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**Overview**: We've introduced many threads in the first few days. All have connections — underlying themes — or else are foundations for upcoming topics.

*It will become more evident as we proceed.*

*So please be patient even if it seems incoherent at times.*

*We need to develop in parallel so we can discuss impact.*

**In this lecture:**

- Follow up on three-body forces and aspects of the EFT example of pointless EFT. => tie these together
- Continue with implications of the more complex features of the NN force (like tensor, spin-orbit) and in the simplest bound system: He deuteron.
Continuation of 3-body force introduction...

- We discussed the examples of the Earth-Moon-Sun system and the interaction of neutral atoms as places where the low-energy theory has three-body forces.

- The general feature is that three-body forces arise from the elimination of degrees of freedom if we included positions of individual masses or interactions between all electrons in two-body only, eliminating these variables (degrees of freedom) in favor of collective coordinates (center-of-mass position) required three-body forces.

- So what about the nuclear case?

  In this diagram two nucleons exchange a boson, maybe a pion, maybe a heavier meson, exciting. 

  \[ N N \rightarrow N^* \] (excited state of nucleon) for a brief period, \[ N^* \] could mean a \( \Delta \), could mean something else.

  Suppose our theory had both \( N \)'s and \( N^* \)'s explicitly and was but no \( p, g, \omega \) (treated as heavy) and \( \rho, \sigma, \omega \) expanded as contact plus derivatives of contacts.

  \[ N \rightarrow N + X \] would be diagrams included in the low-energy theory, but no 3-body, because \( \pi \), etc.

  In two successive 2-body interactions.

  \[ \text{But now we eliminate } N^* \rightarrow \text{This is a 3-body force} \]

  \[ 3 \text{-body if can't be broken into successive two-body interactions} \]
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- When is it good to replace the N* excitation? When we don't resolve that it was excited.

  By the uncertainty principle, if we excite it, a virtual state, it can last for $\Delta t = \hbar/\Delta E$, which is short if $\Delta E$ is large. $\Rightarrow$ Broad endpoint are close enough so they are not resolved $\Rightarrow$ replace by contact + and derivatives.

  So this is a danger if $M_\Delta - M_N \approx 300$ MeV, then it will break down much sooner than for energy differences $\Delta 5000$ MeV (such as heavier meson exchanges).

  We will keep coming back to this.

  Expansion parameter $Q/(M_N M_\Delta)$ may be smaller than we want!

- How about a process like:

  \[ \begin{array}{c}
  \text{time} \\
  \uparrow \\
  \text{time} \\
  \downarrow \\
  \hline \end{array} \]

  So the idea is that a nucleon emits a plan that becomes a nucleon-anti-nucleon pair at A. The anti-nucleon annihilates with a 3rd nucleon at B emitting a pion absorbed by a 3rd nucleon.

Class:

- In the previous case, we had $\Delta E \geq M_\Delta - M_N$. What is it here?

  Initially $3M_N + \text{kinetic}$, in the middle an extra $3M_N$
  $\Rightarrow \Delta E \geq 3M_N$, which is large $\Rightarrow \Delta t$ is small

  \[ \begin{array}{c}
  \text{time} \\
  \uparrow \\
  \text{time} \\
  \downarrow \\
  \hline \end{array} \]

  is a 3-body force that is a good approximation.

Important:

- $\Gamma$ or $\Sigma$ may be good or bad or incomplete models — maybe it requires quarks and gluons to describe. As long as

  \[ \begin{array}{c}
  \text{time} \\
  \uparrow \\
  \text{time} \\
  \downarrow \\
  \hline \end{array} \]

  contains all allowed (by symmetries) vertices, then we don't care — we will be model independent with our EFT!
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- Moral: whether we have a 3-body force or not and how large a contribution depends on our choice of degrees of freedom.

