Independent Particle Model:

1 particle (or hole)
outside closed shell
(very few nuclei)

Multi-particle systems

Recall mean field approximation

\[ H = H_{\text{IPM}} + H_{\text{Residual}} \]

Residual Interactions

Effects not included in independent particle model potential
Residual Interactions

Need to consider a more complete Hamiltonian:

\[ H = H_0 + H_{\text{residual}} \]

\( H_{\text{residual}} \) reflects interactions not in the single particle potential.

**NOT** a minor perturbation. In fact, these residual interactions determine almost everything we know about most nuclei.

Start with 2- particle system, that is, a nucleus “doubly magic + 2”.

\( H_{\text{residual}} \) is \( H_{12}(r_{12}) \)

Consider two identical valence nucleons with \( j_1 \) and \( j_2 \).

Two questions: What total angular momenta \( j_1 + j_2 = J \) can be formed? What are the energies of states with these \( J \) values?
Coupling of two angular momenta

\[ j_1 + j_2 \quad \text{All values from:} \quad j_1 - j_2 \text{ to } j_1 + j_2 \quad (j_1 \neq j_2) \]

Example: \( j_1 = 3, j_2 = 5 \): \( J = 2, 3, 4, 5, 6, 7, 8 \)

\textbf{BUT:} For \( j_1 = j_2 \): \( J = 0, 2, 4, 6, \ldots (2j - 1) \) \quad \text{(Why these?)}
How can we know which total J values are obtained for the coupling of two identical nucleons in the same orbit with total angular momentum j? Several methods: easiest is the “m-scheme”.

<table>
<thead>
<tr>
<th>$j_1 = 7/2$</th>
<th>$j_2 = 7/2$</th>
<th>$M$</th>
<th>$J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>$m_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7/2</td>
<td>5/2</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7/2</td>
<td>3/2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>7/2</td>
<td>1/2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>7/2</td>
<td>−1/2</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>7/2</td>
<td>−3/2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>7/2</td>
<td>−5/2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>7/2</td>
<td>−7/2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5/2</td>
<td>3/2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5/2</td>
<td>1/2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>5/2</td>
<td>−1/2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>5/2</td>
<td>−3/2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5/2</td>
<td>−5/2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3/2</td>
<td>1/2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3/2</td>
<td>−1/2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3/2</td>
<td>−3/2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>−1/2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

* Only positive total $M$ values are shown. The table is symmetric for $M < 0$. 

Table 5.1: $m$ scheme for the configuration $(7/2)^2 J$*
Residual Interactions—Diagonal Effects

Consider 2 particles, in orbits $j_1, j_2$ coupled to spin $J$, and interacting with a residual interaction, $V_{12}$.

2 Identical Nucleons

$| j_1, j_2, J >$  $| j_1, j_2, J >$

$-----------------------------------------------$

$| j_1, j_2, J >$

$J_5$

$J_4$

$J_3$

$J_2$

$J_1$

Differ in angle between orbital planes

NO RESIDUAL INTERACTION
Typical spectra of nuclei with 2 “valence” particles outside doubly magic core. Universal result: Ground state always $0^+$.

Why? Can we obtain such simple results by considering residual interactions?
What are Energies of 2-particle configurations

\[ \Delta E \left( j_1 \ j_2 \ J \right) = \langle j_1 \ j_2 \ J \ M \ \ | \ H_{12} \ | \ j_1 \ j_2 \ J \ M \rangle \]

\[ = \frac{1}{\sqrt{2J+1}} \langle j_1 \ j_2 \ J \ | \ |H_{12}| \ | j_1 \ j_2 \ J \rangle \]

Separate radial and angular coordinates

\[ \Psi = \frac{1}{r} R_{nl} \ (r) \ Y_{lm} (\theta, \phi) \]

where \[ \frac{d^2 R_{nl}}{dr^2} - \frac{l(l+1)}{r^2} R_{nl} + \frac{2m}{\hbar^2} (E_{nl} - V) R_{nl} = 0 \]

\( R_{nl} \) depends on potential – but generally not very much.

Now, what is \( H_{\text{resid}} \)?

