Lecture 2: Chiral Perturbation Theory
QCD in the light quark (up & down) sector (QCD-light) has two mass scales

\[ M(\text{GeV}) \]

\[ \begin{array}{c}
\Lambda_{\text{QCD}} \\
m_N \\
m_\rho \\
m_\pi \\
m_{u,d}
\end{array} \]
In a generic physical system, there are often many scales involved. However, for a specific problem under consideration, it may depend on physics only on a particular scale.

One can “integrate out” physics at other scales and focus on the dynamics on the degrees of freedom relevant to that scale: Effective (Field) Theory

Many example:

- Fluid Dynamics
- Multiple expansion in Electrodynamics
- Nuclear Physics
- ...
When quarks are massless, $N_f$ flavor of QCD lagrangian has $U_L(N_f) \times U_R(N_f)$ chiral symmetry.

Each quark has a left-handed and right-handed components,

$$
\psi_L f = \frac{1}{2} (1 - \gamma_5) \psi_f \quad \psi_R f = \frac{1}{2} (1 + \gamma_5) \psi_f .
$$

The left and right-handed fields do not couple to each others in the massless limit. Each fields can rotate independently producing a symmetry group $U_L(N_f) \times U_R(N_f)$

$$
\mathcal{L}_q = \mathcal{L}_q(\psi_L) + \mathcal{L}_q(\psi_R) , \quad \begin{pmatrix} u'_{L,R} \\ d'_{L,R} \end{pmatrix} = U_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix}
$$
U_L(N_f) \times U_R(N_f) contains two U(1) symmetries: vector and axial: Vector U(1) is related to baryon number, and the axial U(1) is broken by anomaly.

The anomaly is a phenomenon that a classical symmetry is broken by quantum fluctuations, and was first discovered by Adler, Bell, and Jackiw.

The remaining chiral symmetry SU_L(N_f) \times SU_R(N_f) is broken spontaneously to SU(N_f) flavor symmetry (isospin) discussed in the previous lecture.

SSB: spontaneous symmetry breaking.
The representation of SU(2) group [angular momentum algebra] contains dimensions, 1, 2, 3… (2j+1),…

Therefore the representation of the chiral group $SU_L(2) \times SU_R(2)$ can be labeled by $(2j_1+1, 2j_2+1)$.

The left-handed quark field is $(2,1)$ and the right-handed quark field is $(1,2)$.

Isospin representations come from adding the two reps.

The quark mass terms

$H_1 = m_u \bar{u}u + m_d \bar{d}d$

can be decomposed into

$H_1 = m_u (\bar{u}_L u_R + \bar{u}_R u_L) + m_d (\bar{d}_L d_R + \bar{d}_R d_L)$.

It is a $(2\text{-bar}, 2) + (2, 2\text{-bar})$, not invariant under chiral symmetry.
A simple example is a particle moving in a double well.

When the mid-barrier is finite, the ground state is always symmetric in $x \rightarrow -x$.

However, when the height is going to infinity, the ground state is degenerate, and the physical ground state is for the particle in either wells, a broken symmetry state.

Another example of SSB is spontaneous magnetization of a piece of magnet.
In the case of the SSB of a continuous symmetry, there are massless Goldstone bosons produced as result. This is because everywhere in the space, one can choose a different vacuum (vacuum degeneracy), there is no energy difference between the different choices.

Pion would have been the massless Goldstone boson associated with the chiral symmetry breaking of \( SU_L(2) \times SU_R(2) \), if the quark masses were zero.

The pion interactions must be derivative-coupled because in the long wavelength limit, the interactions vanish, because the long-wavelength pion approaches the vacuum.
When SSB happens, there is an order parameter which characterizes the symmetry breaking.

The physical vacuum no longer invariant under chiral symmetry, rather, it is a sum of chiral reps,

$$|\text{vac}\rangle \geq |(1,1)\rangle + |(2,2)\rangle + |(3,3)\rangle + ...$$

the chiral representation must have isospin 0, so $j_1=j_2$.

Therefore, there is a non-zero chiral condensate in the physical vacuum, which characterizes the scale at which SSB happens

$$\langle 0|\bar{u}u + \bar{d}d|0\rangle$$
Easiest way to see that pions are derivatively coupled is to introduce a $U$ fields that transforms as $(2,2)$ of the chiral group.

$$U \rightarrow LUR^{-1}$$

which contains the Goldstone boson field.

Construct lagrangian that are invariant under the chiral transformation.

After SSB, $U$ is related to the pion field,

$$U = \sigma e^{i \pi^a(x) \cdot \tau^a / f_\pi}$$

Or we simply write this $U = \sigma \Sigma$, here $\Sigma$ is a non-linear realization of chiral symmetry.
When the energy of the pion is low, derivatives are small compared to the scale of SSB. Therefore, one can make expansion in $\frac{\partial}{f_\pi}$.

