QCD at nonzero chemical potential and the sign problem

INT lectures 2012

V: complex Langevin dynamics

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Where are we?

complex weight:

- straightforward importance sampling not possible
- overlap problem

various possibilities:

- preserve overlap as best as possible
- use approximate methods at small $\mu$
- do something radical:
  - rewrite partition function in other dof
  - explore field space in a different way
  - ...
Overlap problem

- configurations differ in an essential way from those obtained at $\mu = 0$ or with $|\det M|$.

- cancelation between configurations with ‘positive’ and ‘negative’ weight.

![Graph of $\Re e(\rho(x))$ versus $x$.](image-url)
Complex integrals

- consider simple integral

\[ Z(a, b) = \int_{-\infty}^{\infty} dx \, e^{-S(x)} \quad S(x) = ax^2 + ibx \]

- complete the square/saddle point approximation: into complex plane

- lesson: don’t be real(istic), be more imaginative

radically different approach:

- complexify all degrees of freedom \( x \rightarrow z = x + iy \)

- enlarged complexified space

- new directions to explore
Complexified field space

dominant configurations in the path integral?

real and positive distribution $P(x, y)$: how to obtain it?

⇒ solution of stochastic process

complex Langevin dynamics

Parisi 83, Klauder 83
Gaussian integral

consider complex Gaussian integral

\[
Z(a, b) = \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2} ax^2 - ibx} = \sqrt{\frac{2\pi}{a}} e^{-\frac{1}{2} b^2 / a}
\]

complex action \( S^*(b) = S(-b^*) \) [assume \( a > 0 \) and real]

phase quenched theory

\[
Z_{pq} = \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2} ax^2} = Z(a, 0) = \sqrt{\frac{2\pi}{a}}
\]

sign problem: average phase factor

\[
\langle e^{-ibx} \rangle_{pq} = \frac{Z(a, b)}{Z(a, 0)} = e^{-\frac{1}{2} b^2 / a}
\]
Gaussian integral

average phase factor: one degree of freedom only

\[ \langle e^{-ibx} \rangle_{pq} = \frac{Z(a, b)}{Z(a, 0)} = e^{-\frac{1}{2}b^2/a} \]

sign problem only bad when \( b \) gets large

for \( N \) degrees of freedom \( x_j, j = 1, \ldots, N \)

\[ \langle e^{-ib\sum_j x_j} \rangle_{pq} = e^{-\frac{1}{2}Nb^2/a} \]

limits \( b \to 0, N \to \infty \) do not commute

severe sign problem for all \( b \neq 0 \) in \( N \to \infty \) limit

mimicks nonzero \( \mu \) problem
Gaussian integral

\[ Z(a, b) = \int dx \, e^{-\frac{1}{2}ax^2 - ibx} \]

\[ \langle x^2 \rangle = -2 \frac{\partial \ln Z}{\partial a} = \frac{a - b^2}{a^2} \]

goal: compute numerically without importance sampling

first take \( b = 0 \):

- use analogy with Brownian motion

  particle in a fluid: friction \((a)\) and kicks \((\eta)\)

- Langevin equation

  \[ \frac{d}{dt} x(t) = -ax(t) + \eta(t) \]
  \[ \langle \eta(t)\eta(t') \rangle = 2\delta(t - t') \]

\text{Parisi & Wu 81}
Gaussian integral

- **Langevin equation**
  \[ \dot{x}(t) = -ax(t) + \eta(t) \]

- **analytical solution**
  \[ x(t) = e^{-at}x(0) + \int_{0}^{t} ds \eta(s) e^{-a(t-s)} \]

- **correlator** [take \( x(0) = 0 \), no i.c. dependence]
  \[ \langle x^2(t) \rangle = \int_{0}^{t} ds \int_{0}^{t} ds' \langle \eta(s)\eta(s') \rangle e^{-a(2t-s-s')} \]

