Nuclei as Mesoscopic Systems

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Mesoscopic system

• Quantum many-body system between microscopic (few-body) and macroscopic (thermodynamic limit)
• Quantum many-body system with identifiable individual quantum states, while sufficiently large to reveal regularities of statistical nature.
• Emergence of complexity.
A rich variety of mesoscopic systems

- Nano-wires
- Quantum dots
- Helium drops
- Atomic clusters
- Quantum computers

http://www.pa.msu.edu/~tomanek/
• **Quantum Dot**: 5 metallic gates fabricated on the surface of a GaAs; two dimensional electron gas inside.

• quantum dot can be seen as a cavity in which electrons bounce at the boundaries similar to a billiard table.
Nanotubes

http://pages.unibas.ch/phys-meso/
The nuclear world: the rich variety of natural mesoscopic phenomena

• Predicted: 6000 - 7000 particle-stable nuclides
• Observed: 2932
• even-even 737; odd-A 1469; odd-odd 726.
• Lightest $^2\text{H}_1$ (deuteron), Heaviest $^{294}_{118}(?)_{176}$
• No gamma-rays known 785.
• Largest number of levels known (578) $^{40}_{20}\text{Ca}_{20}$
• Largest number of transitions known 1319 $^{53}_{25}\text{Mn}_{28}$
• Highest multipolarity of electromagnetic transition E6 in $^{53}_{26}\text{Fe}_{27}$, $19/2^- (3040 \text{ keV}) \rightarrow 7/2^- (\text{g.s.}); 2.58 \text{ min}$
• Result of 100 years of research 182000 citations in Brookhaven database, 4500 new entries per year.
Single-Particle Motion

- Symmetry, surface and shells
- Shells and supershells
- Single-particle modes and magic numbers
- Symmetry and chaos
- Classical periodic orbits
Salt Clusters, transition from small to bulk

- Symmetry
- Surface
- “Shells”
Shell Structure in atoms

Nuclear Magic Numbers, nucleon packaging, stability, abundance of elements

From W.D. Myers and W.J. Swiatecki, Nucl. Phys. 81, 1 (1966)
Mean field

Nucleon in a box

single-particle levels

Nuclear Woods-Saxon solver
http://www.volya.net/ws/

Shell gaps N=2,8,20,

oscillator square well Woods-saxon
Level density in the Woods-Saxon Potential: N=1000, 2000, and 3000

Supershells

R.B. Balian, C. Block Ann. Phys. 69 (1971) 76
Supershells and classical periodic orbits

![Fourier Transform Graph](image)

- Fourier Transform
- $N=3000$
- Peaks at $171\text{A}$ and $184\text{A}$
Nucleon in the potential well

Quantum Billiard

- Shell Model
  Levels in nuclei
Chaotic motion

- Non-symmetric shape
  - Shape changes
Periodic orbitals and shell structure

In the realm of chaos

• Why some nuclei are more stable than others?
• Why are there shell effects?
Single-nucleon motion in deformed potential
Quantum chaos
Distribution of energy spacing between neighboring states

- **Regular motion**
  - Analog to integrable systems
  - No level repulsion
  - Poisson distribution
    \[ P(s) = \exp(-s) \]

- **Chaotic motion**
  - Classically chaotic
  - Level repulsion
  - GOE (Random Matrix)
    \[ P(s) = s \exp(-\pi s^2/4) \]
Evolution of shells

• Melting of shell structure
• Shells in deformed nuclei
• Shells in weakly bound nuclei
• Is the mean field concept valid
Shell structure in extreme limits

Melting of shell structure

Shells in nuclei far from stability

J. Dobaczewski et al., PRC53, 2809 (1996)

T=0 and T=0.4 ev,
Deformation and shell gaps

![Graph showing energy levels and deformation for quantum dots and atomic nuclei.]
Mesoscopic many-body complexity

• Complexity and Chaos
  – Typical level density
  – Chaotization process, geometric chaoticity
  – Random matrix theory
  – Enhancement of weak perturbations

• Collective Motion
  – Pairing and superconductivity
  – Phase transitions
  – Giant resonances
  – Fission

• Shapes
  – Shape change transitions
  – Rotations

• Thermodynamics and phase transitions
  – Features of small systems
  – Thermalization and level density
  – Yang-Lee theory, roots of partition functions
Many-nucleons, two-body scatterings

Nucleons in the box collide (interact)
• Jump from level to level
• Many-body dynamics

Even more complicated motion
Quantum billiards and neutron resonances n + $^{232}$Th

Transmission spectrum of a 3D-stadium billiard

![Graph showing transmission spectrum with dB scale and frequency axis ranging from 3.00 to 3.25 GHz.]