Many choices possible. Let’s start with simplest. Nuclear force is short range and attractive. So, take \( \delta \)-force
\[ V_s = \frac{-V_0}{r_1 r_2} \delta (r_1, r_2) \delta (\cos \Theta_1, \cos \Theta_2) \delta (\Phi_1, \Phi_2) \]

in spherical coordinates

Need to evaluate the matrix element (ME) of the form

\[ \left\langle \Psi | \delta'' | \Psi \right\rangle = \left\langle \frac{1}{r} R_{nl} | V_{\delta_r} \frac{1}{r} R_{nl} \right\rangle \times \left\langle Y_{l m} (\phi, \phi) | V_{\delta_{\Theta,\phi}} | Y_{l m} (\phi, \phi) \right\rangle \]

First factor is just a constant independent of J,

\[ i.e., \text{does not depend on J in } |j_1 j_2 J\rangle. \]

So energy shifts for different J’s are independent of the form of the radial wave functions and hence of the radial form of the potential !!

\[ \Rightarrow \text{Great simplification} \]

\[ \Rightarrow \text{Typical of many results - radial effects disappear} \]
How can we understand the energy patterns that we have seen for two – particle spectra with residual interactions? Easy – involves a very beautiful application of the Pauli Principle.

Need 2 ideas only

- Nuclear force (including residual interactions) is
  - Short range and attractive
  - Pauli Principle
Physical Interpretation

$J_{7/2}$  $J$ depends on angle between the two orbital planes

$d_{5/2}$

Interaction strongest when the 2 particles are closest to each other

\[ i.e., \text{when the orbits are co-planar} \]

\[ \Rightarrow \text{strongest interaction either for} \]

\[ J_{\min} \quad \text{or} \quad J_{\max} \]

Which one?

Consider $L, S$ composition of state $J$

\[ \bar{L} = \bar{l}_1 + \bar{l}_2 \quad S = \frac{1}{2} \pm \frac{1}{2} = 1 \text{ or } 0 \]
Pauli Principle

Fermions:
No two fermions can occupy the same state/place

Wave functions must be totally antisymmetric

$$\Psi (\vec{r}) = -\Psi (-\vec{r})$$
\[\vec{r} = \vec{r}_2 - \vec{r}_1\]

$$\therefore$$ If particles are at the same place ----- \(\vec{r} = 0\) ----- then \(\Psi(0) = -\Psi(0)\)
\[\Rightarrow \Psi(0) = 0\]
so PP is satisfied

We split wave functions into 2 parts - spatial part \(L\), and spin part \(S\). PP \(\Rightarrow\)

\[\Psi_{tot} = \Psi_{spat} \times \Psi_{spin} = \text{Anti-sym}\]
This is the most important slide: understand this and all the key ideas about residual interactions will be clear !!!!!

<table>
<thead>
<tr>
<th>PP:</th>
<th>Key Physics Ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_{\text{spatial}}$</td>
<td>$\Psi_{\text{spin}}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$S$</td>
</tr>
<tr>
<td>$S$</td>
<td>$A$</td>
</tr>
</tbody>
</table>

- $S = \frac{1}{2} + \frac{1}{2} = 1 = \text{Sym}$
- $S = \frac{1}{2} - \frac{1}{2} = 0 = \text{A-Sym}$

$\Psi_{\text{spat}} (A) \times \Psi_{\text{spin}} (S = 1)$

$\Psi_{\text{spat}} (S) \times \Psi_{\text{spin}} (S = 0)$

### $S = 1$ case

- $\Psi_{\text{spat}} = A$
- $\Psi (r_{12}) = -\Psi (-r_{12})$

For $\delta$ force, which only acts at $r_{12} = 0$

$\Psi (r_{12} = 0) = 0$ ! !

So, at the ONLY place where a $\delta$-int acts, the wave fct. vanishes—i.e., No effect of $\delta$ fct int on $S = 1$ states ! ! !

### $S = 0$ case

- $\Psi_{\text{spat}} = S$

No restriction on $\Psi (r_{12} = 0)$, hence $\delta$-int can have big effect ! ! !
Equivalent Orbits

\[ J_1 = J_2 \] \[ | J_2, J \rangle \]

\[ J = 0, 2, 4, \ldots \ldots \ldots \ldots \ldots 2j - 1 \]

e.g. \[ | g_{9/2}, J \rangle \] \[ J = 0, 2, 4, 6, 8 \]

\[ \frac{1}{2} \uparrow \downarrow \frac{1}{2} \]
\[ 4 \uparrow \downarrow 4 \]
\[ J = 0 \]
\[ \text{S} = 0 \]

\[ \frac{1}{2} \uparrow \downarrow \frac{1}{2} \]
\[ 4 \uparrow \downarrow 4 \]
\[ J = 8 \]
\[ \text{S} = 1 \]

\[ \text{S} = 0 \text{ lowered } \Rightarrow \text{J} = 0 \text{ lowest} \]

\[ \text{0}^+ \text{ lowest all e - e nuclei have 0}^+ \text{ g.s.} \]

\[ \Delta E (J^2, J) \text{ or } -V_0 \]
\[ \frac{(2j + 1)^2}{2} \left( \begin{array}{ccc} J & J & J \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right)^2 \]