- Noise averaged correlator, use \( \langle \eta(s)\eta(s') \rangle = 2\delta(s - s') \)
  \[ \lim_{t \to \infty} \langle x^2(t) \rangle = \frac{1}{a} \]

- No importance sampling, solution of stochastic process
Fokker-Planck equation

- associated distribution $\rho(x, t)$

$$\langle O(x(t)) \rangle_\eta = \int dx \, \rho(x, t) O(x)$$

noise average distribution average

- Langevin eq for $x(t)$ $\iff$ Fokker-Planck eq for $\rho(x, t)$

$$\dot{\rho}(x, t) = \partial_x \left( \partial_x + S'(x) \right) \rho(x, t)$$

- stationary solution: $\rho(x) \sim e^{-S(x)}$

review: Damgaard & Hüffel 87
Fokker-Planck equation

- stationary solution typically reached exponentially fast

\[ \dot{\rho}(x, t) = \partial_x \left( \partial_x + S'(x) \right) \rho(x, t) \]

- write \( \rho(x, t) = \psi(x, t)e^{-\frac{1}{2}S(x)} \)

\[ \dot{\psi}(x, t) = -H_{FP}\psi(x, t) \]

- Fokker-Planck hamiltonian:

\[
H_{FP} = Q^\dagger Q = \begin{bmatrix} -\partial_x + \frac{1}{2}S'(x) \end{bmatrix} \begin{bmatrix} \partial_x + \frac{1}{2}S'(x) \end{bmatrix} \geq 0
\]

\[ Q\psi(x) = 0 \quad \iff \quad \psi(x) \sim e^{-\frac{1}{2}S(x)} \]

\[ \psi(x, t) = c_0e^{-\frac{1}{2}S(x)} + \sum_{\lambda>0} c_\lambda e^{-\lambda t} \to c_0e^{-\frac{1}{2}S(x)} \]
Complex Gaussian integral

\[ Z(a, b) = \int dx \, e^{-S(x)} \quad S(x) = \frac{1}{2} ax^2 + ibx \]

\( b \neq 0 \):

- analytically: complete the square shift in the complex plane \( x \rightarrow x + \frac{ib}{a} \)
- achieve the same with Langevin equation “complexify” \( x \rightarrow z = x + iy \)

\[ \dot{x} = -\text{Re} \, \partial_z S(z) + \eta = -ax + \eta \]
\[ \dot{y} = -\text{Im} \, \partial_z S(z) = -ay - b \]

with \( S(z) = S(x + iy) \)
Complex Gaussian integral

solution: \[ x(t) = x(0)e^{-at} + \int_0^t ds \ e^{-a(t-s)} \eta(s) \]
\[ y(t) = [y(0) + b/a]e^{-at} - b/a \]

correlators:
\[ \langle x^2(t) \rangle = x^2(0)e^{-2at} + (1 - e^{-2at}) / a \to 1/a \]
\[ \langle x(t)y(t) \rangle = x(0)e^{-at} ([y(0) + b/a]e^{-at} - b/a) \to 0 \]
\[ \langle y^2(t) \rangle = ([y(0) + b/a]e^{-at} - b/a)^2 \to b^2 / a^2 \]

combination \( x \to x + iy \):
\[ \lim_{t \to \infty} \langle [x(t) + iy(t)]^2 \rangle = \langle x^2 - y^2 + 2ixy \rangle = \frac{1}{a} - \frac{b^2}{a^2} = \frac{a - b^2}{a^2} \]
correct!
Distribution

associated distribution $P(x, y; t)$ in complex plane

- real and positive distribution (if it exists)

$$\langle O(x + iy)(t) \rangle = \int dx dy P(x, y; t)O(x + iy)$$

Langevin eq for $x(t)$ and $y(t)$

Fokker-Planck eq for $P(x, y; t)$

Fokker-Planck equation:

$$\dot{P}(x, y; t) = \left[ \partial_x (\partial_x + \text{Re} \partial_z S) + \partial_y \text{Im} \partial_z S \right] P(x, y; t)$$