$T = 4.2$ K

Spectrum of neutron resonances in $^{232}$Th + n

![Graph showing neutron resonance spectrum with energy axis ranging from 100 to 3000 eV.]

- Great similarities between the two spectra: universal behaviour
Chaotic motion in nuclei

"Cold" (low excitation) rare-earth nuclei

High-Energy region, Nuclear Data Ensemble Slow neutron resonant data
Haq. et.al. PRL 48, 1086 (1982)
Pairing interaction in nuclei

\[ \Delta = 12A^{-1/2} \]

mass number $A$

neutron pairing gap

$N=8$

20 28 50 82 126
Rotation

![Graph showing Gamma-ray energy (keV) vs. Counts, with labels for 152Dy and potential energy curves indicating super-deformation and hyper-deformation stages.](image)
Evidence of nuclear superfluidity
Pairing Hamiltonian

- Pairing on degenerate time-conjugate orbitals
  \[ |1\rangle \leftrightarrow |\bar{1}\rangle \quad |j\tilde{m}\rangle = (-1)^{j-m}|j - m\rangle \]
- Pair operators \( P = (a_1a_1)_{J=0} \) (J=0, T=1)
- Number of unpaired fermions is seniority \( s \)
- Unpaired fermions are untouched by \( H \)

\[
H = \sum_1 \epsilon_1 N_1 - \sum_{12} G_{12} P_1^\dagger P_2
\]
Approaching the solution of pairing problem

• Approximate
  – BCS theory
    • HFB+correlations+RPA
  – Iterative techniques

• Exact solution
  – Richardson solution
  – Algebraic methods
  – Direct diagonalization + quasispin symmetry\(^1\)

BCS theory

**Trial wave-function**

\[ |0\rangle = \prod_{\nu} \left( u_\nu - v_\nu a_\nu^\dagger a_\nu^\dagger \right) |0\rangle, \text{ where } \begin{array}{c} |u_\nu|^2 + |v_\nu|^2 \\ \text{empty occupied} \end{array} = 1 \]

**Minimization of energy determines**

\[ |v_\nu|^2 = \frac{1}{2} \left( 1 - \frac{\epsilon_\nu - \mu}{e_\nu} \right), \quad |u_\nu|^2 = \frac{1}{2} \left( 1 + \frac{\epsilon_\nu - \mu}{e_\nu} \right) \]

**Gap equation**

\[ \Delta_\nu = \frac{1}{4} \sum_{\nu'} G_{\nu \nu'} \frac{\Delta_{\nu'}}{e_{\nu'}} \text{, where } e_\nu = \sqrt{(\epsilon_\nu - \mu)^2 + \Delta_\nu^2} \]
Low-lying states in paired systems

- **Exact treatment**
  - No phase transition and $G_{\text{critical}}$
  - Different seniorities do not mix
  - Diagonalize for pair vibrations

- **BCS treatment**

<table>
<thead>
<tr>
<th></th>
<th>$G &lt; G_{\text{critical}}$</th>
<th>$G &gt; G_{\text{critical}}$</th>
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<tbody>
<tr>
<td>Ground state</td>
<td>Hartree-Fock</td>
<td>BCS</td>
</tr>
<tr>
<td>Elementary excitations</td>
<td>single-particle excitations</td>
<td>quasiparticle excitation</td>
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<td></td>
<td>$E_{s=2} = 2,\varepsilon$</td>
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<tr>
<td>Collective excitations</td>
<td>HF+RPA</td>
<td>HFB+RPA</td>
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All nucleons are paired
- 1 broken pair
  - $s=2$
  - 2 q.p.
  - $J=2\ldots2j$
  - $J=0\ldots4j$

1 broken pair
- $s=0$
- 0 q.p.
- $J=0$

2 broken pairs
- $s=4$
- 4 q.p.
- $J=0\ldots4j$

- $2\Delta$
- $4\Delta$
Cooper Instability in mesoscopic system

BCS versus Exact solution

![Energy difference per particle vs Pairing strength G](image1)

