Constraints on Proton Structure from Precision Atomic-Physics Measurements

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Introduction

- consider difference between the well-known hyperfine splittings (hfs) in hydrogen and muonium.
- correct for magnetic moment and reduced mass effects.
- the large QED contributions for a pointlike nucleus essentially cancel.
- difference then due solely to proton structure.
- this provides a sum rule that constrains a particular combination of proton form factors and structure functions.
Acknowledgments

- see also comments in J.L Friar and I. Sick, PRL 95, 049101 (2005) and the reply in PRL 95, 049102 (2005).
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Outline

- introduction
- sum-rule derivation
- evaluation
- interpretation
- conclusions
Hyperfine splittings

- **Fermi energy:** \( E_F^N = \frac{8\alpha^3}{3\pi} \frac{\mu_B \mu_N m_e^3}{(1+m_e/m_N)^3} \),

  with \( N = \mu^+ \) or \( p \), \( \mu_B = \frac{e}{2m_e} \), and \( \mu_N = (1 + \kappa_N) \frac{e}{2m_N} \)

- **muonium**

  \[
  E_{\text{hfs}}(e^-\mu^+) = (1 + \Delta_{QED} + \Delta_R^\mu + \Delta_{hvp}^\mu + \Delta_{\text{weak}}^\mu) E_F^\mu
  \]

- **hydrogen**

  \[
  E_{\text{hfs}}(e^-p) = (1 + \Delta_{QED} + \Delta_R^p + \Delta_S^p + \Delta_{hvp}^p + \Delta_{\mu vp}^p + \Delta_{\text{weak}}^p) E_F^p
  \]

- **construct ratio rescaled by \( \mu_N \) and reduced masses**

  \[
  \Delta_{\text{hfs}} \equiv \frac{E_{\text{hfs}}(e^-p)}{E_{\text{hfs}}(e^-\mu^+)} \frac{\mu_\mu}{\mu_p} \frac{(1+m_e/m_p)^3}{(1+m_e/m_u)^3} - 1
  \]

  \[
  = \frac{1 + \Delta_{QED} + \Delta_R^p + \Delta_S^p + \Delta_{hvp}^p + \Delta_{\mu vp}^p + \Delta_{\text{weak}}^p}{1 + \Delta_{QED} + \Delta_R^\mu + \Delta_{hvp}^\mu + \Delta_{\text{weak}}^\mu} - 1
  \]
The atomic side

\[ \Delta S = \Delta_{hfs} + \Delta_{\mu} R + \Delta_{hvp}^\mu + \Delta_{\text{weak}}^\mu \]
\[ - (\Delta_{\mu}^p R + \Delta_{hvp}^p + \Delta_{\mu hvp}^p + \Delta_{\text{weak}}^p) \]
\[ + \Delta_{hfs}(\Delta_{\text{QED}} + \Delta_{\mu} R + \Delta_{hvp}^\mu + \Delta_{\text{weak}}^\mu) \]

- The leading \( \Delta_{\text{QED}} \) cancel.

- The remaining \( \Delta_{\text{QED}} \) can be replaced by the lowest-order approximation, \( \alpha/2\pi \).
The hadronic side

- \( \Delta_S = \Delta_Z + \Delta_{\text{pol}} \)

**Zemach contribution:** \( \Delta_Z = -2\alpha m_e (1 + \delta^{\text{rad}}_Z) \langle r \rangle_Z \), with

\[
\langle r \rangle_Z = -\frac{4}{\pi} \int_0^\infty \frac{dQ}{Q^2} \left[ \frac{G_E(Q^2)G_M(Q^2)}{1+\kappa_p} - 1 \right]
\]

\[
= \int d^3r d^3r' |\vec{r} - \vec{r}'| \rho_E(\vec{r}) \rho_M(\vec{r}')
\]

**Polarization contribution:** \( \Delta_{\text{pol}} = \frac{\alpha m_e}{2\pi m_p(1+\kappa_p)} (\Delta_1 + \Delta_2) \), with

\[
\Delta_1 = \int_0^\infty \frac{dQ^2}{Q^2} \left\{ \frac{9}{4} F_2^2(Q^2) - 4m_p \int_{\nu_{\text{th}}}^\infty \frac{d\nu}{\nu^2} \beta_1 \left( \frac{\nu^2}{Q^2} \right) g_1(\nu, Q^2) \right\},
\]

\[
\Delta_2 = -12m_p \int_0^\infty \frac{dQ^2}{Q^2} \int_{\nu_{\text{th}}}^\infty \frac{d\nu}{\nu^2} \beta_2 \left( \frac{\nu^2}{Q^2} \right) g_2(\nu, Q^2),
\]

\[
\nu_{\text{th}} = m_\pi + (m_\pi^2 + Q^2)/2m_p,
\]

\[
\beta_1(\theta) = 3\theta - 2\theta^2 - 2(2-\theta)\sqrt{\theta(\theta+1)},
\]

\[
\beta_2(\theta) = 1 + 2\theta - 2\sqrt{\theta(\theta+1)}
\]
Radiative correction to Zemach radius

\( \delta_{Z}^{\text{rad}} \) estimated by Bodwin & Yennie, PRD 37, 498 (1988).