- solvable in Gaussian models (like here)
- no generic solutions known
- no semi-positive Fokker-Planck hamiltonian (in contrast to real Langevin/action)
Distribution

distribution $P(x, y)$ in the complex plane

$\begin{align*}
\text{shift in the complex plane:} & \quad y \rightarrow -\frac{b}{a} \\
\text{Langevin process} & \quad \text{“finds” distribution:} \\
& \quad P(x, y) \sim e^{-ax^2/2} \delta(y + \frac{b}{a})
\end{align*}$
More interesting Gaussian integral

final Gaussian example:

- \( S = \frac{1}{2} (a + ib) x^2 \)
- \( \langle x^2 \rangle = \frac{1}{a + ib} \)

- coupled Langevin equations

\[
\dot{x} = -ax + by + \eta \\
\dot{y} = -ay - bx
\]

- solve and find correlators when \( t \to \infty \)

\[
\langle x^2 \rangle = \frac{1}{2a} \frac{2a^2 + b^2}{a^2 + b^2} \\
\langle y^2 \rangle = \frac{1}{2a} \frac{b^2}{a^2 + b^2} \\
\langle xy \rangle = -\frac{1}{2} \frac{b}{a^2 + b^2}
\]

- correlator \( \langle z^2 \rangle = \langle x^2 - y^2 + 2i xy \rangle = \frac{a - ib}{a^2 + b^2} = \frac{1}{a + ib} \)

correct!
More interesting Gaussian integral

distribution \( P(x, y) \) in the complex plane

\[ b = 0.01 \quad b = 1 \quad b = 10 \]

Langevin process “finds” this distribution

original weight \( e^{-S} \) is complex

this distribution is real and positive
Equilibrium distributions

complex weight $\rho(x)$        real weight $P(x, y)$

main premise:

$$\int dx \rho(x)O(x) = \int dxdy P(x, y)O(x + iy)$$

if equilibrium distribution $P(x, y)$ is known analytically:

shift variables

$$\int dxdy P(x, y)O(x + iy) = \int dx O(x) \int dy P(x - iy, y)$$

$$\Rightarrow \rho(x) = \int dy P(x - iy, y)$$

correct in Gaussian examples

hard to verify in numerical studies!
Discretization

most cases not analytically solvable
numerical solution of Langevin equation

- discretize stochastic equation (Itô calculus)

\[
\begin{align*}
x_{n+1} & = x_n + \epsilon K_n^R + \sqrt{\epsilon} \eta_n \\
y_{n+1} & = y_n + \epsilon K_n^I
\end{align*}
\]

- drift terms

\[
K_n^R = -\text{Re} \frac{\partial S}{\partial z} \quad \quad \quad \quad K_n^I = -\text{Im} \frac{\partial S}{\partial z}
\]

- noise

\[
\langle \eta_n \eta_{n'} \rangle = \delta_{nn'}
\]

- use adaptive stepsize if necessary
Stochastic quantization

adapt to field theory

Parisi & Wu 81, Parisi, Klauder 83

- path integral \[ Z = \int D\phi e^{-S} \]

- Langevin dynamics in “fifth” time direction

\[ \frac{\partial \phi(x, t)}{\partial t} = -\frac{\delta S[\phi]}{\delta \phi(x, t)} + \eta(x, t) \]

- Gaussian noise

\[ \langle \eta(x, t) \rangle = 0 \quad \langle \eta(x, t)\eta(x', t') \rangle = 2\delta(x - x')\delta(t - t') \]

- compute expectation values \( \langle \phi(x, t)\phi(x', t) \rangle \), etc

- study converge as \( t \to \infty \)
Phase transitions and the Silver Blaze

can complex Langevin dynamics handle:

- a severe sign problem?
- the thermodynamic limit?
- phase transitions?
- the Silver Blaze problem?  