![Pairing gap vs Pairing strength G](image2)
Statistical treatment of pairing

- Microcanonical\[ \hat{\rho}(E, N) = \delta(E - \hat{H})\delta(N - \hat{N}) \]
- Canonical\[ \hat{\rho}(\beta, N) = \exp(-\beta \hat{H}) \delta(N - \hat{N}) \]
- Grand canonical\[ \hat{\rho}(\beta, \mu) = \exp\left(-\beta(\hat{H} - \mu \hat{N})\right) \]

Partition functions
\[ Z = \text{Tr}(\hat{\rho}) \quad \text{and} \quad \hat{w} = \frac{\hat{\rho}}{Z}. \]

Statistical averages
\[ \langle \hat{O} \rangle = \frac{\text{Tr}(\hat{O} \hat{\rho})}{\text{Tr}(\hat{\rho})} = \text{Tr}(\hat{O} \hat{w}) \]

Entropy
\[ S = -\langle \ln(\hat{w}) \rangle = -\text{Tr}(\hat{w} \ln \hat{w}) \]
Is there thermalization?

\[ \rho(E) = \rho_0 e^{-\frac{(E-E_0)^2}{2\sigma^2}} \]
Pairing phase diagram
Microcanonical ensemble and thermodynamic limit

![Graphs showing entropy vs. excitation energy for different system sizes (N = 30, 50, 100) with microcanonical and canonical ensembles.]

$G = 1$
Statistical approach to mesoscopic system

Excitation energy [MeV]

N=12, half occupied ladder

Temperature [MeV] Excitation energy [MeV]
Phase Transition in Mesoscopic System

**Complex roots** - similar to charges
Appear symmetrically, never exactly on real axis

**Energy** - similar to potential
Roots become poles
Macroscopic accumulation of poles creates **charged surface**

\[ Z(B_j) = 0, \quad B_j = \beta_j + i\tau_j \]
\[ Z(\beta) = \Omega \prod_j \left( 1 - \frac{\beta}{B_j} \right) \left( 1 - \frac{\beta}{B_j^*} \right). \]
\[ E(\beta) = -\frac{\partial}{\partial \beta} \ln(Z) = \sum_j \left( \frac{1}{B_j - \beta} + \frac{1}{B_j^* - \beta} \right). \]
\[ C_V = \beta^2 \frac{\partial^2 \ln Z}{\partial \beta^2} = \beta^2 \sum_j \left( \frac{1}{(B_j - \beta)^2} + \frac{1}{(B_j^* - \beta)^2} \right). \]

http://www.cco.caltech.edu/~phys1/java/phys1/EField/EField.html
Classification of phase transitions zeros in the complex temperature plane

Main Characteristics

- Angle of approach
  \[ \nu = \arctan \frac{\beta_2 - \beta_1}{\tau_2 - \tau_1} \]

- Congestion of roots
  \[ |\beta_{i+1} - \beta_i| \sim \tau_i^{-\alpha} \]

Classification

First order \[ \nu = 0 \quad \alpha = 0 \]
Second order \[ 0 < \alpha \leq 1 \]
Higher order \[ 1 < \alpha \]
End of lecture
continue reading to learn more…

• Invariant correlational entropy
• Phase diagrams
• Open mesoscopic quantum systems
• Superradiance, quasi-stationary states in continuum
Invariant Correlational Entropy

- Parameter-driven equilibration (pairing strength)
- Averaged density matrix

\[ |\alpha\rangle = \sum_k C_k^\alpha (G) |k\rangle \]

\[ \hat{\rho} = \frac{1}{\delta G} \int_G^{G+\delta G} \hat{\rho}(G) \]

\[ \rho_{k k'}^\alpha = \langle k | \alpha \rangle \langle \alpha | k' \rangle \]

- ICE

\[ I^\alpha = -\text{Tr} (\overline{\rho^\alpha} \ln \overline{\rho^\alpha}) \]

Advantages
- Basis independent
- Explore individual quantum states
- Needs no heat bath
- No equilibration, thermalization and particle number conservation issues.
- Probe sensitivity of states to noise in external parameter(s)
- Phase transitions \(\rightarrow\) peaks in ICE
Mg Phase diagram

strength of $T=1$ pairing

$\lambda_{np} = 0.5$

$\lambda_{np} = 1$

$\lambda_{np} = 2$

realistic nucleus

strength of $T=0$ pairing
Exotic nuclei: Halo Nucleus $^{11}\text{Li}$

$^{11}\text{Li}$ is halo, it is as big a lead

Two neutrons in $^{11}\text{Li}$ are moving on decaying orbitals!