Karshenboim, PLA 225, 97 (1997) calculated analytically for dipole form factors:

\[
\delta_{Z}^{\text{rad}} = \frac{\alpha}{3\pi} \left[ 2 \ln\left( \frac{\Lambda^2}{m_e^2} \right) - \frac{4111}{420} \right].
\]

with \( \Lambda^2 = 0.71 \text{ GeV}^2 \), this yields \( \delta_{Z}^{\text{rad}} = 0.0153 \).
Recoil corrections in muonium

- muonium: \( \Delta R^\mu = \Delta_{\text{rel}}^\mu + \Delta_{\text{rad}}^\mu. \)

- relativistic recoil
  [Bodwin & Yennie, PRD 37, 498 (1988)]:

  \[
  \Delta_{\text{rel}}^\mu = \frac{1}{1+\kappa_\mu} \left[ \frac{-3\alpha}{\pi} \frac{m_\mu m_\mu}{m_\mu^2 - m_e^2} \ln \frac{m_\mu}{m_e} + \alpha^2 \frac{m_e}{m_\mu} \left( 2 \ln \frac{1}{2\alpha} - 6 \ln 2 + \frac{65}{18} \right) \right]
  \]


  \[
  \Delta_{\text{rad}}^\mu = \frac{1}{1+\kappa_\mu} \left[ \frac{\alpha^2}{\pi^2} \frac{m_e}{m_\mu} \left( -2 \ln^2 \frac{m_\mu}{m_e} + \frac{13}{12} \ln \frac{m_\mu}{m_e} + \frac{21}{2} \zeta(3) + \zeta(2) + \frac{35}{9} \right) 
  + \frac{\alpha^3}{\pi^3} \frac{m_e}{m_\mu} \left( -\frac{4}{3} \ln^3 \frac{m_\mu}{m_e} + \frac{4}{3} \ln^2 \frac{m_\mu}{m_e} - \left[ 4\pi^2 \ln 2 + \frac{29}{12} \right] \ln \frac{m_\mu}{m_e} + 47.7213 \right) 
  + \alpha^2 \left( \frac{m_e}{m_\mu} \right)^2 \left( -6 \ln 2 - \frac{13}{6} \right) \right]
  \]
Recoil corrections in hydrogen

- Bodwin & Yennie, PRD 37, 498 (1988):
  \[ \Delta p_R = -1.55 \text{ ppm}. \]
- finite-size corrections → +5.68(1) ppm.
- radiative recoil corrections
  [Karshenboim, PLA 225, 97 (1997)] → 5.77(1) ppm.
- Volotka et al, EPJD 33, 23 (2005):
  - re-evaluation of finite-size corrections → 5.86 ppm.
  - forced \( G_M \) to reproduce their \( \langle r \rangle_Z \) → 6.01 ppm.
  - chose \( \Delta p_R = 5.86(15) \text{ ppm} \).
Atomic inputs

  \( E_{\text{hfs}}(e^- p) = 1\,420.405\,751\,766\,7(9) \text{ MHz}. \)

- W. Liu et al., PRL 82, 711 (1999):
  \( E_{\text{hfs}}(e^- \mu^+) = 4\,463.302\,765(53) \text{ MHz}. \)

- S. Eidelman et al., PLB 592, 1 (2004):
  \( m_p = 938.272\,029(80) \text{ MeV}, \quad m_{\mu} = 105.658\,369(9) \text{ MeV}, \)
  \( m_e = 0.510\,998\,918(44) \text{ MeV}, \quad \alpha^{-1} = 137.035\,999\,11(46). \)

- G.W. Bennett et al., PRL 92, 161802 (2004):
  \( \kappa_{\mu} = 0.001\,165\,920\,8(6). \)

  \( m_{\mu}/m_e = 206.768\,2838(54), \quad m_p/m_e = 1836.152\,672\,61(85). \)

- P.J. Mohr, private communication:
  \( \mu_{\mu}/\mu_p = 3.183\,345\,20(20), \text{ free of muonium hfs.} \)
Cross-check with QED

- **muonium:**
  \[
  \Delta_{\text{QED}} = \frac{E_{\text{hfs}}(e^-\mu^+)}{E_F^\mu} - 1 - (\Delta_R^\mu + \Delta_{hvp}^\mu + \Delta_{\text{weak}}^\mu)
  \]
  \[
  = 1136.12(13) \text{ ppm.}
  \]