...  

study in a model with a phase diagram with similar features as QCD at low temperature

⇒ relativistic Bose gas at nonzero $\mu$
Relativistic Bose gas at nonzero $\mu$

- scalar O(2) model with global symmetry

- continuum action

\[
S = \int d^4x \left[ |\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu (\phi^* \partial_4 \phi - \partial_4 \phi^* \phi) + \lambda |\phi|^4 \right]
\]

- complex scalar field, $d = 4$, $m^2 > 0$

- $S^*(\mu) = S(-\mu^*)$ as in QCD
Relativistic Bose gas at nonzero $\mu$

- scalar O(2) model with global symmetry
- lattice action

$$S = \sum_x \left[ \left( 2d + m^2 \right) \phi_x^* \phi_x + \lambda (\phi_x^* \phi_x)^2 
- \sum_{\nu=1}^{4} \left( \phi_x^* e^{-\mu \delta_{\nu,4}} \phi_{x+\hat{\nu}} + \phi_{x+\hat{\nu}} e^{\mu \delta_{\nu,4}} \phi_x \right) \right]$$

- complex scalar field, $d = 4$, $m^2 > 0$

$$S^*(\mu) = S(-\mu^*)$$ as in QCD
Relativistic Bose gas at nonzero $\mu$

tree level potential in the continuum

$$V(\phi) = (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4$$

condensation when $\mu^2 > m^2$, SSB

when $T = 0$ and $\mu < \mu_c$:

$\mu$ independence

Silver Blaze problem

$\langle \phi \rangle = 0$

$\langle \phi \rangle \neq 0$
Relativistic Bose gas at nonzero $\mu$

- write $\phi = (\phi_1 + i\phi_2)/\sqrt{2} \Rightarrow \phi_a$ ($a = 1, 2$)
- complexification $\phi_a \rightarrow \phi^R_a + i\phi^I_a$
- complex Langevin equations

$$\frac{\partial \phi^R_a}{\partial t} = -\text{Re}\frac{\delta S}{\delta \phi_a} \bigg|_{\phi_a \rightarrow \phi^R_a + i\phi^I_a} + \eta_a$$

$$\frac{\partial \phi^I_a}{\partial t} = -\text{Im}\frac{\delta S}{\delta \phi_a} \bigg|_{\phi_a \rightarrow \phi^R_a + i\phi^I_a}$$

- straightforward to solve numerically, $m = \lambda = 1$
- lattices of size $N^4$, with $N = 4, 6, 8, 10$
Relativistic Bose gas

Field modulus squared

\[ |\phi|^2 \rightarrow \frac{1}{2} \left( \phi_R^2 - \phi_I^2 \right) + i\phi_R \phi_I \]

Silver Blaze!
Relativistic Bose gas

Field modulus squared

\[ |\phi|^2 \rightarrow \frac{1}{2} \left( \phi_a^R R^2 - \phi_a^I I^2 \right) + i \phi_a^R \phi_a^I \]

Second order phase transition in thermodynamic limit
Relativistic Bose gas

Density \( \langle n \rangle = \frac{1}{\Omega} \frac{\partial \ln Z}{\partial \mu} \)
Relativistic Bose gas

density $\langle n \rangle = (1/\Omega) \partial \ln Z / \partial \mu$

second order phase transition in thermodynamic limit
Silver Blaze and the sign problem

Silver Blaze and sign problems are intimately related

- phase quenched theory $Z_{pq} = \int D\phi |e^{-S}|$

physics of phase quenched theory:

- chemical potential appears only in mass parameter (in continuum notation)

$$V = (m^2 - \mu^2) \phi^2 + \lambda \phi^4$$

- dynamics of symmetry breaking, no Silver Blaze

in QCD: phase quenched = finite isospin
onset at $\mu = m_\pi/2$ instead of $m_B/3$
Silver Blaze and the sign problem

density

\[ \text{complex quenched} \]

\[ e^{i\varphi} = \frac{e^{-S}}{|e^{-S}|} \] does precisely what is expected
How severe is the sign problem?

