Light nuclei in the
Fermionic Molecular Dynamics approach

Thomas Neff
6th ANL/MSU/JINA/INT
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Overview

Effective Nucleon-Nucleon interaction: Unitary Correlation Operator Method
Roth, Neff, Feldmeier, arXiv:1003.3624

- Short-range Correlations
- Correlated Interaction
- \textit{ab initio} Few- and Many-Body Calculations

Many-Body Method:
Fermionic Molecular Dynamics

- Model
- Beryllium Isotopes
- Cluster States in $^{12}$C
Argonne V18 (T=0) spins aligned parallel or perpendicular to the relative distance vector

- strong repulsive core: nucleons can not get closer than $\approx 0.5$ fm

- strong dependence on the orientation of the spins due to the tensor force

$V_{NN}$ [MeV] vs $r_{12}$ [fm]

- central correlations

- tensor correlations
Argonne V18 (T=0)
spins aligned parallel or perpendicular to the relative distance vector

- strong repulsive core: nucleons can not get closer than $\approx 0.5$ fm
- **central correlations**

- strong dependence on the orientation of the spins due to the tensor force
- **tensor correlations**

The nuclear force will induce **strong short-range correlations** in the nuclear wave function.
Unitary Correlation Operator Method

**Correlation Operator**

- induce short-range (two-body) central and tensor correlations into the many-body state

\[ \tilde{C} = \tilde{C}_\Omega \tilde{C}_r = \exp[-i \sum_{i<j} g_{\Omega,ij}] \exp[-i \sum_{i<j} g_{r,ij}] , \quad \tilde{C}^\dagger \tilde{C} = 1 \]

- correlation operator should conserve the symmetries of the Hamiltonian and should be of finite-range, correlated interaction **phase shift equivalent** to bare interaction by construction

**Correlated Operators**

- correlated operators will have contributions in higher cluster orders

\[ \tilde{C}^\dagger \tilde{Q} \tilde{C} = \hat{Q}^{[1]} + \hat{Q}^{[2]} + \hat{Q}^{[3]} + \ldots \]

- two-body approximation: correlation range should be small compared to mean particle distance

**Correlated Interaction**

\[ \tilde{C}^\dagger (\tilde{T} + \tilde{V}) \tilde{C} = \tilde{T} + \tilde{V}_{\text{UCOM}} + \tilde{V}^{[3]}_{\text{UCOM}} + \ldots \]
Central and Tensor Correlations

\[ \zeta = \zeta_\Omega \zeta_r \]

\[ p = p_r + p_\Omega \]

\[ p_r = \frac{1}{2} \left\{ \frac{r}{r} \left( \frac{r}{r} p \right) + \left( \frac{p_r}{r} \right) \frac{r}{r} \right\} , \quad p_\Omega = \frac{1}{2r} \left\{ l \times \frac{r}{r} - \frac{r}{r} \times l \right\} \]
Central and Tensor Correlations

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\[ \mathbf{p}_r = \frac{1}{2} \left\{ \mathbf{r} \left( \mathbf{r} \cdot \mathbf{p} \right) + \left( \mathbf{p}_r \right) \mathbf{r} \right\}, \quad \mathbf{p}_\Omega = \frac{1}{2r} \left\{ \mathbf{l} \times \mathbf{r} - \mathbf{r} \times \mathbf{l} \right\} \]

Central Correlations

\[ \zeta_r = \exp \left\{ -\frac{i}{2} \left\{ \mathbf{p}_r s(r) + s(r) \mathbf{p}_r \right\} \right\} \]

probability density shifted out of the repulsive core

\[ S = 0, \quad T = 1 \]
Central Correlations

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- probability density shifted out of the repulsive core

Tensor Correlations

\[ \zeta_\Omega = \exp \left\{ -i \vartheta(r) \left\{ \frac{3}{2} (\sigma_1 \cdot p_\Omega)(\sigma_2 \cdot r) + \frac{3}{2} (\sigma_1 \cdot r)(\sigma_2 \cdot p_\Omega) \right\} \right\} \]