For \[ J = 0 \] \[ \Delta E \propto -V_0 \frac{2j + 1}{2} \] \[ \Rightarrow \Delta E \text{ larger for larger } j: \text{ Why???} \]
Geometrical Interpretation

for $|j^2 J=0\rangle$ being lowest

Pauli Principle is ~ repulsive interaction!
\( V_{12}(\delta) = -V_0 \delta (r_1 - r_2) = \frac{-V_0}{r_1 - r_2} \delta (r_1 - r_2) \delta (\cos \theta_1 - \cos \theta_2) \delta (\Phi_1 - \Phi_2) \)

\[
\Delta E(j_1 j_2 J) = -V_0 F_R(n_1 l_1 n_2 l_2) A(j_1 j_2 J)
\]

where

\[
F_R(n_1 l_1 n_2 l_2) = \frac{1}{4\pi} \int \frac{1}{r^2} R_{n_1 l_1}^2(r) R_{n_2 l_2}^2(r) dr
\]

and

\[
A(j_1 j_2 J) = (2j_1 + 1)(2j_2 + 1) \left( \begin{array}{c}
\frac{j_1}{2} \\
\frac{j_2}{2} \\
j
\end{array} \right) \left( \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array} \right)^2
\]

(if \( l_1 + l_2 - J \) is even)

= 0

(if \( l_1 + l_2 - J \) is odd)

(Non-equivalent orbits)

\[
\Delta E(j^2 J) = -V_0 F_R(n l) A(j^2 J)
\]

where

\[
F_R(n l) = \frac{1}{4\pi} \int \frac{1}{r^2} R_{n l}^4(r) dr
\]

and

\[
A(j^2 J) = \frac{(2j+1)^2}{2} \left( \begin{array}{c}
j \\
j \\
j
\end{array} \right)^2
\]

(J even)

(Equivalent orbits)
Simple treatment of residual interactions accounts for universal fact that even-even nuclei have $0^+$ ground states.

Note that the $0^+$ level is lowered more for higher $j$ orbits.

**NOTE:** $R_{4/2} < 2.0$
Lowering of $0^\pm$ States

\[ \Delta E (j^2 J) \propto - V_0 \frac{(2j+1)^2}{2} \left( \begin{array}{l} j \ 1 \ J \\ \frac{1}{2} \ \frac{1}{2} \ 0 \end{array} \right) \]

For $J = 0$

\[ \Delta E (j^2 J = 0) \propto - V_0 \frac{(2j+1)}{2} \]

\[ \Delta E \propto 2j + 1 \]

Energy lowering of $0^+$ is larger for larger $j$

\[
\begin{align*}
4^+ & \quad 8^+ \\
2^+ & \quad 6^+ \\
0^+ & \quad 4^+ \\
\end{align*}
\]

\[
\begin{align*}
d_{5/2}^2 &
\quad g_{9/2}^2 \\
0^+ & \quad 0^+ \\
2^+ &
\end{align*}
\]

Why?
Lowering of $0^+$ States in $|j^2 J\rangle$

$\Delta E \propto 2j + 1$. Why?

Note: $2j + 1 = \#$ magnetic substates

\begin{align*}
\text{low } j & \quad \text{high } j \\
+ j & \quad + j \\
- j & \quad - j
\end{align*}

$\Psi (J, m, \Theta )$ is localized to an angular range\* centered about normal to ang. mom. vector:

spread of $\Psi$ roughly given by angular “distance” to next substate

\* quantum fluctuations

\[ \text{Larger } j \Rightarrow \text{more magnetic substates} \]
\[ \Rightarrow \text{greater localization} \]
\[ \Rightarrow \text{greater spatial overlap in } |j, m\rangle \text{ and } |j, -m\rangle \]
\[ \Rightarrow \text{lower energy} \]
Extending the IPM with residual interactions

• Consider now an extension of, say, the Ca nuclei to \(^{43}\text{Ca}\), with three particles in a \(j = 7/2\) orbit outside a closed shell?