- complex action \( e^{-S} = |e^{-S}|e^{i\varphi} \)
- average phase factor in phase quenched theory

\[
\langle e^{i\varphi} \rangle_{pq} = \frac{Z_{\text{full}}}{Z_{pq}}
\]

\( = e^{-\Omega \Delta f} \rightarrow 0 \)

as \( \Omega \rightarrow \infty \)

- exponentially hard in thermodynamic limit
Lattice gauge theory

- partition function

\[ Z = \int DU \, e^{-S_B} \det M \]

- \( M \) is the fermion matrix

- fermion determinant is complex

\[ [\det M(\mu)]^* = \det M(-\mu^*) \]
SU(3) lattice gauge theory

Langevin update for link variables $U_{x\nu}$:

$$U_{x\nu}(t+\epsilon) = R_{x\nu}(t) U_{x\nu}(t) \quad R_{x\nu} = \exp \left[ i \lambda_a \left( \epsilon K_{x\nu a} + \sqrt{\epsilon} \eta_{x\nu a} \right) \right]$$

- drift term

$$K_{x\nu a} = -D_{x\nu a} S_{\text{eff}}[U] \quad S_{\text{eff}} = S_B + S_F \quad S_F = -\ln \det M$$

- noise

$$\langle \eta_{x\nu a} \rangle = 0 \quad \langle \eta_{x\nu a} \eta_{x'\nu'a} \rangle = 2 \delta_{xx'} \delta_{\nu\nu'} \delta_{aa'}$$

real action: $\Rightarrow K^\dagger = K \Leftrightarrow R^\dagger R = 1 \Leftrightarrow U \in \text{SU}(3)$

complex action: $\Rightarrow K^\dagger \neq K \Leftrightarrow R^\dagger R \neq 1 \Leftrightarrow U \in \text{SL}(3, \mathbb{C})$
Heavy dense QCD

- **Bosonic action:** Standard SU(3) Wilson action

\[ S_B = -\beta \sum_P \left( \frac{1}{6} [\text{Tr} \ U_P + \text{Tr} \ U_P^{-1}] - 1 \right) \]

- **Determinant** \( \text{det} \ M \) for Wilson fermions

**Fermion matrix:**

\[ M = 1 - \kappa \sum_{i=1}^{3} \text{space} - \kappa \left( e^\mu \Gamma_+ U_{x,4} T_4 + e^{-\mu} \Gamma_- U_{x,4}^{-1} T_4 \right) \]
Heavy dense QCD

- hopping expansion:

\[
det M \approx det \left[ 1 - \kappa \left( e^{\mu \Gamma_{+4} U_{x,4} T_4} + e^{-\mu \Gamma_{-4} U_{x,4}^{-1} T_4} \right) \right]
\]

\[
= \prod_x det \left( 1 + h e^{\mu/T} P_x \right)^2 det \left( 1 + h e^{-\mu/T} P_x^{-1} \right)^2
\]

with \( h = (2\kappa)^{N_\tau} \) and (conjugate) Polyakov loops \( P_x^{(-1)} \)

- static quarks propagate in temporal direction only: Polyakov loops

- full gauge dynamics included

GA & Stamatescu 08
first results on $4^4$ lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$

low-density phase $\Rightarrow$ high-density phase
(conjugate) Polyakov loops

results on $4^4$ lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$

low-density “confining” phase $\Rightarrow$ high-density “deconfining” phase
SU(3) $\rightarrow$ SL(3, $\mathbb{C}$)

- complex Langevin dynamics: no longer in SU(3)
- instead $U \in$ SL(3, $\mathbb{C}$)
- in terms of gauge potentials $U = e^{i\lambda_a A_a/2}$
- $A_a$ is now complex
- how far from SU(3)?