Two “valence” states are possible
Nuclear reaction theory
Quantum billiards with particle-leaks

- Due to finite lifetime states acquire width (uncertainty in energy $\Gamma = h/\tau$)
- Internal complex motion $\Leftrightarrow$ Radiation and decay?
Superradiance, collectivization by decay

**Dicke coherent state**
N identical two-level atoms coupled via common radiation

Single atom $\gamma$

Coherent state $\Gamma \sim N\gamma$

**Analog in nuclei**
Interaction via continuum
Trapped states $\Rightarrow$ self-organization

$\gamma \sim D$ and few channels
- Nuclei far from stability
- High level density (states of same symmetry)
- Far from thresholds
Shape vibration and GDR

\[ (2^+ \times 3^-)1^- \]

\[ B(E1) \]
Simple example: Two-spin system

- Two interacting spins \( H^o = \alpha \vec{s}_1 \cdot \vec{s}_2 \)
  - Spherical symmetry (triplet and singlet states)
- Magnetic field \( H^B = \epsilon \vec{s}_1^z + \epsilon \vec{s}_2^z = \epsilon S^z \)
  - Preserves spherical symmetry \( W = -i \frac{\gamma}{4} (\vec{s}_1^z - s) \)
- First spin in \( s^z=1/2 \) state decays
  - Reduces symmetry (\( S^z \) is preserved but not \( S^2 \))
• Hamiltonian for $S^z=0$

$$\mathcal{H} = -\frac{\alpha}{4} + \frac{1}{2} \left( \begin{array}{cc} -i\gamma & \alpha \\ \alpha & 0 \end{array} \right)$$

• Complex energies

$$\mathcal{E}_\pm = -\frac{\alpha}{4} \pm \frac{1}{2} \sqrt{\alpha^2 - \left(\frac{\gamma}{2}\right)^2} - i\frac{\gamma}{4}$$

Features of open system
• Incompatible symmetries
• Many-body versus single-spin properties
• Interaction of two resonances
• Superradiance and separation of states
• “Phase transition”
$^{11}\text{Li}$ an example of interacting resonances

$^{11}\text{Li}$ is stable, it is held by interaction of resonances

\[ \mathcal{H} = \mathcal{H}^0 + V - iW/2 \]

$^{11}\text{Li}$ is borromean, if one nucleon is removed it becomes unstable

$E_1$ $E_2$

Threshold

Real $V$

$V = 0$

$V \neq 0$

$E = $ $E_1$ $E_2$

Imaginary $W$

$W = 0$

$W \neq 0$
Scattering and cross section near threshold

Scattering Matrix

\[ S^{ab} = (s^a)^{1/2}(\delta^{ab} - T^{ab})(s^b)^{1/2} \]

where \( s^a = \exp(i\delta_a) \)

is smooth scattering phase

\[ T^{ab} = \sum_{12} A_1^{a*} \left( \frac{1}{E - \mathcal{H}} \right)_{12} A_2^b \]

Solution in two-level model

\[ T(E) = \frac{E(\gamma_1 + \gamma_2) - \gamma_1 \varepsilon_2 - \gamma_2 \varepsilon_1 - 2uA_1A_2}{(E - \varepsilon_+)(E - \varepsilon_-)} \]

Cross section

\[ \sigma(E) = \frac{\pi}{k^2} |S(E) - 1|^2 \]
Single-particle decay in many-body system

Evolution of complex energies $E = E - i \frac{\Gamma}{2}$ as a function of $\gamma$

- Assume energy independent $W$
- Assume one channel $\gamma = A^2$
- System 8 s.p. levels, 3 particles
- One s.p. level in continuum $e = \varepsilon - i\gamma/2$

Total states $8!/(3! \times 5!) = 56$; states that decay fast $7!/(2! \times 5!) = 21$