TALENT/INT Course on Nuclear Forces
Exercises and Discussion Questions T3

[Last revised on July 16, 2013 at 10:40:14.]

Tuesday 3: MBPT, operators; Neutron matter and astrophysics

We have again grouped all of the two-minute and discussion questions toward the beginning. But remember to spend only about an hour working on questions and then try some of the other problems as well. When you need a break, go back and try another question!

1. Two minute and discussion questions:

   (a) Why is it important that once we renormalize our Hamiltonian in free space to remove sensitivity to the high-momentum cutoff that no further renormalization is needed at finite density? (What would we have to do to an input chiral EFT interaction in an NCSM calculation is this were not true?)

   (b) Which of the following make neutron and nuclear matter more perturbative: large scattering length, effective range, large cutoffs, short-range tensor forces, Pauli blocking, low density?

   (c) What goes wrong when replacing the two-body potential $V_{\rm NN}$ by the density-dependent two-body interaction $\Gamma^{(2)}$ that includes the $V_{\rm NN}$ plus the contribution from $V_{\rm 3N}$ from normal ordering? Why doesn't this include the 3N force contributions correctly?

   (d) Are there density-dependent Hamiltonians before normal ordering?

   (e) How is it possible that different NN interactions at $\rm N^2LO$ (see the AFDMC results) or different NN interactions at $\rm N^3LO$ (see the perturbative calculations) vary in their perturbativeness, although they're at the same order in the power counting?

   (f) What part(s) of the chiral EFT three-body force contributes to pure neutron matter?

   (g) We found that the leading and subleading 3N forces are repulsive in neutron matter. Should this apply for other 3N forces as well?

   (h) What is a "resolution dependent" observable? Describe an operational test of whether an observable is resolution dependent. Of the following, which are resolution dependent and which are independent of resolution: bound-state energies, excitation spectra, cross sections, ANCs, momentum distributions, spectroscopic factors, charge radii, point proton radii, charge form factors, beta decay rates.

   (i) In an EFT, how do you ensure that operators are consistent with the EFT Hamiltonian? How do you do it when you have a phenomenological Hamiltonian?

   (j) Does every Hermitian operator correspond to an observable?

(k) In the lecture, a graph of the quadrupole moment as a function of the renormalization group showed it wasn't constant. Which scale gives the correct answer? (Be careful of trick questions!)

(l) Why do neutron stars not only consist of neutrons? Why is the proton fraction in neutron stars so small ($Z/(N+Z) \sim 0.1$, smaller than in any atomic nucleus)?

(m) Discuss the neutron star mass-radius plot. What varies along the $M - R$ curve? What happens to neutron stars for $M > M_{\rm max}$ or $R < R(M = M_{\rm max})$?

2. Brueckner-Bethe-Goldstone (BBG) power counting and MBPT.

   (a) Why is it necessary to do summations of ladder diagrams into G-matrices for traditional nucleon-nucleon potentials (such as the local phenomenological potentials)?

   (b) How can you use Weinberg eigenvalues to test if this is necessary for softened potentials (either from chiral EFT with a lower cutoff or RG-softened interactions)?

   (c) Why is the MBPT expansion *still* non-perturbative in terms of G-matrices (rather than "bare" interactions) for traditional potentials but not for softer potentials?

   (d) What is the expansion parameter for the hole-line expansion? Do we need this for soft potentials?

   (e) Soft potentials were known long ago but were abandoned because the energy of nuclear matter didn't have a minimum at "saturation density" (about the density inside a heavy nucleus). Can you guess what changes that result in our modern calculations with low-momentum potentials?

3. For the neutron matter calculations, we showed bands at second- and (particle-particle/hole-hole) third-order in MBPT for either a free single-particle spectrum $\varepsilon_{\mathbf{p}} = p^2/2m$ or a Hartree-Fock spectrum $\varepsilon_{\mathbf{p}} = p^2/2m + \Sigma_{\rm HF}(p)$, with Hartree-Fock self-energy $\Sigma_{\rm HF}$. How is this a measure of uncertainty of the many-body calculation?

4. Hellmann-Feynman (or Feynman-Hellmann) theorem. If λ is a parameter in the Hamiltonian H_λ (could be a mass or a coupling constant or a parameter you add by hand), then

$$\frac{dE(\lambda)}{d\lambda} = \langle\Psi(\lambda)|\frac{\partial \widehat{H}_\lambda}{\partial\lambda}|\Psi(\lambda)\rangle \quad \text{where} \quad \widehat{H}_\lambda|\Psi(\lambda)\rangle = E(\lambda)|\Psi(\lambda)\rangle \ .$$

   (a) Prove it. (Hint: use $\langle\Psi(\lambda)|\Psi(\lambda)\rangle = 1$.) [Note: The story is that this was proven by Feynman in his undergraduate thesis at MIT in 1939.]