- tensor force admixes other angular momenta
Central and Tensor Correlations

Central Correlations

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\[ p_r = \frac{1}{2} \left\{ \frac{r}{r} \left( \frac{r}{r} p \right) + \left( p \frac{r}{r} \right) \frac{r}{r} \right\}, \quad p_\Omega = \frac{1}{2r} \left\{ I \times \frac{r}{r} - \frac{r}{r} \times I \right\} \]
two-body densities

\[ \rho^{(2)}_{S,T}(r_1 - r_2) \quad S = 1, M_S = 1, T = 0 \]

central correlator \( C_r \)
shifts density out of the repulsive core

tensor correlator \( C_\Omega \)
aligns density with spin orientation

Realistic Effective Interaction

Unitary Correlation Operator Method

two-body densities

$\rho_{S,T}^{(2)}(r_1 - r_2) \quad S = 1, M_S = 1, T = 0$

$C_r$ shifts density out of the repulsive core

$C_\Omega$ aligns density with spin orientation

both central and tensor correlations are essential for binding

$\langle T \rangle$  
$\langle H \rangle$  
$\langle V \rangle$


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bared interaction has strong off-diagonal matrix elements connecting to high momenta.
Correlated Interaction in Momentum Space

bare interaction has strong off-diagonal matrix elements connecting to high momenta

correlated interaction is more attractive at low momenta

off-diagonal matrix elements connecting low- and high- momentum states are strongly reduced

Correlated Interaction in Momentum Space

bare interaction has **strong off-diagonal** matrix elements connecting to high momenta

correlated interaction is **more attractive** at low momenta

**off-diagonal matrix elements** connecting low- and high- momentum states are **strongly reduced**

similar to $V_{\text{low-}k}$

• choose tensor correlation range or SRG flow parameter $\alpha$ such that need for three-body forces is minimized

**different perspective**: don’t try to reproduce the results with the bare interaction but consider $V_{UCOM}$ as a realistic potential
Convergence dramatically improved compared to bare interaction

Binding energy close to experiment

Spectra with $V_{\text{UCOM}}$ are of similar quality than with other modern NN forces

Roth, Neff, Feldmeier, arXiv:1003.3624
Hartree-Fock calculations

- Hartree-Fock calculations in spherical $12\,\hbar\omega$ harmonic oscillator basis

- UCOM interaction is less attractive in higher partial waves

- Problems with saturation indicate need for three-body forces

Roth, Neff, Feldmeier, arXiv:1003.3624
Fermionic Molecular Dynamics

Motivation
FMD Wave Functions
Nucleon-Nucleon Interaction
Mean-Field Calculations
Projection After Variation, Variation After Projection and Multiconfiguration
Exotica: Special Challenges

Fermionic Molecular Dynamics

**Fermionic**

Slater determinant

\[ |Q\rangle = \mathcal{A} \left( |q_1\rangle \otimes \cdots \otimes |q_A\rangle \right) \]

- antisymmetrized $A$-body state

Feldmeier, Schnack, Rev. Mod. Phys. **72** (2000) 655
Fermionic Molecular Dynamics

Fermionic
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Molecular

single-particle states

\[ \langle x | q \rangle = \sum_i c_i \exp\left\{ -\frac{(x - b_i)^2}{2a_i} \right\} \otimes |\chi^i, \chi'^i\rangle \otimes |\xi\rangle \]

- Gaussian wave-packets in phase-space (complex parameter \( b_i \) encodes mean position and mean momentum), spin is free, isospin is fixed
- width \( a_i \) is an independent variational parameter for each wave packet
- use one or two wave packets for each single particle state

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Feldmeier, Schnack, Rev. Mod. Phys. 72 (2000) 655
FMD
Fermionic Molecular Dynamics

**Fermionic**
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see also

**Antisymmetrized Molecular Dynamics**
H. Horiuchi, Y. Kanada-En’yo

Feldmeier, Schnack, Rev. Mod. Phys. **72** (2000) 655
Evaluation of Matrix Elements

> non-orthogonal basis, use inverse overlap matrix

**One-Body Operators**

\[
\frac{\langle Q \mid \mathcal{T}^{[1]} \mid Q \rangle}{\langle Q \mid Q \rangle} = \sum_{k,l}^A \langle q_k \mid \mathcal{T}^{[1]} \mid q_l \rangle o_{lk}
\]

