this is just one time step.

- A 2-body operators generated
- Is this a problem?

If yes, we need to be able to truncate \( \Rightarrow \) need hierarchy

Also need to be able to calculate with minimal (usually 3-body)

SRG \( \Rightarrow \) Alternative: Pick a different reference state \( \Rightarrow \) reshuffles what is many-body!
11/1/2013

SRG technology is to evolve 3-body forces:

Three methods not exist:

1. evolve in a discrete harmonic oscillator basis
   (Eric Tjonnessen)

   \[ \Rightarrow \text{applied to No-Core Shell Model (tomorrow)} \]

2. evolve in a partial-wave momentum basis
   (Kai Hebeler)

   \[ \Rightarrow \text{separate evolution of 2 and 3 body parts} \]

3. evolve in a hyperspherical basis

   \[ \Rightarrow \text{good features, visualization} \]

- more later on finite comparisons
- Recent: 4-body evolution (see Angelo Colci talk from Trento)

Oscillator evolution is in a 3-body Jacobi basis:

- generalization of center-of-mass and relative

\[ \mathcal{L} = \frac{1}{2} \left( \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 \right) \]

- potential doesn't depend

\[ \mathbf{p}_1 - \mathbf{p}_2 \]

- relative between 1 and 2:

\[ \mathbf{r}_3 = \sqrt{3} \left( \mathbf{r}_1 + \mathbf{r}_2 \right) \]

- relative between 3 and com

\[ \text{com of } 4 \text{ and } 5 \]

- hard part:

\[ \text{must anti-symmetrize} \]

- H0 basis:

\[ \left| \xi \right> = \left| \mathcal{N}_{a} - \mathcal{S}_{a} \right> \left| J_2 \right> \left| T \right> \]

\[ \left| J_2 \right> \left| T \right> \]

- momentum space evolution

\[ \left| p q \gamma \right| \Rightarrow \left| p q \gamma \right| \]

\[ \left| p q \right| \]

\[ \left| \left( L S \right) \right| \left( T_i \right) \left( T_f \right) \]