High Temperature and Density in Lattice QCD:
Strong-coupling, high temperature limit and the Potts model paradigm

http://www.physics.utah.edu/~detar/int12/

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Why study high $T$ and high density QCD?

- Early universe
- Dense stars
Why study high $T$ and high density QCD?

- Heavy ion collisions
- Intrinsic field theory interest

[Thanks CERN image]
Phase structure in $T$ and $\mu$

- Confinement lost at high temperature or density
- “Quark-gluon plasma”
- Phase transition or crossover?
- Speculative phase diagram
Phase structure in $m_{ud}$ and $m_s$

- Whether there is a phase transition depends on quark masses
- Speculative phase diagram

![Diagram showing phase structure with $N_f = 2$, $N_f = 2+1$, $N_f = 3$, and $N_f = 1$.]

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Open questions

- What happens at high density?
- Does the critical end-point exist?
- Is it experimentally accessible?
- Is there a first order phase transition at very small nonzero $m_{ud}$?
Lattice QCD to the rescue!

- These are nonperturbative questions
- LQCD is the only nonperturbative *ab initio* method we have.
- BUT: Lattice QCD is based on equilibrium thermodynamics
- Heavy ion collisions are dynamical.
- Relevant only where thermal equilibrium is a good approximation.
- Good for dense stars, early universe, and some stages in heavy ion collisions.
Outline of lectures

1. Strong-coupling, high temperature limit and the Potts model paradigm.
2. Deconfining transition in QCD
3. Chiral symmetry restoration in QCD
4. Connection with phenomenology
Classic Wilson action

- Path integral: \( T = 1/(N_T a) \)

\[
Z_W = \text{Tr} \exp(-H/T) = \int [dU][d\psi d\bar{\psi}] \exp(-S)
\]

- \( S = S_G + S_F \)

\[
S_G = \frac{6}{g^2} \sum_{x, \mu < \nu} \left[ 1 - \text{Re} \text{Tr} U_P(x; \mu, \nu)/3 \right]
\]

\[
S_F = \sum_x \bar{\psi}(x)\psi(x)
\]

\[
- \kappa \sum_{x, \mu} \left[ \bar{\psi}(x)(1 + \gamma_\mu)U_\mu(x)\psi(x + \hat{\mu}) + \bar{\psi}(x + \hat{\mu})(1 - \gamma_\mu)U_\mu^\dagger(x)\psi(x) \right]
\]

- Conserved (Noether) current

\[
J_\mu(x) = \kappa \left[ \bar{\psi}(x)(1 + \gamma_\mu)U_\mu(x)\psi(x + \hat{\mu}) - \bar{\psi}(x + \hat{\mu})(1 - \gamma_\mu)U_\mu^\dagger(x)\psi(x) \right]
\]
External point-charge source

- Introduce an external point charge $g$ in the fundamental representation, moving along the world line $C$.
- Continuum representation

$$\delta S = -ig \oint_C \lambda^a A^\mu_a dx^\mu$$

- For gauge invariance $C$ must be closed.
- Lattice representation ($\gamma_{-\mu} = -\gamma_\mu; \ U_{-\mu}(x) = U_\mu^\dagger(x - \hat{\mu})$)

$$Z = \int [dU][d\psi d\bar{\psi}] \exp(-S)L_C$$

$$L_C = \text{Tr} \prod_{x,\mu \in C} (1 + \gamma_\mu) U_{x,\mu}$$
Static charge

- The static charge worldline $C$ is fixed at $x$, moving forward only in $\tau$.
- Product of time-like links, closing by periodicity in imaginary time:
  \[ L_C \propto \text{Tr} \prod_{\tau} U_{x,\tau;0} \]
- Called a “Polyakov loop” (also, sometimes “Wilson line”).
Gauge field at strong coupling, high $T$


- Taking soluble limits often provides insight into the workings of a theory.
- Anisotropic lattice ($a_t \neq a_s$)

\[
S_G = \frac{6a_s}{a_t g^2} \sum_{x,i} [1 - \text{Tr} \, U_P(x; 0, i)/3] + \frac{6a_t}{a_s g^2} \sum_{x,i>j} [1 - \text{Tr} \, U_P(x; i, j)/3]
\]

- High temperature: $N_t = 1$ so $a_t = 1/T$ and $a_t/a_s \ll 1$. Drop the space-space term.
- We have only $U(x, 0)$ and $U(x, i)$

\[
\text{Tr} \, U_P(x; 0, i) = \text{Tr} \, U_0(x, 0) U_i(x) U_0^\dagger(x + \hat{i}) U_i^\dagger(x)
\]