**Two-Body Operators**

\[
\frac{\langle Q \mid \mathcal{V}^{[2]} \mid Q \rangle}{\langle Q \mid Q \rangle} = \frac{1}{2} \sum_{k,l,m,n}^A \langle q_k, q_l \mid \mathcal{V}^{[2]} \mid q_m, q_n \rangle (o_{mk}o_{nl} - o_{ml}o_{nk})
\]

\[
o = n^{-1} = \left(\langle q_i \mid q_j \rangle\right)^{-1}
\]
**Interaction Matrix Elements**

(One-body) Kinetic Energy

\[
\langle q_k | T | q_l \rangle = \langle a_k b_k | T | a_l b_l \rangle \langle \chi_k | \chi_l \rangle \langle \xi_k | \xi_l \rangle
\]

\[
\langle a_k b_k | T | a_l b_l \rangle = \frac{1}{2m} \left( \frac{3}{a_k^* + a_l} - \frac{(b_k^* - b_l)^2}{(a_k^* + a_l)^2} \right) R_{kl}
\]

(Two-body) Potential

- fit radial dependencies by (a sum of) Gaussians

\[
G(x_1 - x_2) = \exp \left\{ - \frac{(x_1 - x_2)^2}{2\kappa} \right\}
\]

- Gaussian integrals

\[
\langle a_k b_k, a_l b_l | G | a_m b_m, a_n b_n \rangle = R_{km} R_{ln} \left( \frac{\kappa}{\alpha_{klmn} + \kappa} \right)^{3/2} \exp \left\{ - \frac{\rho_{klmn}^2}{2(\alpha_{klmn} + \kappa)} \right\}
\]

- analytical formulas for matrix elements
Effective two-body interaction

- FMD model space can’t describe correlations induced by residual medium-long ranged tensor forces
  - use long ranged tensor correlator – “low cutoff” to partly account for that
- no three-body forces, missing spin-orbit strength, radii tend to be too small
  - add phenomenological two-body correction term with a momentum-dependent central and (isospin-dependent) spin-orbit part (about 15% contribution to potential)
  - fit correction term to binding energies and radii of “closed-shell” nuclei ($^4$He, $^{16}$O, $^{40}$Ca), ($^{24}$O, $^{34}$Si, $^{48}$Ca)

Outlook:
  - use three-body or density dependent two-body force instead of two-body correction term
Minimization

- minimize Hamiltonian expectation value with respect to all single-particle parameters \( q_k \)

\[
\min_{\{q_k\}} \frac{\langle Q | H - T_{cm} | Q \rangle}{\langle Q | Q \rangle}
\]

- this is a Hartree-Fock calculation in our particular single-particle basis
- the mean-field may break the symmetries of the Hamiltonian
Projection After Variation (PAV)

- mean-field may break symmetries of Hamiltonian
- restore inversion, translational and rotational symmetry by projection on parity, linear and angular momentum

\[ P^\pi = \frac{1}{2} \left( 1 + \pi \Pi \right) \]

\[ P'_{MK} = \frac{2J + 1}{8\pi^2} \int d^3\Omega \, D^{\dagger}_{MK}(\Omega) \tilde{R}(\Omega) \]

\[ P^P = \frac{1}{(2\pi)^3} \int d^3X \, \exp\{ -i(\tilde{P} - P) \cdot X \} \]
**PAV, VAP and Multiconfiguration**

**Projection After Variation (PAV)**
- mean-field may break symmetries of Hamiltonian
- restore inversion, translational and rotational symmetry by projection on parity, linear and angular momentum

\[
P^\pi = \frac{1}{2}(1 + \pi \Pi)
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**Variation After Projection (VAP)**
- effect of projection can be large
- full Variation after Angular Momentum and Parity Projection (VAP) for light nuclei
- perform VAP in GCM sense by applying constraints on radius, dipole moment, quadrupole moment or octupole moment and minimizing the energy in the projected energy surface for heavier nuclei

\[
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Multiconfiguration Calculations

- diagonalize Hamiltonian in a set of projected intrinsic states

\[
\left\{ |Q^{(a)}\rangle , \quad a = 1, \ldots, N \right\}
\]

\[
P^{\pi}_{\sim} = \frac{1}{2} (1 + \pi \Pi)
\]

\[
P^{J}_{MK} \sim = \frac{2J + 1}{8\pi^2} \int d^3\Omega D^{J}_{MK} (\Omega) \tilde{R}(\Omega)
\]

\[
P^{P} = \frac{1}{(2\pi)^3} \int d^3X \exp\{-i(P - P) \cdot X\}
\]
Beryllium Isotopes

Questions

- $\alpha$-clustering, halos in $^{11}\text{Be}$ and $^{14}\text{Be}$, $N = 8$ shell closure?