This expansion is called the chiral expansion.

Taking into account the non-zero pion mass, $m_\pi/f_\pi$ is another small expansion parameter.

Chiral perturbation theory (ChiPT) carries out systematic expansion in these small parameters. Since the theory uses symmetry and SSB, test of ChiPT is usually considered as a test of QCD itself. [“If ChiPT does not work, QCD is in trouble.”]
Pion is massless in the chiral limit. Therefore, its non-zero mass must come from the non-zero quark mass.

One can show, 

\[ m_\pi^2 = - (m_u + m_d) \langle 0 | \bar{u}u + \bar{d}d | 0 \rangle / f_\pi^2 \]

which is linear in quark mass and also related to the chiral condensate!
The simplest lagrangian for pure pion involves the kinetic energies and pion mass, second-order in small parameter

\[
L = \frac{f_\pi^2}{4} \text{Tr}\left[\partial_\mu \Sigma \partial^\mu \Sigma^+\right] + \frac{f_\pi^2}{4} m_\pi^2 \text{Tr}\left[\Sigma + \Sigma^+\right] + O(m_\pi^4)
\]

the dependence in the pion mass is analytical in sense that it is a Taylor expansion.

Higher-order term can also be written down, involving more unknown constants, called chiral constants

\[
L_4 \text{Tr}(D^\mu \Sigma^\dagger D_\mu \Sigma) \text{Tr}(\chi^\dagger \Sigma + \chi \Sigma^\dagger) + L_5 \text{Tr}(D^\mu \Sigma^\dagger D_\mu \Sigma)(\chi^\dagger \Sigma + \chi \Sigma^\dagger)
\]
\[
+ L_6 (\text{Tr}(\chi^\dagger \Sigma + \chi \Sigma^\dagger))^2 + L_7 (\text{Tr}(\chi^\dagger \Sigma - \chi \Sigma^\dagger))^2
\]
\[
+ L_8 \text{Tr}(\chi^\dagger \Sigma \chi^\dagger \Sigma + \chi \Sigma^\dagger \chi \Sigma^\dagger) + H_2 \text{Tr}(\chi^\dagger \chi)
\]
Expand the pion lagrangian to the first non-trivial order

\[ \mathcal{L}_{\pi\pi}^{(2)} = \frac{1}{2} (\partial_{\mu} \pi) \right (\partial_{\mu} \pi) + \frac{1}{6f_{\pi}^2} [((\partial_{\mu} \pi \cdot \pi)^2 - \pi^2 (\partial_{\mu} \pi)^2] + ... \]

There is no unknown parameter!

Taking into account the pion mass effects as well,

\[ \mathcal{M} = -f_{\pi}^{-2} \left( \delta_{ab} \delta_{cd} (s - m_{\pi}^2) + \delta_{ac} \delta_{bd} (t - m_{\pi}^2) + \delta_{ad} \delta_{bc} (u - m_{\pi}^2) \right) \]

Scattering length in isospin 0 and 2 sectors,

\[ a_0 = \frac{7m_{\pi}}{32\pi f_{\pi}^2} = 0.16m_{\pi}^{-1} \]
\[ a_2 = -\frac{2m_{\pi}}{32\pi f_{\pi}^2} = -0.046m_{\pi}^{-1} \]

Experimentally,

\[ a_0 = 0.26 \pm 0.5 \]
\[ a_2 = -0.028 \pm 0.012 \]
Chiral Physics in the Nucleon

$1/m_N$

$1/m_\pi$ “pion cloud”
Since only the long distance part of nucleon physics is related to the pion and is calculable using the ChiPT, the short distance physics are parameterized by the so-called low-energy constants. There are large number of such low-energy constants.

The predictive power of ChiPT comes from distinctive contributions of the pion, non-analytic contributions from the pion mass.
Loop calculations depend on the pion propagator:

\[
\frac{i}{k^2 - m_\pi^2 + i\varepsilon}
\]

with loop momentum-\(k\) to be integrated over. The integrations can generate non-analytic dependence on \(m_\pi^2\).

What are they? \(\frac{1}{m_\pi^n}\) (\(n>0\)), \(m_\pi^{2n+1}\), \(\ln(m_\pi)\), ...

There dependence usually comes from IR divergences.

Dependence on \(m_\pi^2\) in the counter terms are analytic because they are treated in perturbative expansion.
\[ \frac{g_A}{f_\pi} \vec{\sigma} \cdot \vec{k} \tau^a \]

g_A: neutron decay constant, dimension 0
f_\pi: pion decay constant, dimension 1
Heavy-Baryon Chiral Perturbation Theory

Get rid of the hadron mass scale, $m_N \to \infty$.