- The trace takes its maximum value of 3 when $U_0(x, 0) = z(x) I \in \mathbb{Z}(3)$, the center of $SU(3)$: $\{1, \exp(\pm 2\pi i/3)\}$. 
Gauge field at strong coupling, high $T$

- Approximate the integral over the gauge fields by a sum over $Z(3)$

$$Z = \int \prod_{x,\mu} [dU_\mu(x)] \exp(S_G) \rightarrow \sum_{z_x} \exp \left[ \frac{6a_s}{g^2a_t} \sum_{x,i} \text{Re} \left(z_x^*z_{x+i} \hat{i}\right) \right]$$

- The theory becomes the three-state, 3D Potts model.
- Global $Z(3)$ Symmetry: $z_x \rightarrow Yz_x$ for $Y \in Z(3)$. 
Gauge field at strong coupling, high $T$

- Again

\[ Z = \sum_{z_x} \exp \left[ \frac{6a_s}{g^2 a_t} \sum_{x,i} \text{Re} \left( z_x^* z_{x+i} \right) \right] \]

- In a spin system, we would replace $6a_s/(g^2 a_t) = J/T_{\text{Potts}}$
- So $T_{\text{Potts}} \propto g^2$ at fixed $a_t/a_s$.
- At fixed $a_t/a_s$ we vary $a_t = 1/T_{\text{QCD}}$ by varying $g^2$.
- As $g^2 \to 0$ asymptotic freedom says $a_t \to 0$ so $T_{\text{QCD}}$ increases while $T_{\text{Potts}}$ decreases.
- At low $T_{\text{Potts}}$ the spin system is in a ferromagnetic state.
- The order parameter is the magnetization $\langle z \rangle$.
- There is a first order magnetization phase transition. The ordered phase corresponds to high $T_{\text{QCD}}$.
- The order parameter corresponds to $\text{Tr} \, U_0(x)$.
- More generally, it is the “Polyakov loop”.

\[ L(x) = P \exp \left[ \int i g A_0(x, \tau) d\tau \right] \]
Chemical potential

- Conserved charges $Q_f$ are flavor number (or baryon number).
- Grand canonical ensemble

$$Z_W = \text{Tr} \exp \left( -\frac{H}{T} + \sum_f \mu_f \frac{Q_f}{T} \right)$$
Chemical potential

- Conserved charge density

\[ \rho_f(x) = \kappa [\bar{\psi}_f(x)(1+\gamma_0)U_0(x)\psi_f(x) + \hat{0}) - \bar{\psi}_f(x+\hat{0})(1-\gamma_0)U_0^\dagger(x)\psi_f(x)] \]

- So we add to the action

\[ \mu_f Q_f / T = \mu_f \int d\tau Q_f = \int d^4x \mu_f \rho_f(x) \]

- Note this term is just like the time-like kinetic term in the action except for a sign. We get a factor \((1 + a\mu)\) for forward hopping and \((1 - a\mu)\) for backward. It is more natural to use \(e^{\pm \mu a}\).

- So we replace

\[ \bar{\psi}(x)(1 + \gamma_0)U_0(x)\psi(x + \hat{0}) \rightarrow \bar{\psi}(x)(1 + \gamma_0)U_0(x)\psi(x + \hat{0})e^{\mu a} \]

\[ \bar{\psi}(x + \hat{0})(1 - \gamma_0)U_0^\dagger(x)\psi(x) \rightarrow \bar{\psi}(x + \hat{0})(1 - \gamma_0)U_0^\dagger(x)\psi(x)e^{-\mu a} \]

- Note the fermion determinant \(\det M(\mu)\) is not real now.
Chemical potentials and Monte Carlo

- At nonzero chemical potential $\text{det}[M(\mu)]$ is complex.
- Can’t be used as a Monte Carlo probability weight.
- Phase oscillations grow with the volume of the system $V$.
- Can’t take the thermodynamic limit $V \to \infty$. 

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Fermions at strong coupling, large mass, high $T$

- Wilson action. Anisotropic ($a_t \neq a_s$). Chemical potential $\mu$.

\[
S_F = \sum_x \bar{\psi}(x)\psi(x) \\
- \kappa \sum_x [\bar{\psi}(x)(1 + \gamma_0)U_0(x)e^{-\mu a_t}\psi(x + \hat{0}) + \bar{\psi}(x + \hat{0})(1 - \gamma_0)U_0^\dagger(x)e^{\mu a_t}\psi(x) + \bar{\psi}(x + \hat{i})(1 - \gamma_i)U_i^\dagger(x)\psi(x)]
\]

\[
\frac{1}{\kappa} = 6\frac{a_t}{a_s} + 2 + 2Ma_t
\]

- High temperature: $N_t = 1$ and $a_t/a_s = 1/(a_s T) \to 0$. Drop the space-like term.