Calculation

- FMD wave functions with two Gaussians per sp-state
- mean field, variation after projection, variation after multiconfiguration mixing
- VAP and multiconfiguration-VAP configurations with mean proton distance as generator coordinate

Observables

- energies
- charge and matter radii, electromagnetic transitions

Results still preliminary!
Beryllium Isotopes

Mean field

[Diagrams of various isotopes showing distributions]
• create configurations by variation after parity and angular momentum projection

➢ large gain in binding energy compared to mean-field result

➢ intrinsic states show pronounced cluster structure. Parameters of $^4$He and $^3$He clusters are close to those of the free clusters
Beryllium Isotopes
Variation after Projection

\( ^{10}\text{Be} \)

\( ^{10}\text{Be} - p \)
\( ^{10}\text{Be} - n \)

\( ^{11}\text{Be} \)

\( ^{11}\text{Be} - p \)
\( ^{11}\text{Be} - n \)

\( \text{VAP } 0^+ \)

\( \text{VAP } 1/2^- \)

\( \text{VAP } 0^+ \)

\( \text{VAP } 1/2^+ \)

\( \text{VAP } 0^+ \)

\( \text{VAP } 5/2^+ \)
Beryllium Isotopes

Variation after Projection

\[ ^{12}\text{Be} \]

- "p^2"  
  VAP \( 0^+ \)

- "d^2"  
  VAP \( 0^+ \)

- "s^2"  
  VAP \( 0^+ \)

\[ ^{14}\text{Be} \]

- "d^2"  
  VAP \( 0^+ \)

- "s^2"  
  VAP \( 0^+ \)
Mean proton distance

\[ R_{pp}^2 = \frac{1}{Z^2} \left( \sum_{i<j} (r_i - r_j)^2 \right) \]

\( R_{pp} \) as a measure of \( \alpha \)-cluster distance

Beryllium Isotopes

Mean proton distance as generator coordinate

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Beryllium Isotopes

Mean proton distance as generator coordinate

\[ E_{\text{VAP}}(R_{pp}) \text{ [MeV]} \]

\[ R_{pp} \text{ [fm]} \]

\[ 11^\text{Be} - "p", "s" \text{ and } "d"\text{-configurations} \]

- "s"- and "d"-configurations will mix in 1/2\(^+\) state
- energy surfaces for "p" and "s" similar to those in \(^{10}\text{Be}\)
- "d" surface has minimum at larger cluster distance \(\rightarrow d\)-configuration has a polarized \(^{10}\text{Be}\) core
large correlation energies due to cluster structure
loosely bound systems gain most by configuration mixing
"almost correct" level ordering in $^{11}\text{Be}$

- $^{12}\text{Be}$ ground state dominated by $p^2$ configuration, sizeable admixture of $s^2$ and $d^2$ configurations which strongly mix
Beryllium Isotopes
Charge Radii

Zakova, Neff, et al., J. Phys. G, accepted for publication
### Beryllium Isotopes