consider

$$\frac{1}{N} \text{Tr} \; U^\dagger U \left\{ \begin{array}{ll}
= 1 & \text{if } U \in \text{SU}(N) \\
\geq 1 & \text{if } U \in \text{SL}(N, \mathbb{C})
\end{array} \right.$$
\[ \frac{1}{3} \text{Tr} \ U^\dagger U \geq 1 \quad = 1 \quad \text{if} \quad U \in \text{SU}(3) \]
One-dimensional QCD

- exactly solvable
  - Gibbs 86, Bilic & Demeterfi 88

- phase quenched: transition at $\mu = \mu_c$, full: no transition

\[
\text{severe sign problem when } |\mu| > |\mu_c|
\]

- chiral condensate:
  - write as integral over spectral density

\[
\Sigma = \int d^2 z \frac{\rho(z; \mu)}{z + m}
\]

\[
\mu_c = \text{arcsinh } m
\]

- $\rho(z; \mu)$ complex and oscillatory
  - Ravagli & Verbaarschot 07

- condensate independent of $\mu$: Silver Blaze

- solve with complex Langevin
  - GA & Splittorff 10
One-dimensional QCD

- exact results reproduced
- discontinuity at $\mu_c = 0$ in thermodynamic limit $n \to \infty$

\[ \mu_c = \text{arcsinh } m \]

- sign problem severe when $|\mu_c| < |\mu|$
- condensate independent of $\mu$: Silver Blaze
One-dimensional QCD

elegant analytical solution in thermodynamic limit:

- **original distribution:**

  \[ \rho(x) \sim e^{n(\mu - \mu_c)} e^{inx} \]

  when \( n \to \infty \)

- **real distribution sampled by complex Langevin:**

\[
P(x, y) = \begin{cases} 
1 & \mu - \mu_c < y < \mu + \mu_c \\
0 & \text{elsewhere} 
\end{cases}
\]
Troubled past

1. numerical problems: runaways, instabilities
   ⇒ adaptive stepsize
   no instabilities observed, works for SU(3) gauge theory
   GA, James, Seiler & Stamatescu 09
   a la Ambjorn et al 86

2. theoretical status unclear
   ⇒ detailed analysis, identified necessary conditions
   GA, FJ, ES & IOS 09-12

3. convergence to wrong limit
   ⇒ better understood but not yet resolved
   in progress
Instabilities: heavy dense QCD

Adaptive time step during the evolution

Occasionally very small stepsize required
Can go to longer Langevin times without problems
Analytical understanding

consider expectation values and Fokker-Planck equations

one degree of freedom $x$, complex action $S(x)$, $\rho(x) \sim e^{-S(x)}$

• wanted: $\langle O(x, t) \rangle_\rho = \int dx \, \rho(x, t)O(x)$

\[
\partial_t \rho(x, t) = \partial_x \left( \partial_x + S'(x) \right) \rho(x, t)
\]

• solved with CLE:

\[
\langle O(x + iy, t) \rangle_P = \int dx dy \, P(x, y; t)O(x + iy)
\]

\[
\partial_t P(x, y; t) = \left[ \partial_x \left( \partial_x - K_x \right) - \partial_y K_y \right] P(x, y; t)
\]

with $K_x = -\text{Re}S'$, $K_y = -\text{Im}S'$

• question: $\langle O(x + iy, t) \rangle_P = \langle O(x, t) \rangle_\rho$ ?
Analytical understanding

question: $\langle O(x + iy, t) \rangle_P = \langle O(x, t) \rangle_\rho$ as $t \to \infty$?

answer: yes, provided some conditions are met:

- distribution $P(x, y)$ should drop off fast enough in $y$ direction
- partial integration without boundary terms possible
- actually $O(x + iy)P(x, y)$ for large enough set $O(x)$

⇒ distribution should be sufficiently localized

- can be tested numerically via criteria for correctness

$$\langle LO(x + iy) \rangle = 0$$

with $L$ Langevin operator
apply these ideas to 3D SU(3) spin model

- earlier solved with complex Langevin
  - Karsch & Wyld 85
  - Bilic, Gausterer & Sanielevici 88
  - however, no detailed tests performed