Physics at scale $m_N$ is not really calculable in chiral perturbation theory, should be included in the counter terms.

Relativistic Chiral Perturbation Theory

Contain partial high-order contributions.

Better or Worse? Don’t know. They provide some idea on the size of higher-order corrections.
At leading order (one-loop), two powers of $\frac{1}{4\pi f_\pi}$.

Since the contribution must have a dimension of mass.

\[ \delta m_N \sim \frac{m_\pi^3}{(4\pi f_\pi)^2} \]

Since this is nonanalytic, the coefficient is calculable! ($3/\pi^2$).
It contributes $-15$ MeV to the nucleon mass.

There is a $m_\pi^2$ contribution, proportional to the $\sigma$-term.
 Isovector charge radius

- $<r^2>$ has a mass-dimension $-2$.
- Leading pion-loop has a factor of $1/(4\pi f_\pi)^2$.
- Therefore, the chiral contribution goes like

$$\frac{1}{(4\pi f_\pi)^2} \ln\left(\frac{m_\pi^2}{\Lambda^2}\right)$$

which diverges as $m_\pi \to 0$. (coefficient $(5g_A^2 + 1)$)

just like the charge radius of the electron in QED!

- Isoscalar charge radius is regular as $m_\pi \to 0$.
- Small neutron charge radius is an accident!
Magnetic moment has a mass dimension $-1$.

Leading pion-loop has a factor of $1/(4\pi f_\pi)^2$.

Thus the nonanalytical chiral contribution, $\frac{m_\pi}{(4\pi f_\pi)^2}$

Coefficient is $-2\pi g_A^2$

A significant contribution at physical pion mass.
Low-energy Scattering off the Nucleon

- Compton Scattering and Sum Rules
  - Real Photon
  - Virtual Photon
  - Doubly Virtual Photon
- Pion-photo and electroproduction
- Pion scattering
- ....
Two Compton amplitudes:

$$T^{[\mu\nu]}(P, q, S) = -i \varepsilon^{\mu\nu\alpha\beta} q_\alpha \left[ S_\beta S_1(\nu, Q^2) + (M \nu S_\beta - S \cdot q P_\beta) S_2(\nu, Q^2) \right].$$

$S_1$ and $S_2$ at low energy can be calculated in CHIPT.
Unsubtracted dispersion relations

\[ S_1(\nu, Q^2) = 4 \int_{Q^2/2M}^{\infty} \frac{d\nu' \nu' G_1(\nu', Q^2)}{\nu'^2 - \nu^2} , \]

\( G_1 \) is the spin-dependent structure function.

Expand at small \( \nu \),

\[ S_1(\nu, Q^2) = \sum_{n=0,2,4,...} \nu^n S_1^{(n)}(Q^2) , \]

Dispersion sum rules valid at all \( Q^2 \)

\[ S_1^{(n)}(Q^2) = 4 \int_{Q^2/2M}^{\infty} \frac{d\nu}{\nu^{n+1}} G_1(\nu, Q^2) \quad (n = 0, 2, 4, ...) , \]

X. Ji & J. Osborne, JPG27, 127 (2001)
\[ S_1(0, Q^2) = 4 \int_{Q^2/2M}^{\infty} \frac{d\nu}{\nu} G_1(\nu, Q^2). \]

- At low-\(Q^2\), \(S_1(0,Q^2)\) can be calculated in CHIPT
- At \(O(p^3)\), \(S_1(0,Q^2)\) is zero
- At \(O(p^4)\): Ji, Kao, Osborne, PLB472,1(2000)
Data: M. Amarian et.al. PRL89, 242301 (2002)

Bernard, Hemmert, Metzner (2002)

Ji, Kao, Osborne (2000)
Parity-Violating Photo-Pion Production

- Can be used to measure the parity-violating pion-nucleon coupling

\[ \mathcal{L}^{PV} = -i\hbar^{(1)}_{\pi NN} \pi^+ p^\dagger n + h.c. + \cdots, \]

- Calibration:

\[ \sigma \quad (\mu\text{b}) \]

\[ E_\gamma \quad (\text{MeV}) \]
FIG. 2. Feynman diagrams contributing to the parity-violating amplitudes at LO ($\mathcal{O}(1)$) and NLO ($\mathcal{O}(p)$) in $\gamma p \to \pi^+ n$.

\[
A_\gamma (\omega_{th}, \theta) = \frac{\sqrt{2} f_\pi (\mu_p - \mu_n)}{g_A m_N} h_{\pi NN}^{(1)},
\]

~ $2 \times 10^{-7}$