   (b) It is usually sufficient in practice to approximate $dE(\lambda)/d\lambda$ by a simple difference formula. Explain which of the following is better to use (and why):

$$\frac{dE(\lambda)}{d\lambda} \approx \frac{E(\lambda+\epsilon) - E(\lambda)}{\epsilon} \qquad \text{or} \qquad \frac{dE(\lambda)}{d\lambda} \approx \frac{E(\lambda+\epsilon/2) - E(\lambda-\epsilon/2)}{\epsilon} \ ,$$

   where ϵ is a small (but nonzero) value.

(c) Show that if you want to know the expectation value of *any* operator $\widehat{O}$, you can add it to the Hamiltonian: $\widehat{H} \to \widehat{H} + \lambda\widehat{O}$ and then use

$$\langle\Psi|\widehat{O}|\Psi\rangle = \left.\frac{dE(\lambda)}{d\lambda}\right|_{\lambda=0} .$$

(d) Explain how to use the Hellmann-Feynman theorem to find the expectation value of part of the three-body force in a nucleus or nuclear matter if g_3 is the coupling constant for this part.

5. Suppose $\widehat{O} = a^\dagger_q a_q$ for fixed $q = |\boldsymbol{q}|$.

(a) Prove that the one-body part of an operator like $\widehat{O}$ does not change during SRG evolution with $G_s = T_{\text{rel}}$.

(b) What will the $\widehat{O}$ operator look like (in form, not detail) after evolving?

(c) Will there be 3-body operator contributions to the deuteron?

(d) If we take the matrix element of the unevolved $\widehat{O}$ in the unevolved deuteron wave function, the entire contribution will, of course, be from the one-body operator $\widehat{O}$. If we now take the matrix element of the evolved $\widehat{O}$ in a deuteron wave function with both evolved to $\lambda \ll q$, what type of operator will give the leading contribution (and how will the full result compare to before evolution)?

6. We can extend the microscopic results for neutron matter to matter containing both neutrons and protons. To this end we use for the energy per particle ϵ of asymmetric nuclear matter an expression that interpolates between the properties of symmetric nuclear matter and neutron matter. For ϵ we take the kinetic energy plus an expression for the interaction energy that is quadratic in the neutron excess $1 - 2x$:

$$\begin{aligned} \epsilon(\bar{n}, x) &= T_0 \biggl(\frac{3}{5} \left[x^{5/3} + (1-x)^{5/3}\right] (2\bar{n})^{2/3} \\ &\quad - \left[(2\alpha - 4\alpha_L)x(1-x) + \alpha_L\right] \bar{n} + \left[(2\eta - 4\eta_L)x(1-x) + \eta_L\right] \bar{n}^\gamma \biggr) , \end{aligned} \tag{1}$$

where $\bar{n} = n/n_0$ and $x = n_p/n$ denote the density in units of the saturation density and the proton fraction, respectively. $T_0 = (3\pi^2 n_0/2)^{2/3}\hbar^2/(2m) = 36.84\,\text{MeV}$ is the Fermi energy of symmetric nuclear matter at the saturation density. The parameters α, η, α_L and η_L can be determined from the saturation properties of symmetric nuclear matter combined with results for neutron matter. For $\gamma = 4/3$ and empirical saturation properties of symmetric nuclear matter,

$$\epsilon(\bar{n} = 1, x = 1/2) = -16\,\text{MeV} \quad \text{and} \quad \frac{\partial\epsilon}{\partial\bar{n}}(\bar{n} = 1, x = 1/2) = 0 , \tag{2}$$

this results in $\alpha = 5.87$ and $\eta = 3.81$. (Show this.) The fit to the neutron matter calculations discussed in the lecture gives central values for $\alpha_L = 1.4$ and $\eta_L = 0.9$. The proton fraction x for matter in beta equilibrium is determined by minimizing in x, the energy per particle ϵ plus the contribution from electrons $\frac{E_e}{N_e} = \frac{3}{4}\, x\, (3\pi^2 x n)^{1/3}$. (This neglects the difference of neutron and proton rest masses.)

(a) Determine the proton fraction for matter in beta equilibrium at saturation density.

(b) Determine the central values for the symmetry energy S_v and its density derivative L,

$$S_v = \frac{1}{8}\frac{\partial^2 \epsilon(\bar{n}, x)}{\partial x^2}\bigg|_{\bar{n}=1, x=1/2} \quad \text{and} \quad L = \frac{3}{8}\frac{\partial^3 \epsilon(\bar{n}, x)}{\partial \bar{n}\partial x^2}\bigg|_{\bar{n}=1, x=1/2} . \tag{3}$$