- But this includes when our cut-off eliminates nucleons from our low-energy theory.

\[
\begin{align*}
\text{Nucleons with momentum } &\quad p < \Lambda_1 \\
\text{Nucleons with } p < \Lambda_2 \\
\text{Nucleons with } \Lambda_1 < p < \Lambda_2 \\
\text{Plus longer range effects of nucleons with } \Lambda_1 < p < \Lambda_2 \\
\end{align*}
\]

\[\Rightarrow \text{even with just nucleons, two-body interactions become 3-body if we eliminate degrees of freedom (in this case by lowering } \Lambda)\]

- This is the same principles we had with } C_0(\Lambda) \text{ in the pinless theory:}

\[
\begin{align*}
\text{C_0 cutoff} &\quad \Lambda = \Delta \Lambda \text{ cutoff} \\
\text{C_0} &\quad \Lambda \leq \Delta \Lambda \\
\text{C_0} &\quad \Lambda \leq \Delta \Lambda \\
\end{align*}
\]

\[
\Rightarrow C_0(n) \frac{m}{2\pi} \int \frac{d^3 p}{p^2 + i\epsilon} \rightarrow \int_0^{\Delta \Lambda} d\Lambda \int_0^{\Delta \Lambda} d\Lambda \frac{C_0(n) m}{2\pi} \int \frac{d^3 p}{p^2 + i\epsilon}
\]

\[\text{Small}
\]

\[
\Rightarrow \text{The } x^x \text{ contribution for } \Lambda_c - \Lambda_c < q < \Lambda_c \text{ looks like a constant:}
\]

\[
\Rightarrow \text{ Change } C_0 \text{ to compensate } x \Delta C_0 = C_0(\Delta) \frac{m}{2\pi} (1 + \Lambda_n) \text{ or } \Delta C_0 = m^2 \frac{2\pi}{2\pi} (C_0(\Delta))
\]

\[\text{which is the RG equation from Achim's lecture! (sign does work!)}
\]

\[\text{(sign does work if } \Delta \Lambda < 0, \text{ then } C_0 \text{ decreases)}\]
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- How do we estimate truncation errors?
  - c.f. Legendre plots: the need to know how big the coefficients are.
  - For the natural (flow-breakdown) pioneerless theory \((\Lambda \approx M_{\Pi})\)
    "breakdown scale"

\[
\chi = \chi_0 + \chi_0^2 + \chi_0^3 + \chi_0^4 + \chi_0^5 + \chi_0^6
\]

\[
iT(k, \cos \theta) = -ic_0 - \frac{m}{\hbar \pi} C_0 k^2 + \frac{m}{\hbar \pi} C_0 k^2 + \frac{iZ}{\hbar \pi} C_0 k^2 - \frac{iZ}{\hbar \pi} (R \cos \theta)
\]

reproduces:

\[
\frac{-4\pi a_0}{m} \left[ 1 - i\alpha k + \left( \frac{a_0^2}{2} - \frac{a_0^3}{3!} \right) k^2 - \frac{4\pi a_0}{m} R \cos \theta + \frac{a_0^3}{3!} \right]
\]

\[
C_0 = \frac{4\pi}{m} a_0, \quad C_0 = \frac{4\pi}{m} a_0, \quad C_0 = \frac{4\pi}{m} a_0, \quad C_0 = \frac{4\pi}{m} a_0
\]

*Power counting*: diagrams contribute \(4^P\) where \(\nu = 5 - \frac{3}{2} E + \sum \sum (2i + 3n - 5) V_{ij}\)

where \(i = \# \text{ of derivatives}\)

- \(n = 2 \text{ for 2-body term, } 3 \text{ for 3-body term, ...}\)
- \(E = \# \text{ external lines}\)

\[
\left\{ \begin{array}{l}
\text{every extra } \text{ 3-body part!}
\end{array} \right.
\]

\[
C_{2i} \sim \frac{4\pi}{m} \frac{1}{(\Lambda)^{2i+1}}, \quad D_{2i} \sim \frac{4\pi}{m} \frac{1}{(\Lambda)^{2i+4}}
\]

\[
\Rightarrow \text{ estimates of how big coefficients are}
\]

\[
\Rightarrow \text{ just dimensional analysis with } \Lambda \text{ as momentum or length}
\]