• How do the three particle angular momenta, \(j\), couple to give final total \(J\) values?

• If we use the m-scheme for three particles in a \(7/2\) orbit the allowed \(J\) values are \(15/2, 11/2, 9/2, 7/2, 5/2, 3/2\).

• For the case of \(J = 7/2\), two of the particles must have their angular momenta coupled to \(J = 0\), giving a total \(J = 7/2\) for all three particles.

• For the \(J = 15/2, 11/2, 9/2, 5/2,\) and \(3/2\), there are no pairs of particles coupled to \(J = 0\).

• Since a \(J = 0\) pair is the lowest configuration for two particles in the same orbit, that case, namely total \(J = 7/2\), must lie lowest!!
Treat as 20 protons and 20 neutrons forming a doubly magic core with angular momentum $J = 0$. The lowest energy for the 3-particle configuration is therefore $J = 7/2$. 

Note that the key to this is the results we have discussed for the 2-particle system !!
But, these were simple cases. As the number of valence nucleons grows, the number of ways of making states of a given J grows hugely. Those “basis states” will mix. How many states do we need to mix? What are the resulting structures? How difficult a calculation is this? Consider a couple of simple cases and a more typical one.

The Need for Simplification in Multiparticle Spectra

Example: How many 2+ states?

\# nucl.

<table>
<thead>
<tr>
<th>2</th>
<th>$d_{5/2}^2$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$d_{5/2} g_{7/2}$</td>
<td>≥ 7</td>
</tr>
</tbody>
</table>

\[\begin{align*}
&d_{5/2}^2 J = 2, g_{7/2} J = 0, \quad d_{5/2}^2 J = 0, g_{7/2} J = 2, \\
&d_{5/2}^2 J = 4, g_{7/2} J = 2; J = 2, \\
&d_{5/2}^2 J = 2, g_{7/2} J = 4; J = 2, \\
&d_{5/2}^2 J = 4, g_{7/2} J = 6; J = 2, \\
&d_{5/2} g_{7/2} J = 1, d_{5/2} g_{7/2} J = 1; J = 2, \\
&d_{5/2}^2 J = 4, g_{7/2} J = 4; J = 2.
\end{align*}\]

For \[^{154}_{62}\text{Sm}\]

- Cl. sh.: 50 82
- $N_p = 12$  $N_n = 10$
- 12 val. π in 50 – 82
- 10 val. ν in 82 – 126

\begin{align*}
82 & \quad \text{s}_{1/2} \\
82 & \quad \text{d}_{3/2} \\
82 & \quad \text{h}_{11/2} \\
82 & \quad \text{d}_{5/2} \\
82 & \quad \text{g}_{7/2} \\
50 & \quad \pi \\
82 & \quad \nu
\end{align*}

How many 2+ states subject to Pauli Principle limits?

\[3 \times 10^{14} \text{ !!!}
\]

$^{154}\text{Sm}$ 2+ states within the valence shell space
So, with even just a few valence nucleons, such calculations become intractable by simple diagonalization. But yet, nuclei show very simple patterns despite the complexity and chaotic behavior one might expect. Emergence of collective behavior.

How can we understand emergent collective collectivity?
Two approaches

a) Advanced methods at the level of nucleons and their interactions – See Vary lectures next week

b) Collective models that look at the many-body system as a whole, with its shapes, oscillations, quantum numbers, selection rules, etc.

We will follow this route but then return to ask what the microscopic drivers of structural evolution and emergent collectivity are.
The key concept for Collectivity – Coherent motion of many nucleons. Lowering of collective states

Lowering of one state. Note that the components of its wave function are all equal and in phase

Please think about this carefully – it is one of the most important concepts in all of many-body physics

Consequences of this: Lower energies for collective states, and enhanced transition rates.
First consider nuclei with a moderate number of valence nucleons (~ 6-16).