- The fermion matrix is diagonal in space-time with values on each spatial site

\[
1 - \kappa(1 + \gamma_0)Ze^{-\mu a_t} - \kappa(1 - \gamma_0)Z^*e^{\mu a_t}
\]
For large mass $\Rightarrow$ small $\kappa$ the fermion determinant becomes

$$\exp \left[ h_0(\kappa, \mu) + h(\kappa, \mu) \sum_x \text{Re} z_x + ih'(\kappa, \mu) \sum_x \text{Im} z_x \right]$$

For small $\kappa$ we have

$$h(\kappa, \mu) \approx 24\kappa \cosh(a_t \mu) \quad h'(\kappa, \mu) \approx 24\kappa \sinh(a_t \mu)$$
The quark mass corresponds to an external real magnetic field. The chemical potential introduces an imaginary magnetic field

$$H = -J \sum_{x,i} \text{Re} \left( z_x^* z_{x+i} \right) - \sum_x [h \text{Re} z_x - ih' \text{Im} z_x]$$

for $h \approx 24\kappa \cosh(at)$ and $h' \approx 24\kappa \sinh(at)$.

The real external field weakens the phase transition and eventually destroys it.

The imaginary field further weakens it.
In mean field theory we consider the statistical mechanics of a single site, assuming that the neighbors of the site take on the same mean value. So for the Potts model we have a single-site partition function

\[ Z(\bar{z}) = \sum_z \exp\left[ -\frac{H(z, \bar{z})}{T_{\text{Potts}}} \right] \]

where the single-site \( H(z, \bar{z}) \) is obtained from the full \( H \) by setting all spins to the mean value \( \bar{z} \), except for one site, which carries variable spin \( z \).

We then impose self-consistency by calculating the output mean value of the spin on the single site and requiring that it equal the input mean value.

Do this for the 3D 3-state Potts model with \( h = h' = 0 \), and show that there is one real solution for low \( J/T \) and three real nonzero solutions for sufficiently high \( J/T \). (In the latter case, the middle one happens to be unstable.) Then show that the transition is first order.

Maple, Mathematica, or gnuplot can help with the numerics here.
Magnetization vs. inverse Potts temperature with external field $h$.

The first order phase transition disappears with increasing field.
3D flux tube model of QCD


- Sites: $Z(3)$ charges $n_x \in \{0, 1, -1\}$
- $+1 = $ quark; $-1 = $ antiquark.
- Links: $Z(3)$ flux $\ell_{x,i} \in \{0, 1, -1\}$
- Gauss’ Law
  \[
  \sum_i (\ell_{x,i} - \ell_{x,-i}) \mod 3 = n_x
  \]
- Hamiltonian
  \[
  H = \sigma \sum_{x,i} |\ell_{x,i}| + m \sum_x |n_x|.
  \]
Flux tube model at nonzero chemical potential


- Grand canonical partition function \( (N = \sum_x n_x) \)
  \[
  Z = \sum_{n_x,\ell_x,i} \exp\left[-(H - \mu N)/T\right]
  \]

- There is no complex phase problem at nonzero density here.
- The fluxtube model is equivalent to the \( Z(3) \) Potts model
- Not difficult to show. Use the \( Z(3) \) identity
  \[
  \frac{1}{3} \sum_z z^\ell = \delta_{\ell,0}
  \]
  to replace the Gauss’ Law Kronecker delta in the partition function.
- Lesson: changing from a field basis to a color singlet basis cures the complex phase problem in the high-\( T \) strong-coupling limit.
Flux tube model at nonzero chemical potential

- In the field basis the complex phase comes from the imbalance between forward time-like and backward time-like hopping, combined with the presence of complex time-like gauge links.

\[ \bar{\psi}(x)(1+\gamma_0)U_0(x)e^{-\mu at}\psi(x+\hat{0}) + \bar{\psi}(x+\hat{0})(1-\gamma_0)U_0^\dagger(x)e^{\mu at}\psi(x) \]

- Integration over the time-like links enforces Gauss' Law at each lattice site.

- Changing from the field basis to the hadron basis eliminates the complex phase.

- With $SU(3)$, it is much more difficult to formulate the path integral with a basis change because of the infinite number of $SU(3)$ irreps.

- Moreover, there will still be a fermion sign problem, just as with electrons in condensed matter physics.

- Finally, strong coupling, large mass doesn’t capture chiral symmetry.