**"naive"** \(\Rightarrow\) only \(\Lambda\) and natural: remaining coefficient is close to 1.

So even if \(\nu\) didn’t determine \(C_0\) and \(\Lambda\), we can estimate the contribution as \(\frac{4\pi}{m} \frac{1}{(\Lambda)^P}\) (since \(2i = 2 \text{ derivatives}\)).

\[
\Rightarrow \text{ truncation error } + \text{ we know that it will fail for } k \sim \Lambda
\]
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- In our pionless example we didn't mention spin, because we said the interaction was spin independent, but this doesn't mean spin doesn't play a role.
  - Because the wave function must be antisymmetric, if the spin part is symmetric \( |s_{1/2}, s_{1/2} \rangle \) or \( |s_{1/2}, -s_{1/2} \rangle \), then the wave function of two neutrons (isospin space symmetric \( T=1 \)) would vanish for separation \( \mathbf{R} = 0 \) (antisymmetry)
  - So the matrix element of \( G_{\delta} \) vanishes: \( \langle \mathbf{r}_1 / \mathbf{r}_2 | \delta_{i j} | \mathbf{r}_1 / \mathbf{r}_2 \rangle = 0 \)
  - So we really need to keep track of the spins as well on the legs of \( X \)

- What about including spin at leading order in the pionless EFT?
  - Consider neutrons only. A general term consistent with symmetries is
    \[
    V_{L0} = C_5 + C_7 \sigma_1 \cdot \sigma_2
    \]
    so we might expect two \( S \)-wave scattering lengths.
  - In the lagrangian \( L_{L0} = \cdots \frac{1}{2} g_{(1/2)}^\alpha (\mathbf{r} \cdot \mathbf{J}) \frac{1}{2} g_{(1/2)}^{\alpha \beta} (\mathbf{r} \cdot \mathbf{J}) \)
    \[
    = -\frac{1}{2} g_{(1/2)}^\alpha (\mathbf{r} \cdot \mathbf{J}) \frac{1}{2} g_{(1/2)}^{\alpha \beta} (\mathbf{r} \cdot \mathbf{J}) = \frac{1}{2} (\mathbf{r} \cdot \mathbf{J}) \mathbf{S} \cdot \mathbf{S}
    \]
  - If you did the Fierz rearrangement exercise, you would have found that these two terms are not independent
  - Thus, it's only one combination, again because of antisymmetry and contact interactions.

- In Achim's W1b notes, there is a nice alternative way to show this, which I'll repeat here.
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We can include antisymmetry in the potential by including the exchange term (with the appropriate minus sign).

\[ V_{\text{antisym}} = (1 - \hat{P}_{12}) V \]

where \( \hat{P}_{12} \) is the exchange operator \( \hat{P}_{12} = \hat{P}_{21} = \hat{P}_{\text{spin}} \).

And \( \hat{P}_{\text{spin}} = \frac{1 + \vec{s}_1 \cdot \vec{s}_2}{2} \) (if you've never seen this, act with \( \hat{P}_{\text{spin}} \) on spin up's to verify).

\[ V_{\text{antisym}} = (1 - \hat{P}_{\text{spin}})(C_s + C_T \hat{\sigma}_4 \hat{\sigma}_5) \quad \text{(no \( R, \vec{r}_1' \))} \]

\[
\begin{align*}
\text{use } (\hat{\sigma}_4 \hat{\sigma}_5) = & \frac{1}{2} \left[ (C_{-3} - 3C_T)(\hat{\sigma}_4 \hat{\sigma}_5) \right] = \begin{cases} 0 & s = 1 \\
2(C_{-3} - 3C_T) & s = 0 \end{cases} \\
\text{as before from Eqs.} \\
\text{see Alex tomorrow!}
\end{align*}
\]

So any choices of \( C_s, C_T \) for which \( C_{-3} - 3C_T \) is the same will give the same result \( \Rightarrow \) only one independent constant.