These nuclei retain the spherical shapes of nuclei near closed shells but are “soft” -- they can take on oscillatory vibrational motion. The lowest lying such excitation is a small amplitude surface quadrupole oscillation with angular momentum $2^+$

\[2^+ \quad J = 2 \text{ one “phonon” vibrational excitation}\]

\[0^+ \quad \]
### Table 6.1 m scheme for two-quadrupole phonon states*

<table>
<thead>
<tr>
<th>$J_1 = 2$</th>
<th>$J_2 = 2$</th>
<th>$M = \sum m_i$</th>
<th>$J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>$m_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

*Only positive total $M$ values are shown; the table is symmetric for $M < 0$. The full set of allowable $m_i$ values giving $M \geq 0$ is obtained by the conditions $m_1 \geq 0$, $m_2 \leq m_1$.***
Types of collective structures
Few valence nucleons of each type:
The spherical vibrator

\[ E(J) = n (\hbar \omega_0) \]
\[ R_{4/2} = 2.0 \]

Gamma-ray transitions:
Selection rule: Can destroy only one phonon
Deformed Nuclei

• What is different about non-spherical nuclei?

• They can ROTATE !!!

• They can also VIBRATE
  – For axially symmetric deformed nuclei there are two low lying vibrational modes called $\beta$ and $\gamma$

• So, levels of deformed nuclei consist of the ground state, and vibrational states, with rotational sequences of states (rotational bands) built on top of them.
Rotational states built on (superposed on) vibrational modes

Ground or equilibrium state

Rotor

\[ E(I) \propto \left( \frac{\hbar^2}{2I} \right) I(I+1) \]

\[ R_{4/2} = 3.33 \]
Transition rates (half lives of excited levels) also tell us a lot about structure

\[ B(E2: 0^{+}_1 \rightarrow 2^{+}_1) \propto \langle 2^{+}_1 \| E2 \| 0^{+}_1 \rangle^2 \]
The more configurations that mix, the stronger the $B(E2)$ value and the lower the energy of the collective state. Fundamental property of collective states.

**Coherence and Transition Rates**

Consider simple case of $N$ degenerate levels: $2^+$

$$\Delta E = (N - 1)V$$

$$\Psi = a\varphi_1 + a\varphi_2 + \cdots + a\varphi_N$$

where $a = \frac{1}{\sqrt{N}}$

$$\left( \sum_{i=1}^{N} a_i^2 = \frac{N}{N} = 1 \right)$$

---

Consider transition rate from $2_1^+ \rightarrow 0_1^+$

$$B(E2; 2_1^+ \rightarrow 0_1^+) = \frac{1}{2J_i + 1} \left\langle 0_1^+ \parallel E2 \parallel 2_1^+ \right\rangle^2$$

$$\left\langle 0_1^+ \parallel E2 \parallel 2_1^+ \right\rangle = \left\langle 0_1^+ \parallel E2 \parallel \Psi \right\rangle = a \sum_{i=1}^{N} \left\langle 0_1^+ \parallel E2 \parallel \varphi_i \right\rangle$$

Assume all $\left\langle 0_1^+ \parallel E2 \parallel \varphi_i \right\rangle$ matrix elements equal.

$\therefore \left\langle 0_1^+ \parallel E2 \parallel 2_1^+ \right\rangle = Na \left\langle 0_1^+ \parallel E2 \parallel \varphi_i \right\rangle = NaW = \sqrt{N}W$

$\therefore B(E2) \propto NW^2$

**Transition rate enhanced by factor of $N$**

$\therefore$ Enhanced transition rates are a signature of collectivity, along with low $2_1^+$ energies. Lower $E(2_1^+)$, higher $B(E2)$ →
An algebraic approach
Collective behavior superposed on shell structure
IBA, a symmetry-based model (Iachello and Arima)

Drastic simplification of shell model

• Valence nucleons, in pairs as bosons
• Only certain configurations. Only pairs of nucleons coupled to angular momentum 0(s) and 2(d). Why?
• Simple Hamiltonian in terms of s and d boson creation, destruction operators – simple interactions
• **Group theoretical** underpinning
• **Why?** Because it works. And extremely parameter-efficient
Shell Model Configurations

Fermion configurations

Roughly, gazillions!!
Need to simplify

The IBA

Boson configurations
(by considering only configurations of pairs of fermions with $J = 0$ or $2$.)
Modeling a Nucleus

Why the IBA is the best thing since baseball, a jacket potato, aceto balsamico, Mt. Blanc, raclette, pfannekuchen, baklava, ….

\[ ^{154}\text{Sm} \rightarrow \text{Shell model} \rightarrow 3 \times 10^{14} \text{ 2}^+ \text{ states} \]

Need to truncate

IBA assumptions

1. Only valence nucleons
2. Fermions \(\rightarrow\) bosons
   
   \[ J = 0 \text{ (s bosons)} \]
   
   \[ J = 2 \text{ (d bosons)} \]

IBA: 26 2\(^+\) states

Is it conceivable that these 26 basis states are correctly chosen to account for the properties of the low lying collective states?