⇒ test reliability of complex Langevin using developed tools

- analyticity in $\mu^2$:
  - from imaginary to real $\mu$
  - Taylor series

- criteria for correctness

- comparison with flux formulation

Gattringer & Mercado 12
SU(3) spin model

3-dimensional SU(3) spin model: \( S = S_B + S_F \)

\[
S_B = -\beta \sum_{<xy>} \left[ P_x P_y^* + P_x^* P_y \right]
\]

\[
S_F = -h \sum_x \left[ e^{\mu} P_x + e^{-\mu} P_x^* \right]
\]

- SU(3) matrices: \( P_x = \text{Tr} \, U_x \)
- gauge action: nearest neighbour Polyakov loops
- (static) quarks represented by Polyakov loops
- complex action \( S_F^*(\mu) = S_F(-\mu^*) \)

effective model for QCD with static quarks, centre symmetry
SU(3) spin model

- phase structure

- effective model for QCD with static quarks
SU(3) spin model

real and imaginary potential:
first-order transition in $\beta - \mu^2$ plane, $\langle P + P^* \rangle / 2$

negative $\mu^2$: real Langevin — positive $\mu^2$: complex Langevin
SU(3) spin model

real chemical potential

immediate splitting between $\langle P \rangle$ and $\langle P^* \rangle$: no Silver Blaze

\[ \beta = 0.125, \ h = 0.02, \ 10^3 \]

inset: lines from first-order Taylor expansion
SU(3) spin model

stepsizdependence
left: $\langle P \rangle$ (top) and $\langle P^* \rangle$ (bottom) at $\mu = 3$
right: criteria for correctness $\langle LO \rangle = 0$

improved stepsize algorithm to eliminate linear dependence
criteria satisfied as stepsize $\epsilon \to 0$
SU(3) spin model

comparison with result obtained using flux representation

- CL: finite stepsize errors in lowest-order algorithm
- improved algorithm removes discrepancy in critical region
SU(3) spin model

complex Langevin passes all the tests: why?

- localized distribution: fast decay in imaginary direction
- real manifold is stable under small fluctuations
- Haar measure plays essential role

⇒ Haar measure contribution to complex drift restoring
Stabilizing drift

- Haar measure contribution to complex drift restoring
- controlled exploration of the complex field space

employ this: generate Jacobian by field redefinition

\[ Z = \int dx \ e^{-S(x)} \quad x = x(u) \quad J(u) = \frac{\partial x(u)}{\partial u} \]

\[ = \int du \ e^{-S_{\text{eff}}(u)} \quad S_{\text{eff}}(u) = S(u) - \ln J(u) \]

\[ \text{drift:} \quad K(u) = -S_{\text{eff}}'(u) = -S'(u) + \frac{J'(u)}{J(u)} \]

which field redefinition?

singular at \( J(u) = 0 \) but restoring in complex plane
Fun with complex Langevin

Gaussian example: defined when \( \text{Re}(\sigma) = a > 0 \)

\[
Z = \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2} \sigma x^2} \quad \sigma = a + ib \quad \langle x^2 \rangle = \frac{1}{\sigma}
\]

what if \( a < 0 \)? flow in complex space for \( a = -1, b = 1 \):

left: highly unstable right: after transformation

attractive fixed points

left: highly unstable

right: after transformation \( x(u) = u^3 \)
Fun with complex Langevin

do CLE in the $u$ formulation and compute $\langle x^2 \rangle = \langle u^6 \rangle$

\[ \langle x^2 \rangle = \frac{1}{\sigma} = \frac{a - ib}{a^2 + b^2} \]

take also negative $a$

CLE finds the analytically continued answer to negative $a$!

clearly needs more exploration — potential for stabilization — affects convergence
XY model

three-dimensional XY model at nonzero $\mu$

$$S = -\beta \sum_{x} \sum_{\nu=0}^{2} \cos (\phi_x - \phi_{x+\nu} - i\mu\delta_{\nu,0})$$