Your choice, e.g., \( C_T = 0 \). So what we had with \( C_T = C_s \) was actually general at 10 for neutrons only. Also, see scattering length.

Now with spin and isospin: \( \{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \} \), \( \{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \} \), \( \{ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \} \)

Anti-symmetry says only 2 independent. Choose which: (Fierz ambiguity)

Conventional choice: \( V_{NN}^{14} = C_s + C_T \hat{\sigma}_4 \hat{\sigma}_5 \) See Alex tomorrow!

At \( N = 0 \), 14 possible, but only 7 linearly independent. Usual choice:

\[ V_{NN}^{14} = C_2 \frac{1}{2} (\hat{r} \cdot \hat{k}) + C_4 \hat{r} \cdot \hat{k} + C_5 \frac{1}{2} (\hat{r} \cdot \hat{k}) \hat{\sigma}_4 \hat{\sigma}_5 + C_6 \hat{r} \cdot \hat{k} \hat{\sigma}_4 \hat{\sigma}_5 \]

\[ + i C_7 \frac{1}{2} (\hat{\sigma}_4 + \hat{\sigma}_5) \cdot (\hat{r} \times \hat{k}) \Rightarrow \text{spin-orbit interaction (Exercise)} \]

\[ + C_8 \hat{\sigma}_4 \cdot (\hat{r} \times \hat{k}) \hat{\sigma}_4 \hat{\sigma}_5 \hat{r} \hat{k} \Rightarrow \text{lead to tensor interaction} \]
Where does the tensor interaction in pionless EFT come from?

- One source is the pion, and the pion tensor interaction has important effects on nuclear structure.

- Let's do a quick derivation of the one-pion exchange potential starting from the interacting Hamiltonian density

\[ \hat{H}_{\text{int}} = \frac{g_A}{2} \mathbf{N}^T \mathbf{\sigma} \cdot (\mathbf{\nabla} \phi) \mathbf{N} \] (lots of hidden indices!)

To a quantized form

\[ \hat{H}_{\text{int}} = -i \frac{g_A}{2\pi} \int \frac{d^4k}{(2\pi)^4} \mathbf{b}^\dagger (k,m_\pi^2) \mathbf{b} (k,m_\pi^2) \hat{\mathbf{\sigma}} \cdot \hat{q} (r) \]

\[ \mathbf{b} (k,m_\pi^2) = \sum_{i=1,2,3} \mathbf{b}_i (k,m_\pi^2) \]

\[ \mathbf{b}_i (k,m_\pi^2) = \mathbf{B}_{i}^\dagger (k,m_\pi^2) \mathbf{B}_{i} (k,m_\pi^2) \]

\[ \mathbf{B}_{i} (k,m_\pi^2) = \mathbf{N}_{m_\pi,n_i} \]

\[ \mathbf{N}_{m_\pi,n_i} = \text{create pion in isospin state } i \]

\[ \mathbf{b}^\dagger (k,m_\pi^2) = \text{pion absorbed} \]

\[ \hat{\mathbf{\sigma}} \cdot \hat{q} (r) \]

Time-ordered perturbation theory (2nd order)

\[ \langle \mathbf{R} | W_{\text{eff}}^{(1)} | \mathbf{F}_k \rangle = \sum_{n_1} \langle \mathbf{R} | \mathbf{H}_{\text{int}} | n_1 \rangle \langle n_1 | \mathbf{H}_{\text{int}} | \mathbf{F}_k \rangle \]

\[ \langle \mathbf{R} | \mathbf{H}_{\text{int}} | n_1 \rangle = -\frac{1}{w_q} (\frac{g_A}{2\pi}) \partial_q \cdot \partial_q (\frac{-\hat{q} \cdot Q}{w_q}) \partial_q \cdot \partial_q \]

\[ \langle n_1 | \mathbf{H}_{\text{int}} | \mathbf{F}_k \rangle = \text{sum over } \hat{q} \cdot Q (\frac{w_q}{\partial_q \cdot \partial_q}) \partial_q \cdot \partial_q \]
Putting it together, this is a local potential (no particle exchange at long distance):

\[ V_{\omega\pi E}(Q, k) = V_{\omega\pi E}(q = k - Q) \]

\[ = -\frac{g^2}{(2\pi Z)^2} \frac{\sigma_+ \cdot q \sigma_+ \cdot q_+}{2\pi^2} \frac{q_2}{m^2} \]

\[ = -\frac{g^2}{4\pi^2} \frac{\sigma_+ \cdot q \sigma_+ \cdot q_+}{q^2 + m^2} \]