IBA: Truncation of Shell Model with Group Theory structure
IBA has a deep relation to Group theory

That relation is based on the operators that create, destroy $s$ and $d$ bosons

\[ s^+, s, \quad d^+, d \quad \text{operators} \]

\[ N_B = n_s + n_d = s^s = d^d \]

\[ \text{Ang. Mom. 2} \quad d^\mu, d^\mu \quad \mu = 2, 1, 0, -1, -2 \]

Hamiltonian is written in terms of $s$, $d$ operators

\[ H = H_s + H_d + H_{\text{int}}(s^s, s^d, d^s, d^d) \]

Since boson number, $N_B$, is conserved for a given nucleus, $H$ can only contain “bilinear” terms: 36 of them.

\[ s^s, s^d, d^s, d^d \]

Group is called $U(6)$

$U(6)$ has three subgroups corresponding to different shapes
Concepts of group theory
First, some fancy words with simple meanings: Generators, Casimirs, Representations, conserved quantum numbers, degeneracy splitting

**Generators** of a group: Set of operators, $O_i$ that close on commutation.

$$[O_i, O_j] = O_i O_j - O_j O_i = O_k \quad i.e., \quad \text{their commutator gives back 0 or a member of the set}$$

For IBA, the 36 operators $s^\dagger s, d^\dagger s, s^\dagger d, d^\dagger d$ are generators of the group $U(6)$.

**ex:**

$$\left[ d^\dagger s, s^\dagger s \right] |n_d n_s\rangle = \left( d^\dagger ss^\dagger s - s^\dagger sd^\dagger s \right) |n_d n_s\rangle$$

$$= d^\dagger s n_s |n_d n_s\rangle - s^\dagger s d^\dagger s |n_d n_s\rangle$$

$$= \left( n_s - s^\dagger s \right) d^\dagger s |n_d n_s\rangle$$

$$= \left( n_s - s^\dagger s \right) \sqrt{n_d + 1} \sqrt{n_s} |n_d + 1, n_s - 1\rangle$$

$$= \sqrt{n_d + 1} \sqrt{n_s} \left[ n_s - (n_s - 1) \right] |n_d + 1, n_s - 1\rangle$$

$$= \sqrt{n_d + 1} \sqrt{n_s} |n_d + 1, n_s - 1\rangle$$

$$= d^\dagger s |n_d n_s\rangle$$

**or:**

$$\left[ d^\dagger s, s^\dagger s \right] = d^\dagger s$$
Concepts of group theory

First, some fancy words with simple meanings: Generators, Casimirs, Representations, conserved quantum numbers, degeneracy splitting

**Generators** of a group: Set of operators, $O_i$ that close on commutation.

\[ [O_i, O_j] = O_i O_j - O_j O_i = O_k \quad i.e., \quad \text{their commutator gives back 0 or a member of the set} \]

For IBA, the 36 operators $s^\dagger s$, $d^\dagger s$, $s^\dagger d$, $d^\dagger d$ are generators of the group $U(6)$.

**Generators:** define and conserve some quantum number.

Ex.: 36 Ops of IBA all conserve total boson number $N = s^\dagger s + d^\dagger \tilde{d} = n_s + n_d$

**Casimir:** Operator that commutes with all the generators of a group. Therefore, its eigenstates have a specific value of the q.# of that group. The energies are defined solely in terms of that q. #. $N$ is Casimir of $U(6)$.

**Representations** of a group: The set of degenerate states with that value of the q. #.

A **Hamiltonian** written solely in terms of Casimirs can be solved analytically
**Sub-groups:**

Subsets of generators that commute among themselves.

