TALENT/INT Course on Nuclear Forces
Exercises and Discussion Questions W1

[Last revised on July 3, 2013 at 11:06:35.]

Wednesday 1: Nuclear forces 2; Renormalization and universality

We have grouped all of the two-minute and discussion questions at the beginning, as usual. However, for today you should only spend about an hour working on questions and then try some of the other problems as well. When you need a break, go back and try another question! You might also go back to the T1 problems and do the hydrogen atom as a warm-up for EFT.

1. Two-minute and discussion questions on pionless EFT:

   (a) What gives rise to scale and scheme dependence of nuclear forces? (Of course to answer you must be clear what scale and scheme dependence means!)

   (b) Discuss how the C_0 contact interaction regulated with a sharp cutoff looks in coordinate space. Start from the limit of infinite cutoff. What happens as high-momentum modes are cut off from the Fourier transformation?

   (c) If we don't include an effective range in $T_0(k)$ (i.e., we set it to zero), why is there an "induced" effective range from the regulator (i.e., from putting in a cutoff on our interaction)?

   (d) Compare the running of C_0 from the RG equation with the QCD running coupling. What are similarities and what are differences?

   (e) Why can we use a Hamiltonian that has the high-energy (UV) physics completely wrong and still correctly predict the low-energy observables?

   (f) At which order do tensor forces enter in pionless EFT? Use this to estimate the contribution of the ${}^3D_1$ channel to the deuteron. For this estimate, take as momentum scale the deuteron binding momentum $k_B = \sqrt{m_N B_d}$ and for the breakdown scale of pionless EFT the pion mass m_π.

   (g) Discuss what happens to the C_0 contact interaction across a Feshbach resonance.

2. Two-minute and discussion questions on three-body forces:

   (a) What is the definition of a three-body force?

   (b) Based on the discussion of the origin of three-body forces and the analogies, what would you say about the possibility of four-body forces? Even higher-body forces?

   (c) What evidence is there that different 3NF's are needed for different NN interactions? Can you point to a figure shown in lecture that supports your answer?

(d) Check that you understand how to apply the power-counting formula for ν to the various diagrams considered in class (see the lecture notes). What value of ν would a tree-level (no internal integrations) three-body interaction have?

(e) How can you tell experimentally if there are three-body forces? Why is this not really a well-posed question until additional information is supplied?

(f) Can you have a three-body contact term among neutrons only?

3. Let's think about coordinate space with *unregulated* interactions.

   (a) What is the coordinate-space potential corresponding to $\langle \mathbf{k}|V_{\text{eff}}|\mathbf{k}'\rangle = C_0$?

   (b) What happens if you try to solve the coordinate-space Schrödinger equation for this potential?

   (c) Repeat with $C_2(k^2 + k'^2)$. Is the problem better or worse?

   (d) How can you change the potential to fix the "problem"?

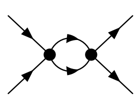
4. In the lecture, the leading-order coupling $C_0(\Lambda)$ was derived using sharp cutoffs on momenta $\theta(\Lambda - k)$. Here we want to derive the leading-order coupling $C_0(\Lambda)$ when Gaussian regulators $f(k/\Lambda) = \exp(-k^2/\Lambda^2)$ are used instead of sharp cutoffs. In this case the leading-order EFT potential is given by $V_{\text{EFT}}(k', k) = C_0(\Lambda) f(k'/\Lambda) f(k/\Lambda)$.

   (a) What would the integral for $I_0(k, \Lambda)$ look like now (with the q integral from 0 to ∞)?

   (b) Evaluate the integral and compare to the sharp-cutoff regulated result. How does this change $C_0(\Lambda)$ compared to the result obtained for a sharp cutoff.

   (c) Is the difference in $C_0(\Lambda)$ a scale or scheme dependence?

5. When we considered a second-order loop diagram with C_0 at each of the two vertices, we found a linear divergence:



$$\Longrightarrow \int^{\Lambda_c} \frac{d^3q}{(2\pi)^3} \frac{1}{k^2 - q^2 + i\epsilon} \longrightarrow -\frac{ik}{4\pi} + \frac{\Lambda_c}{2\pi^2} + \mathcal{O}(\frac{k^2}{\Lambda_c}) \,.$$

   Now if we consider the same diagram but with one C_0 vertex and one C_2 vertex (instead of two C_0 vertices), what type of divergence would it have?

6. Illustration of Fierz transformations and NN contact interactions. Consider spin-1/2 particles with only one "flavor" (e.g., interacting neutrons). Then allowed 2-body contact interactions (four-fermion operators) are of the form $(N^\dagger \Gamma N)(N^\dagger \Gamma N)$, where the N's and $N^\dagger$'s' are (anti-commuting) two-component spinor field operators and the Γs are 2×2 matrices that are appropriately contracted so that the combination is a scalar.