- **hydrogen:**
  \[
  \Delta_{\text{QED}} = \frac{E_{\text{hfs}}(e^-p)}{E_F^p} - 1 - (\Delta_R^p + \Delta_S^p + \Delta_{hvp}^p + \Delta_{\mu vp}^p + \Delta_{\text{weak}}^p)
  \]

- **mix:**
  \[
  \Delta_{\text{QED}} = \frac{E_{\text{hfs}}(e^-p)}{E_F^p} - 1 - (\Delta_R^\mu + \Delta_{hvp}^\mu + \Delta_{\text{weak}}^\mu + \Delta_{\text{hfs}}^\mu)
  \]
  \[
  - \Delta_{\text{hfs}} \left( \frac{\alpha}{2\pi} + \Delta_R^\mu + \Delta_{hvp}^\mu + \Delta_{\text{weak}}^\mu \right)
  \]
  \[
  = 1136.09(14) \text{ ppm.}
  \]

- consistent with Dupays et al., PRA 68, 052503 (2003) and Volotka et al., EPJD 33, 23 (2005).
Evaluation of atomic side

\( \Delta_{\text{hfs}} = 145.51(4) \) ppm.

\( \Delta_{\mu_R} = -178.34 \) ppm.

more inputs [Volotka et al., EPJD 33, 23 (2005)]:

\( \Delta_{\mu_{hvp}} = 0.05 \) ppm, \( \Delta_{\mu_{\text{weak}}} = -0.01 \) ppm, \( \Delta_{p_{hvp}} = 0.01 \) ppm,

\( \Delta_{p_{\muvp}} = 0.07 \) ppm, \( \Delta_{p_{\text{weak}}} = 0.06 \) ppm.

\( \Delta_S = -38.62(16) \) ppm.

\( \rightarrow \) constraint on \( G_E, G_M, g_1, \) and \( g_2 \) that is better than 1\%.
Interpretation of hadronic side

if use estimate of $\Delta_{\text{pol}} = 1.4(6) \text{ ppm}$ by Faustov and Martynenko [EPJC 24, 281 (2002)], then $\Delta_Z = -40.0(6) \text{ ppm}$ and $\langle r \rangle_Z = 1.043(16) \text{ fm}$.

Griffioen et al.: $\Delta_{\text{pol}} = 0.72(37) \text{ ppm}$
$\rightarrow \Delta_Z = -39.3(4) \text{ ppm}$ and $\langle r \rangle_Z = 1.024(16) \text{ fm}$.

if use estimate of $\langle r \rangle_Z = 1.086(12) \text{ fm}$ by Friar and Sick [PLB 579, 285 (2004)], then $\Delta_{\text{pol}} = 3.05(49) \text{ ppm}$.
\[ \langle r \rangle_Z \text{ from form-factor models} \]

- dipole $\rightarrow 1.025 \text{ fm}$

- fit to standard Rosenbluth separation
  [J. Arrington, PRC 69, 022201(R) (2004), Table I]
  \[ G_E(Q^2), G_M(Q^2)/(1 + \kappa_p) = 1/(1 + p_2 Q^2 + p_4 Q^4 + \cdots) \]
  $\rightarrow 1.081 \text{ fm}$

- fit constrained by polarization transfer data
  [Arrington, Table II] $\rightarrow 1.050 \text{ fm}$
Electric charge radius

\[ r_{E,\text{rms}} = \sqrt{-6 \frac{d}{dQ^2} G_E(Q^2)}|_{Q^2=0} \]

obtain estimates from

- a standard empirical fit: \(0.862(12)\) fm
  [G.G. Simon et al., NPA 333, 381 (1980)]

- Lamb-shift measurements: \(0.871(12)\) fm
  [K. Pachucki, PRA 63, 042503 (2001);
   K. Pachucki and U.D. Jentschura, PRL 91, 113005 (2003); updated by M. Eides, private communication]

- a continued-fraction fit for \(G_E\): \(0.895(18)\) fm
  [I. Sick, PLB 576, 62 (2003)]

- the 2002 CODATA value: \(0.8750(68)\) fm
  [P.J. Mohr and B.N. Taylor, RMP 77, 1 (2005)]
Plot of $\langle r \rangle_Z$ vs $r_{E,\text{rms}}$
Conclusions

- atomic physics provides a very precise constraint on proton structure, to better than 1%.
- the subtraction method removes uncertainties associated with pure QED contributions to hfs.
- the method could also be applied to Lamb shifts, to extract $r_{E,rms}$ with less uncertainty.
- the interpretation of individual structure contributions requires more data and analysis, particularly for
  - $g_1$, $g_2$, and $\Delta_{pol}$.
  - two-photon contributions to electron-proton scattering.