*e.g.*  \( d^\dagger d \)  

25 generators—span \( U(5) \)

They conserve \( n_d \) (# \( d \) bosons)

Set of states with same \( n_d \) are the representations of the group [ \( U(5) \)]

Summary to here:

- **Generators:** commute, define a q. #, conserve that q. #
- **Casimir Ops:** commute with a set of generators
  - ∴ Conserve that quantum #
  - ∴ A Hamiltonian that can be written in terms of Casimir Operators is then diagonal for states with that quantum #

Eigenvalues can then be written **ANALYTICALLY** as a function of that quantum #
Group Structure of the IBA

6-Dim. problem

U(6) →

Magical group theory stuff happens here

Three Dynamic symmetries, nuclear shapes

U(5) vibrator
SU(3) rotor
O(6) γ-soft
Group Structure of the IBA

6-Dim. problem

U(6) ➔

U(5) vibrator
SU(3) Rotor
O(6) γ-soft

Magical group theory stuff happens here

Three Dynamic symmetries, nuclear shapes

Symmetry Triangle of the IBA

R_{4/2} = 2.0
Sph.

R_{4/2} = 2.5
γ-soft

R_{4/2} = 3.33
Def.

U(5) vibrator
SU(3) rotor

O(6)
Let’s illustrate group chains and degeneracy-breaking.

Consider a Hamiltonian that is a function ONLY of: \( s^\dagger s + d^\dagger d \)

Note that \( s^\dagger s = n_s \) and \( d^\dagger d = n_d \) and that \( n_s + n_d = N = \frac{1}{2} \text{val nucleons} \)

That is: \( H = a(s^\dagger s + d^\dagger d) = a(n_s + n_d) = aN \)

H “counts” the numbers of bosons and multiplies by a boson energy, \( a \). The energies depend ONLY on total number of bosons -- the total number of valence nucleons. The states with given \( N \) are degenerate and constitute a “representation” of the group U(6) with the quantum number \( N \). U(6) has OTHER representations, corresponding to OTHER values of \( N \), but THOSE states are in DIFFERENT NUCLEI.

Of course, a nucleus with all levels degenerate is not realistic (!!!) and suggests that we should add more terms to the Hamiltonian. I use this example to illustrate the idea of successive steps of degeneracy breaking related to different groups and the quantum numbers they conserve.
\[ H' = H + b \ d^\dagger d = aN + b \ n_d \]

Now, add a term to this Hamiltonian:

**Now the energies depend not only on N but also on** \( n_d \)

**States of a given** \( n_d \** are now degenerate. They are “representations” of the group U(5). **States with different** \( n_d \** are not degenerate**
\[ H = aN + b \ d^+ d = a \ N + b \ n_d \]

\[ U(6) \quad U(5) \]
Example of a nuclear dynamical symmetry -- $O(6)$

Spectrum generating algebra

Each successive term:

- Introduces a new sub-group
- A new quantum number to label the states described by that group
- Adds an eigenvalue term that is a function of the new quantum number, and therefore
- Breaks a previous degeneracy
Most nuclei do not exhibit the idealized symmetries but rather lie in transitional regions. Mapping the triangle.
What do you do with all the nuclei that do not manifest a symmetry? Need a Hamiltonian that breaks the symmetries.

Truncated form of with just two parameters (+ scale):

\[ H = \varepsilon n_d - \kappa Q \cdot Q \]

\[ Q = e[s^\dagger a + d^\dagger s + \chi (d^\dagger a)^{(2)}] \]

Competition:

<table>
<thead>
<tr>
<th>Term</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon n_d )</td>
<td>Counts quad bosons: vibrator.</td>
</tr>
<tr>
<td>( \kappa Q \cdot Q )</td>
<td>Gives deformed nuclei.</td>
</tr>
<tr>
<td>( \chi )</td>
<td>Determines axial asymmetry</td>
</tr>
</tbody>
</table>

Hence structure is given by two parameters, \( \varepsilon/\kappa \) and \( \chi \).

More complicated forms exist but this is the form usually used. It works extremely well in most cases.
$H$ has two parameters. A given observable can only specify one of them. That is, a given observable has a contour (locus) of constant values within the triangle $R_{4/2} = 2.9$. 
Mapping Structure with Simple Observables – Technique of Orthogonal Crossing Contours

\[
\frac{E(4^+)}{E(2^+)}
\]

\[
\frac{E(0^+)}{E(2^+)} - \frac{E(2^+)}{E(2^+)}
\]
Mapping Structure with Simple Observables – Technique of Orthogonal Crossing Contours
R. Burcu Cakirli et al.
Beta decay exp. + IBA calcs.
Warner, Borner, and Davidson
Complementarity of macroscopic and microscopic approaches. Why do certain nuclei exhibit specific symmetries? Why these evolutionary trajectories?

What will happen far from stability in regions of proton-neutron asymmetry and/or weak binding?