- $\mu$ couples to the conserved Noether charge
- symmetry $S^*(\mu) = S(-\mu^*)$

unexpectedly difficult to simulate with complex Langevin!

numerics shares many features with heavy dense QCD

also studied by Banerjee & Chandrasekharan using worldline formulation

hep-lat/1001.3648
Convergence: XY model

- comparison with known result (world line formulation)
- analytic continuation from imaginary $\mu = i\mu_1$

real $\mu$, complex action:

$$S = -\beta \sum_{x} \sum_{\nu=0}^{2} \cos (\phi_x - \phi_{x+\hat{\nu}} - i\mu\delta_{\nu,0})$$

imaginary $\mu = i\mu_1$, real action:

$$S_I = -\beta \sum_{x} \sum_{\nu=0}^{2} \cos (\phi_x - \phi_{x+\hat{\nu}} + \mu_1\delta_{\nu,0})$$

- real and imag $\mu$ results analytic in $\mu^2$
Convergence: XY model

- comparison with known result (world line formulation)
- analytic continuation from imaginary $\mu = i\mu_I$

![Graph showing action density versus $\mu^2$]

- action density versus $\mu^2$
- $\beta = 0.7$
- ordered phase

- "Roberge-Weiss" transition at $\mu_I = \pi/N_T$
Convergence: XY model

- comparison with known result (world line formulation)
- analytic continuation from imaginary $\mu = i\mu_1$

![Graph showing action density versus $\mu^2$ for different values of $\beta$. The graph features lines labeled complex Langevin, real Langevin, and world line, with a note on Silver Blaze feature at small $\beta$ and $\mu$.]

action density versus $\mu^2$

$\beta = 0.3$

disordered phase

Silver Blaze feature at small $\beta$ and $\mu$
Convergence: XY model

XY model

- comparison with known result (world line formulation)

phase diagram:

relative deviation:

$$\Delta S = \frac{\langle S \rangle_{\text{cl}} - \langle S \rangle_{\text{wl}}}{\langle S \rangle_{\text{wl}}}$$

high $\beta$: ordered

low $\beta$: disordered

- phase boundary from Banerjee & Chandrasekharan

- highly correlated with ordered/disordered phase
Convergence: XY model

- apparent correct results in the ordered phase
- incorrect result in the disordered/transition region

diagnostics:

- distribution $P[\phi_R, \phi_I]$ qualitatively different
- classical force distribution qualitatively different
- complexified dynamics $\neq$ real dynamics when $\mu = 0$

but:

- independent of strength of the sign problem

conclusion: failure not due to sign problem
Summary

many stimulating results: examples where complex Langevin can handle

- sign problem
- Silver Blaze problem
- phase transition
- thermodynamic limit

problems from the 80s:

- instabilities and runaways $\rightarrow$ adaptive stepsize
- convergence: correct result not guaranteed

resolution in progress, important:

- failure does not depend on strength of sign problem
- distinct from all other approaches
Outlook

QCD at nonzero $\mu$ ⇔ sign problem

relevant for QCD phase diagram, heavy-ion collisions, dense objects, . . .

- sign problem has been studied from many perspectives
- ‘well understood’ (overlap, Silver Blaze, . . .)
- no solution for QCD (yet . . .)

sign problem appears not only in QCD

also in many (lower-dimensional, condensed matter) theories

⇒ learn from those models as well
Outlook

some approaches with limited applicability in full QCD:

- overlap preserving reweighting
- Taylor series
- imaginary $\mu$ and analytical continuation
- …

partial or full solutions in not quite QCD:

- strong coupling QCD
- flux representations in spin models (not discussed)
- complex Langevin
- …
Outlook

to do (possible):

determine QCD phase diagram for

- imaginary chemical potential
- isospin chemical potential

⇒ no sign problem

⇒ large-scale numerical project

⇒ intricate phase structure depending on quark masses
Outlook

to do (possible?):
solve sign problem
don’t give up!