Lecture 2: Traditional shell model
Shell structure in nuclei


Relatively expensive to remove a neutron form a closed neutron shell.

Bohr & Mottelson, Nuclear Structure.
Shell structure cont’d

Nuclei with magic N
- Relatively high-lying first $2^+$ exited state
- Relatively low $B(E2)$ transition strength

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1963 Nobel Prize in Physics

Maria Goeppert-Mayer

J. Hans D. Jensen

“for their discoveries concerning nuclear shell structure”
Need spin-orbit force to explain magic numbers beyond 20.

http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/shell.html
Modification of shell structure at the drip lines!

Quenching of 82 shell gap when neutron drip line is approached.

Also observed in lighter nuclei

Caution: Shell structure seen in many observables.

FIG. 3. Spherical single-particle levels for the $A=120$ isobars calculated in the SkP HF model (top) and SkP HFB model (middle) as a function of neutron number. The single-particle canonical HFB energies are given by $\epsilon_k = \langle \Psi_k | H | \Psi_k \rangle$. Solid (dashed) lines represent the orbitals with positive (negative) parity. The bottom portion shows the average neutron and proton gaps defined by $\Delta = \int \Delta(r) \rho(r) d^3r / \int \rho(r) d^3r$.

Traditional shell model

Main idea: Use shell gaps as a truncation of the model space.

• Nucleus \((N,Z) = \text{Double magic nucleus } (N^*, Z^*) + \text{ valence nucleons } (N-N^*, Z-Z^*)\)

• Restrict excitation of valence nuclons to one oscillator shell.
  – Problematic: Intruder states and core excitations not contained in model space.

• Examples:
  • pf-shell nuclei: \(^{40}\text{Ca}\) is doubly magic
  • sd-shell nuclei: \(^{16}\text{O}\) is doubly magic
  • p-shell nuclei: \(^{4}\text{He}\) is doubly magic
Shell model

Example: $^{20}\text{Ne}$
Shell-model Hamiltonian

Hamiltonian governs dynamics of valence nucleons; consists of one-body part and two-body interaction:

\[
\hat{H} = \sum_j \varepsilon_j \hat{a}_j^\dagger \hat{a}_j + \sum_{JT,j_1,j_2,j'_1,j'_2} \langle j_1 j_2 | \hat{\mathcal{V}} | j'_1 j'_2 \rangle_{JT} \hat{A}_{JT,j_1j_2}^\dagger \hat{A}_{JT,j'_1j'_2}
\]

Q: How does one determine the SPE and the TBME?
Empirical determination of SPE and TBME

<table>
<thead>
<tr>
<th>Mass</th>
<th>State</th>
</tr>
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<tbody>
<tr>
<td>6.36</td>
<td>$1/2^+$</td>
</tr>
<tr>
<td>5.94</td>
<td>$3/2^+ 1/2^-$</td>
</tr>
<tr>
<td>5.09</td>
<td>$5/2^- 3/2^-$</td>
</tr>
<tr>
<td>5.22</td>
<td>$5/2^- 3/2^+$</td>
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<tr>
<td>5.38</td>
<td>$3/2^-$</td>
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<tr>
<td>4.554</td>
<td>$3/2^-$</td>
</tr>
<tr>
<td>3.843</td>
<td>$5/2^-$</td>
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<tr>
<td>3.0554</td>
<td>$1/2^-$</td>
</tr>
<tr>
<td>0.8707</td>
<td>$1/2^+$</td>
</tr>
</tbody>
</table>

- Determine SPE from neighbors of closed shell nuclei having mass $A = \text{closed core} + 1$
- Determine TBME from nuclei with mass $A = \text{closed core} + 2$.
- The results of such Hamiltonians become inaccurate for nuclei with a larger number of valence nucleons.
- Thus: More theory needed.
Effective shell-model interaction: G-matrix

- Start from a microscopic high-precision two-body potential
- Include in-medium effects in G-matrix
- Bethe-Goldstone equation

\[ G = V + V \frac{Q_P}{E - H_0} G \]

- Formal solution:

\[ G = \frac{V}{1 - V Q_P / (E - H_0)} \]

- Properties: in-medium effects renormalize hard core.
- But: The results of computations still disagree with experiment.

Further empirical adjustments are necessary

Two main strategies

1. Make minimal adjustments only. Focus on monopole TBME:

\[ V_{T; j_1, j_2} \propto \sum_J (2J + 1) \langle j_1 j_2 | V | j_1 j_2 \rangle_{J_T} \]

- **Rationale:**
  - Monopole operators are diagonal in TBME.
  - Set scale of nuclear binding.
  - Sum up effects of neglected three-nucleon forces.

2. Make adjustments to all linear combinations of TBME that are sensitive to empirical data (spectra, transition rates); keep remaining linear combinations of TBME from G-matrix.

- **Rationale:**
  - Need adjustments in any case.
  - Might as well do best possible tuning.
Two-body G-matrix + monopole corrections

G-matrix and monopole adjustments compared to experiment.

Monopole corrections capture neglected three-body physics.

FIG. 18. The level scheme of $^{49}$Ca obtained with the interactions KB, KB', and KB3, compared to the experimental result.