   (a) Argue why allowing Γ to run over the identity and the Pauli matrices forms a complete set. Therefore we have $(N^\dagger N)(N^\dagger N)$ and $(N^\dagger \boldsymbol{\sigma}_1^a N)(N^\dagger \boldsymbol{\sigma}_2^a N)$. Add the indices to the Γ's and to the N and $N^\dagger$'s and indicate their range (i.e., i runs from 1 to ???).

(b) This result would seem to imply there are two independent low-energy constants to find, one to multiply each of the two terms. However, these terms are not actually independent. Here we'll show how to relate them using Fierz relations. The idea is to change the pattern of contraction by noting that $N_i N_j^\dagger = c_0 \delta_{ij} + \boldsymbol{c} \cdot \boldsymbol{\sigma}_{ij}$, because the left side is a two-by-two matrix and the right side has a complete set with c_0 and $\boldsymbol{c}$ to be determined. Find c_0 and $\boldsymbol{c}$ by multiplying the equation first by the identity and then by $\boldsymbol{\sigma}$ and in each case taking the trace. The only tricky part is that the Ns and $N^\dagger$s *anti-commute*, so you pick up minus signs when you do this trace.

(c) Now write $(N^\dagger N)^2 = N_i^\dagger N_i N_j^\dagger N_j$ and apply the result of the last part to the $N_i N_j^\dagger$ in the middle. This will give you a relation between the two contact terms. Start again with $(N^\dagger \boldsymbol{\sigma} N)^2$ and verify that you get the same result.

(d) Generalize to include isospin as well. That is, allow $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$. Instead of four possible terms, you should show there are only two independent ones.

7. The most general spinless, S-wave zero-range two-body interaction at order k^2 that is consistent with the symmetries discussed in lecture is $V(k, k') = C_2(k^2 + k'^2)$. Now consider the interaction at order k^4.

(a) Which of the following terms: $C_4(k^2 + k'^2)^2/4$, $C_4'(k^4 + k'^4)$, $\widetilde{C}_4(k^2 - k'^2)^2$, $\widetilde{C}_4'(k^2 k'^2)$, $\overline{C}_4(k^4 - k'^4)$ are allowed?

(b) How many of these candidate terms are independent?

(c) Consider $\widetilde{C}_4(k^2 - k'^2)^2$ in 2–2 scattering at tree level (Born approximation or no loops). Evaluate this diagram. [Hint: you shouldn't need to worry about getting the sign correct.] Why is this called an "off-shell" vertex?

(d) Now consider a tree-level 3–3 scattering diagram with one $\widetilde{C}_4$ vertex and one C_0 vertex (include diagram). Show that the internal propagator (or the energy denominator) cancels with the $\widetilde{C}_4$ vertex, leaving a single three-body vertex. Therefore argue that it is possible to trade-off three-body vertices (at least of this type) against off-shell two-body vertices.

(e) Discuss whether this result can be generalized to eliminate all three-body forces by taking advantage of off-shell freedom. (Note: the extent to which this is possible is an open question.)

8. Why is the sky blue? We want to construct an effective theory to understand why the sky is blue. To construct such a theory, we need to analyze the relevant scales of the problem first. The wavelength of visible light ($\sim 500\,\text{nm}$) is much larger than the size of atoms ($\sim 0.1\,\text{nm}$). Thus, the photon is insensitive to the details of the electrically neutral atoms it scatters off. Hence, we can construct the leading-order Hamiltonian simply as that of the Hydrogen atom

$$H_{\text{eff}}^{(0)} = \frac{p^2}{2m} + e\phi\,,$$

where there is no interaction with the photon field.

In order to construct an effective interaction with photons, we only demand that the Hamiltonian $H_{\text{eff}}^{(1)}$ fulfills all symmetry requirements: it must be gauge invariant, scalar under rotations, and even under both parity and time reversal transformations.

(a) Construct all possible operators – relevant for the leading-order interaction – involving $\mathbf{E}$, $\mathbf{B}$, ϕ and $\mathbf{A}$, which are consistent with these symmetry requirements.

(b) We can write the interaction as a perturbation to the free Hamiltonian

$$H_{\text{eff}}^{(1)} = -\frac{1}{2}c_E E^2 - \frac{1}{2}c_B B^2 \,,$$

where c_E and c_B are low-energy constants. Show by dimensional analysis that $c_E = k_E a_0^3$ and $c_B = k_B a_0^3$, where k_E, k_B are dimensionless and a_0 is the Bohr radius.

(c) For photons with four-momentum $q^\mu = (\omega, q)$ and polarization $\hat{\epsilon}$ we can identify $A = \hat{\epsilon}\exp(iq \cdot x)$. Show that

$$\frac{d\sigma}{d\Omega} \propto \left|\langle\text{final}|H_{\text{eff}}^{(1)}|\text{initial}\rangle\right|^2 \sim \omega^4 a_0^6 \,.$$

Why does this explain that the sky is blue? (Hint: consider what scatters more according to the crosssection., blue light or red light?)