#### Electromagnetic transitions

<table>
<thead>
<tr>
<th>$^{10}\text{Be}$</th>
<th>FMD(Multiconfig)</th>
<th>Experiment</th>
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<tbody>
<tr>
<td>$B(E2; 2^+_1 \rightarrow 0^+_1)$</td>
<td>11.27 e²fm⁴</td>
<td>9.2 ± 0.3 e²fm⁴</td>
</tr>
<tr>
<td>$B(E2; 2^+_2 \rightarrow 0^+_1)$</td>
<td>1.00 e²fm⁴</td>
<td>0.11 ± 0.02 e²fm⁴</td>
</tr>
<tr>
<td>$B(E2; 0^+_2 \rightarrow 2^+_1)$</td>
<td>4.99 e²fm⁴</td>
<td>3.2 ± 1.9 e²fm⁴</td>
</tr>
<tr>
<td>$B(E1; 0^+_2 \rightarrow 1^-_1)$</td>
<td>0.013 e²fm²</td>
<td>0.013 ± 0.004 e²fm²</td>
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<td>0.099 ± 0.010 e²fm²</td>
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<td>8.0 ± 3.0 e²fm⁴</td>
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<tr>
<td>$B(E2; 0^+_2 \rightarrow 2^+_1)$</td>
<td>6.50 e²fm⁴</td>
<td>7.0 ± 0.6 e²fm⁴</td>
</tr>
<tr>
<td>$M(E0; 0^+_1 \rightarrow 0^+_2)$</td>
<td>1.05 efm²</td>
<td>0.87 ± 0.03 efm²</td>
</tr>
<tr>
<td>$B(E1; 0^+_1 \rightarrow 1^-_1)$</td>
<td>0.08 e²fm²</td>
<td>0.051 ± 0.003 e²fm²</td>
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</table>

### 11Be-10Be Overlaps

#### Spectroscopic Factors

<table>
<thead>
<tr>
<th></th>
<th>11Be</th>
<th>10Be</th>
<th>l_j</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2⁺</td>
<td>0⁺</td>
<td>s₁/₂</td>
<td>0.937</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2⁺</td>
<td>d₅/₂</td>
<td>0.094</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2⁺</td>
<td>d₃/₂</td>
<td>0.007</td>
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<tr>
<td>5/2⁺</td>
<td>0⁺</td>
<td>d₅/₂</td>
<td>0.543</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2⁺</td>
<td>s₁/₂</td>
<td>0.329</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2⁺</td>
<td>d₅/₂</td>
<td>0.243</td>
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<tr>
<td>1/2⁻</td>
<td>0⁺</td>
<td>p₁/₂</td>
<td>0.805</td>
<td></td>
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<tr>
<td></td>
<td>2⁺</td>
<td>p₃/₂</td>
<td>0.779</td>
<td></td>
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- extended s-wave halo
- $s_{1/2}$ spectroscopic factor larger than results obtained from knockout and transfer reactions
Cluster States in $^{12}\text{C}$

Astrophysical Motivation
- Helium burning: triple alpha-reaction
- Is the Hoyle state a pure $\alpha$-cluster state?
- Other excited $0^+$ and $2^+$ states

Structure
- Compare FMD results to microscopic $\alpha$-cluster model
- Analyze wave functions in harmonic oscillator basis
- No-Core Shell Model Calculations?
Cluster States in $^{12}\text{C}$

**Triple $\alpha$ Reaction**

- AGB star (radius ~ 1-1.5 AU)
- Asymptotic Giant Branch star
- Close-up of core region for a $1 M_\odot$
- Hydrogen-burning shell
- Helium layer
- Helium-burning shell
- Carbon-oxygen core (no fusion)

![Diagram of AGB star and stellar core](image)

![Diagram of triple alpha reaction](image)

\[
\begin{align*}
\text{7.2747} & \quad 3\alpha \\
\text{7.6542} & \quad 0^+ \\
\text{4.4389} & \quad 2^+ \\
\text{7.3666} & \quad \alpha + ^8\text{Be}
\end{align*}
\]

**The Triple Alpha Process (Helium Fusion)**

- $^4\text{He}$
- $^6\text{He}$ (alpha particle)
- $^8\text{Be}$
- $^{12}\text{C}$
- $^{16}\text{O}$

![Diagram of triple alpha process](image)


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Cluster States in $^{12}$C

Microscopic $\alpha$-Cluster Model

Basis States

- describe Hoyle State as a system of 3 $^4$He nuclei

$$\left| \psi_{3\alpha}(R_1, R_2, R_3); JMK\pi \right> = P^J_{MK}P^\pi A \left\{ \left| \psi_\alpha(R_1) \right> \otimes \left| \psi_\alpha(R_2) \right> \otimes \left| \psi_\alpha(R_3) \right> \right\}$$

Volkov Interaction

- simple central interaction
- parameters adjusted to reproduce $\alpha$ binding energy and radius, $\alpha - \alpha$ scattering data and C12 ground state energy
- only reasonable for $^4$He, $^8$Be and $^{12}$C nuclei