Shell-model computations

1. Construct Hamiltonian matrix
2. Use Lanczos algorithm to compute a few low-lying states.
3. Problem: rapidly increasing matrix dimensions

Publicly available programs
- Oxbash (MSU)
- Antoine (Strasbourg)

FIG. 7. (Color in online edition) $m$-scheme dimensions (circles) and total number of nonzero matrix elements (squares) in the pf shell for nuclei with $M=T_z=0$ as a function of neutron number $N$. The dotted and dashed lines serve as guides for the eye.

Spectra and transition strengths suggest that N=28 Nucleus $^{44}\text{S}$ exhibits shape mixing in low excited states $\rightarrow$ erosion of N=28 shell gap.

Semi-empirical interactions for the nuclear shell model

- **pf-shell** ~200 TBME (1990) $10^9$ dimensions

- **sd-shell** 63 TBME (1980s) $10^5$ dimensions

- **p-shell**

  - 1960s

  - At present: pf $g_{9/2}$ shell.

  - Approach also been used across sd and pf shell.
Subshell closure at neutron number N=32 in neutron rich pf-shell nuclei (enhanced energy of excited 2\(^+\) state).

No new N=34 subshell.


FIG. 3. \(E(2_1^+)\) values versus neutron number for the even-even \(^{24}\text{Cr}, \(^{22}\text{Ti},\) and \(^{20}\text{Ca}\) isotopes. Experimental values are denoted by dashes. Shell model calculations using the GXPFI [14] and KB3G [22] interactions are shown as filled circles and crosses, respectively.
Nuclear landscape and consequences.

~ 300 stable nuclei
N/Z~1 for light nuclei
N/Z~1.5 for $^{208}$Pb

~4000-6000 unstable nuclei
decay by $\alpha$, $\beta$, 1p, 2p, 1n, cluster emission, fission...
Modification and quenching of shell structure at the dripline.

**Nuclear Shell Structure**

1949

Nobel Prize 1963

- $P_{3/2}$
- $f_{5/2}$
- $h_{11/2}$
- $h_{11/2}$
- $d_{5/2}$
- $g_{9/2}$
- $f_{7/2}$
- $h_{11/2}$
- $h_{11/2}$
- $d_{3/2}$
- $g_{9/2}$

Around the valley of nuclear stability
$N/Z \sim 1 - 1.6$

Neutron-rich nuclei
$N/Z \sim 3$

FIG. 4 (color online). The experimental [25,26] (data points) and theoretical [13–15] (lines) one- and two-neutron separation energies for the $N=15$–18 oxygen isotopes. The experimental error is shown if it is larger than the symbol size.

25O neutron separation energy: -820 keV
the width was measured to be 90(30) keV giving a lifetime of $t \sim 7\times10^{-21}$ sec

C. Hoffman PRL 100 (2008) 152502
Cluster states near threshold.

\[ \begin{align*}
\text{J. Rotureau (2008)}
\end{align*} \]
Halo structures

Thomas-Ehrmann effect

Spectra and matter distribution modified by the proximity of scattering continuum

J. Rotureau (2008)
Open vs. closed quantum systems.

Open Quantum System.
Coupling with continuum taken into account.

Closed Quantum System.
No coupling with external continuum.
Formation of single particle resonances.

\[ h\psi = (e - i\frac{\Gamma}{2})\psi \]

**bound state:** \( k_n = i\kappa_n \)

**resonance:** \( k_n = \gamma_n - i\kappa_n \)

- Gamow, Z. Phys. 51, 204 (1928)
- Siegert, Phys. Rev. 36, 750 (1939)

\[ u''(r) = \left[ \frac{l(l + 1)}{r^2} + \frac{2\mu}{\hbar^2} V(r) - k^2 \right] u(r) \]

\[ u(r) \sim C_0 \ r^{l+1} , \ r \to 0 \]

\[ u(r) \sim C_+ \ H^+_{l,\eta}(kr) , \ r \to +\infty \ (\text{bound, resonant}) \]

\[ u(r) \sim C_+ \ H^+_{l,\eta}(kr) + C_- \ H^-_{l,\eta}(kr) , \ r \to +\infty \ (\text{scattering}) \]
Gamow Shell Model (2002)

Gamow states and completeness relations

\[ \sum_{n=0}^{\infty} |u_n|^2 + \frac{1}{\pi} \int_{L_{-}} |u(k)\bar{u}(k^*)| dk = 1 \]

**particular case: Newton completeness relation**

\[ \sum_{n=0}^{\infty} |u_n|^2 + \frac{1}{\pi} \int_{R} |u(k)\bar{u}(k^*)| dk = 1 \]

(N. Michel et al, PRL 89 (2002) 042502)

complex-symmetric eigenvalue problem for hermitian hamiltonian
Summary

• Shell model a powerful tool for understanding of nuclear structure.
• Shell quenching / erosion of shell structure observed when drip lines are approached.
• Shell model calculations based on microscopic interactions
  – Adjustments are needed
  – Due to neglected three body forces (?!)
• Effective interactions have reached maturity to make predictions, and to help understanding experimental data.

• Weakly bound and unbound nuclei
  – Berggren completeness relation
  – Bound, resonant and scattering states form basis
  – Gamow shell model
• Toward unification of nuclear structure and reactions