In the exercises for today, you carry out the Fourier transform showing:

\[ \int \frac{d^3q}{(2\pi)^3} \frac{1}{2m^2} e^{iq \cdot r} = \frac{1}{4\pi r} \]

(standard, but remind yourself) and then evaluating the derivatives in \( \sigma \cdot q \). The bottom line is:

\[ V_{\omega\pi E}(r) = \frac{m^2}{12\pi} \left( \frac{1}{(2\pi)^3} \right) \left[ 3T(r)S_{\omega\pi}(r) + Y(r)\sigma_+ \cdot \sigma_2 \right] \]

where:

\[ T(r) = \frac{e^{-m_\pi r}}{m_\pi r} \left( 1 + \frac{2}{3} \frac{m_\pi r}{m_\pi r} + \frac{3}{(m_\pi r)^2} \right) \]

\[ Y(r) = \frac{e^{-m_\pi r}}{m_\pi r} \]

and

\[ S_{\omega\pi}(r) = \left[ (\sigma_+ \cdot r)(\sigma_+ \cdot r) - \frac{1}{2} \sigma_+ \cdot \sigma_2 \right]. \]
Let's talk a bit about the impact of the tensor force on NN bound states.

- **nn** and **pp** have no bound states
- **np** has one shallow bound state (large scattering length)
  \[ \Rightarrow \text{deuteron } T=0 \]
- What's special about \( T=0 \)?

**Deuteron properties:** (measured!)
- binding energy \(-2.3245\pm0.019\) MeV small!
  \[ \Rightarrow \text{rms radius } R_{\text{rms}}^2 = 1.976(21) \text{ fm} \text{! large! (large tail)} \]
- \( J^P=1^+ \) (angular momentum 1, parity +)
- isospin \( T=0, M_T=0 \) (np)
- electric quadrupole moment \( Q_d = 0.085(3) e \text{ fm}^2 \geq 0 \)

- Two spins \( \frac{1}{2} \) nuclei \( \Rightarrow S=0,1 \) \( J-1<\ell<\ell+1 \Rightarrow \ell=0,1,2 \)
  - parity + \( \Rightarrow \ell=0,2 \) \( \Rightarrow \) space symmetric so \( S=1 \)
  - So \( ^3S_1, ^3D_1 \) possible
- Expect \( ^3S_1 \) energetically but \( Q_d \neq 0 \Rightarrow l=2 \) admixture
  \[ \Rightarrow \text{tensor force mixes } ^3S_1-^3D_1, \text{ (recall Tab)} \]
  - Attractive tensor \( \Rightarrow \) extra binding (ft. nn)

**Is attractive tensor in \( T=0 \) consistent with \( Q_d > 0 \)?**
- Like magnetic dipole-dipole
  \[ S = e(3\hat{r} \cdot \hat{r}) e^0(3\hat{r} \cdot \hat{r} - 1) \Rightarrow \text{QD} \Rightarrow \langle \phi^2 > > \langle \phi^0 > \]
  \[ \Rightarrow \text{prolate} \Rightarrow Q_d > 0 \]
  - at attractive repulsive
- Unlike spins \( (m_S=\pm) \) prefer to be oriented head-to-tail

- Wave functions: \( S \)-wave \( n(0) \), \( D \)-wave \( |0\rangle \) ports
  \[ n(0) \rightarrow A \text{e}^{-\gamma r} \]
  \[ \text{binding momentum } x = -2 E_1 \]
  \[ \Rightarrow \text{Asymptotic normalization} \{\text{measured}\} \]