‘BEC’ wave functions

- interpretation of the Hoyle state as a Bose-Einstein Condensate of $\alpha$-particles by Funaki, Tohsaki, Horiuchi, Schuck, Röpke
- same interaction and $\alpha$-cluster parameters used
Cluster States in $^{12}$C

Basis States

- 20 FMD states obtained in Variation after Projection on $0^+$ and $2^+$ with constraints on the radius
- 42 FMD states obtained in Variation after Projection on parity with constraints on radius and quadrupole deformation
- 165 $\alpha$-cluster configurations
  ➼ projected on angular momentum and linear momentum

Interaction

- not tuned for $\alpha-\alpha$ scattering or $^{12}$C properties
Cluster States in $^{12}$C

α-α Phaseshifts

- Phaseshifts calculated with cluster configurations only (dashed lines)
- Phaseshifts calculated with additional FMD VAP configurations in the interaction region (solid lines)
- only cluster configurations included

Similar quality for description of α-α-scattering
Comparison of Cluster States in $^{12}\text{C}$

E$-E_{3\alpha}$ [MeV]

$^{12}\text{C}$

- $2^-$
- $4^+$
- $0^+$
- $2^+$

FMD

Experiment

$\alpha$–cluster
## Cluster States in $^{12}$C

### Comparison

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<th>Exp$^2$</th>
<th>Exp$^3$</th>
<th>FMD</th>
<th>$\alpha$-cluster</th>
<th>‘BEC’$^4$</th>
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<td>$E(0^+_1)$</td>
<td>-92.16</td>
<td></td>
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<td>-92.64</td>
<td>-89.56</td>
<td>-89.52</td>
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<td>$E^*(2^+_1)$</td>
<td>4.44</td>
<td></td>
<td></td>
<td>5.31</td>
<td>2.56</td>
<td>2.81</td>
</tr>
<tr>
<td>$E(3\alpha)$</td>
<td>-84.89</td>
<td></td>
<td></td>
<td>-83.59</td>
<td>-82.05</td>
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<td>$E(0^+_2) - E(3\alpha)$</td>
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<td>0.38</td>
<td>0.26</td>
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<td>$E(0^+_3) - E(3\alpha)$</td>
<td>(3.0)</td>
<td>2.7(3)</td>
<td>3.96(5)</td>
<td>2.84</td>
<td>2.81</td>
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<td>$E(2^+_2) - E(3\alpha)$</td>
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<td>$B(E2, 2^+_1 \rightarrow 0^+_1)$</td>
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Experimental situation for $0^+_3$ and $2^+_2$ states still unsettled

$2^+_2$ resonance at 1.8 MeV above threshold included in NACRE compilation
<table>
<thead>
<tr>
<th></th>
<th>Exp^1</th>
<th>Exp^2</th>
<th>Exp^3</th>
<th>FMD</th>
<th>α-cluster</th>
<th>‘BEC’^4</th>
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<tr>
<td>$E^*(2^+_1)$</td>
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<td>2.77</td>
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</table>

|                  |       |       |       |      |           |         |
| $r_{\text{charge}}(0^+_1)$ | 2.47(2) | 2.53 | 2.54 |
| $r(0^+_1)$       |       | 2.39 | 2.40 | 2.40 |
| $r(0^+_2)$       |       | 3.38 | 3.71 | 3.83 |
| $r(0^+_3)$       |       | 4.62 | 4.75 |
| $r(2^+_1)$       |       | 2.50 | 2.37 | 2.38 |
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| $M(E0, 0^+_1 \rightarrow 0^+_2)$ | 5.4(2) | 6.53 | 6.52 | 6.45 |
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---


Experimental situation for $0^+_3$ and $2^+_2$ states still unsettled.

$2^+_2$ resonance at 1.8 MeV above threshold included in NACRE compilation.

Calculated in bound state approximation.

Include $^8\text{Be}+^4\text{He}$ channels for two-body decay.
Cluster States in $^{12}\text{C}$

Electron Scattering Data

- compare with precise electron scattering data up to high momenta in Distorted Wave Born Approximation
- use intrinsic density

$$\rho(\mathbf{x}) = \sum_{k=1}^{A} \langle \psi | \delta(\mathbf{x}_k - \mathbf{x} - \mathbf{X}) | \psi \rangle$$

Cluster States in $^{12}$C

Important Configurations

- Calculate the overlap with FMD basis states to find the most important contributions to the Hoyle state

\[
\begin{align*}
\langle \cdot | 0^+_1 \rangle &= 0.94 \\
\langle \cdot | 2^+_1 \rangle &= 0.93 \\
\langle \cdot | 0^+_2 \rangle &= 0.72 \\
\langle \cdot | 0^+_2 \rangle &= 0.71 \\
\langle \cdot | 0^+_2 \rangle &= 0.61 \\
\langle \cdot | 0^+_2 \rangle &= 0.61 \\
\langle \cdot | 3^-_1 \rangle &= 0.83 \\
\langle \cdot | 0^+_3 \rangle &= 0.50 \\
\langle \cdot | 0^+_3 \rangle &= 0.49 \\
\langle \cdot | 0^+_3 \rangle &= 0.44 \\
\langle \cdot | 0^+_3 \rangle &= 0.41
\end{align*}
\]

FMD basis states are not orthogonal!

loosely bound, gas-like states
Calculate the overlap of FMD wave functions with pure $\alpha$-cluster model space

$$N_{\alpha} = \langle \psi \mid P_{3\alpha} \mid \psi \rangle$$

- $2_2^+$
- $0_3^+$
- $3_1^-$
- $0_2^+$
- $2_1^+$
- $0_1^+$

Hoyle state has 15% non-alpha admixtures
Cluster States in $^{12}\text{C}$
Harmonic Oscillator $N\hbar\Omega$ Excitations


$$\text{Occ}(N) = \langle \psi \mid \delta \left( \sum_i \left( H_i^{\text{HO}} / \hbar\Omega - 3/2 \right) - N \right) \mid \psi \rangle$$
Cluster States in $^{12}$C
Harmonic Oscillator $N\hbar \Omega$ Excitations


$$\text{Occ}(N) = \langle \psi | \delta \left( \sum_i (H_i^{HO}/\hbar \Omega - 3/2) - N \right) | \psi \rangle$$

Cluster Model

0\textsubscript{1}\textsuperscript{+}

2\textsubscript{1}\textsuperscript{+}

3\textsubscript{1}\textsuperscript{-}

0\textsubscript{2}\textsuperscript{+}

0\textsubscript{3}\textsuperscript{+}

2\textsubscript{2}\textsuperscript{+}
Cluster States in $^{12}$C

$\alpha$-cluster states in the No-Core Shell Model?

- compare spectra in NCSM and $\alpha$-cluster model using the Volkov interaction
- bare interaction used in NCSM calculations
  - good agreement for ground state band ($0^+_1$, $2^+_1$, $4^+_1$)
  - very slow convergence for cluster states

### Binding energies

<table>
<thead>
<tr>
<th></th>
<th>$^4$He</th>
<th>$^{12}$C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cluster</td>
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<td>-89.6 MeV</td>
</tr>
<tr>
<td>NCSM</td>
<td>-28.3 MeV</td>
<td>-95.4 MeV</td>
</tr>
</tbody>
</table>

![Graph showing energy levels](image-url)
Cluster States in $^{12}$C

$\alpha$-cluster states in the No-Core Shell Model?

Three-body forces do not help!

**Advantages/Disadvantages of FMD approach**

**FMD vs ab initio**

**Advantages**
- Basis very flexible, clusters and halo structure can be described
- Can be used for light $p/sd$-shell nuclei
- Many observables can be calculated
- Intrinsic states provide an “intuitive” picture of the nucleus

**Disadvantages**
- Interaction has to be soft and given in operator representation
- Does not provide “exact” results for given interaction, not straightforward to check convergence by “increasing model space size”

**FMD vs few-body models**

**Advantages**
- Microscopic - antisymmetrization
- Cluster structure appears naturally, includes polarization effects
- Uses nucleon-nucleon interaction, no need for phenomenological potentials

**Disadvantages**
- Numerical effort, “exact” calculations are not possible
- Not possible to adjust thresholds “by hand”
- Much more difficult to include boundary conditions for resonance or scattering states
Thanks

************

to my Collaborators

S. Bacca, A. Cribeiro, R. Cussons, H. Feldmeier, P. J. Ginsel, B. Hellwig, K. Langanke, R. Torabi, D. Weber

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